A Langlands Program for Covering Groups?

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Abstract. We describe a still largely speculative and conjectural extension of the Langlands program to a class of nonlinear covering groups studied by Brylinski and Deligne. This is an exposition of the work of many people, especially the recent work of Martin Weissman, Peter McNamara and Wen-Wei Li.

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1. Covering Groups.

The classical Langlands program concerns representations and automorphic forms of a connected reductive linear algebraic group G defined over a local or global field F, The class of linear algebraic groups enjoy several good properties:

- Naturality: these groups first arise as connected compact Lie groups which intervene in many walks of mathematical life.
- Algebraicity: even though they started life as topological groups, their theory is in fact totally algebraic in the sense that they are the F-points of an algebraic variety over F.
- Functoriality: Being algebraic in nature, they have good functorial properties, such as with respect to base change.
- Inductivity: Levi factors of parabolic subgroups of reductive groups are reductive. This provides a framework to treat their representation theory inductively.
- Classification: they admit a simple classification by combinatorial data, namely root systems and Dynkin diagrams.

More precisely, pick a maximal torus $T \subset G$ defined over F. To (G, T), we can attach a quadruple:

$$(X(T), \Phi, Y(T), \Phi^{\vee})$$

called the root datum of G:

- $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ is the character group of T;
- $Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ is the cocharacter group of T;

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- $\Phi = \{\text{roots } \alpha \text{ of } (G,T)\} \subset X(T)$;
- $\Phi^\vee = \{\text{coroots } \alpha^\vee \text{ of } (G,T)\} \subset Y(T)$.

Such root data classify the connected reductive linear algebraic groups up to isomorphism over algebraically closed fields.

In this expository paper, we are interested in a larger class of groups called *nonlinear covering groups*. More precisely, a central extension of $G(F)$ by a discrete (finite) group μ is a short exact sequence:

$$1 \longrightarrow \mu \longrightarrow \tilde{G} \xrightarrow{p} G(F) \longrightarrow 1$$

with μ in the centre of $\tilde{G}$. One may consider such extensions in the category of abstract groups, or in the category of topological groups, in which case we require p to be open and continuous. Thus, such groups are in general not algebraic in nature: they are not the F-points of an algebraic group.

1.1. 2-cocycles. Such covering groups are often described using 2-cocycles. Indeed, central extensions of $G(F)$ by μ are classified by the cohomology group:

$$H^2(G(F),\mu) \quad \text{for abstract extensions}$$

or

$$H^2_m(G(F),\mu) \quad \text{for topological extensions}$$

where one considers measurable cochains. Given

$$1 \longrightarrow \mu \longrightarrow \tilde{G} \xrightarrow{p} G(F) \longrightarrow 1,$$

one picks a (measurable) section $j : G(F) \longrightarrow \tilde{G}$ of p, and set

$$c(g,h) = j(g) \cdot j(h) \cdot j(gh)^{-1}.$$

Then c is a 2-cocyle and its class in $H^2_m(G(F),\mu)$ is well-defined.

Conversely, given a 2-cocycle c, $\tilde{G}$ is given as a set by $G(F) \times \mu$, with group law

$$(g,\alpha) \cdot (h,\beta) = (g \cdot h, \alpha \cdot \beta \cdot c(g,h)).$$

1.2. Some examples. Let us look at some examples of such covering groups. Let G be a simple split simply-connected linear algebraic group over $\mathbb{R}$. Then $G(\mathbb{R})$ is a connected real Lie group, but it is not necessarily simply-connected as a real Lie group. If $K \subset G(\mathbb{R})$ is a maximal compact subgroup, then

$$\pi_1(G(\mathbb{R})) = \pi_1(K).$$

Here are some examples:

G	$\mathrm{SL}_n(\mathbb{R})$	$\mathrm{Sp}_{2n}(\mathbb{R})$	$G_2(\mathbb{R})$	$F_4(\mathbb{R})$
K	$\mathrm{SO}_n(\mathbb{R})$	$\mathrm{U}_n(\mathbb{C})$	$(\mathrm{SU}_2 \times \mathrm{SU}_2)/\Delta\mu_2$	$(\mathrm{SU}_2 \times \mathrm{Sp}_6^c)/\Delta\mu_2$
$\pi_1(G)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

When $\pi_1(K)$ is nontrivial, the universal cover $\tilde{G}$ of $G(\mathbb{R})$ is then a nonlinear covering group. It is interesting to note that Harish-Chandra's work on non-abelian harmonic analysis includes such groups.

1.3. Metaplectic Group $\mathrm{Mp}_{2n}(F)$**.** Another classic example of a covering group is the metaplectic group $\mathrm{Mp}_{2n}(F)$ defined by Weil:

$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Mp}_{2n}(F) \xrightarrow{p} \mathrm{Sp}_{2n}(F) \longrightarrow 1$$

It is a 2-fold cover of the symplectic group $\mathrm{Sp}_{2n}(F)$.

This group has a family of distinguished representations: the Weil representations or oscillator representations which intervene in the theory of the quantum harmonic oscillator and provides a representation theoretic framework for the theory of theta functions. It is one of the best understood covering group, but even so, it is not easy to write down a 2-cocycle.

1.4. Steinberg's Group. Yet another example is a group constructed by Steinberg [**St**]. Let

- $G \neq \mathrm{SL}_2$ be a split, absolutely simple and simply connected linear algebraic group over F;
- T be a maximal split torus of G;
- $\Phi = \Phi(G,T)$ the associated root system, assumed to be simply laced for simplicity.
- $\{x_\alpha : \mathbb{G}_a \to U_\alpha : \alpha \in \Phi\}$ a Chevalley pinning (or épinglage).

Then one has:

$$G(F) = \langle x_\alpha(t),\ \alpha \in \Phi,\ t \in F\rangle/(\text{relations (A) and (B)}).$$

where the relations (A) and (B) are:

(A) For $\alpha + \beta \neq 0$,

$$[x_\alpha(t), x_\beta(u)] = \begin{cases} x_{\alpha+\beta}(\pm tu) \text{ if } \alpha+\beta \in \Phi; \\ 0 \text{ else.} \end{cases}$$

(B) If we set

$$w_\alpha(t) = x_{-\alpha}(t) \cdot x_\alpha(-1/t) \cdot x_{-\alpha}(t)$$
$$h_\alpha(t) = w_\alpha(t) \cdot w_\alpha(-1),$$

then

$$h_\alpha(s \cdot t) = h_\alpha(s) \cdot h_\alpha(t) = h_\alpha(t) \cdot h_\alpha(s).$$

Let E be the abstract group generated by the $x_\alpha(t)$'s subject to (A) only, so that there is a short exact sequence:

$$1 \longrightarrow C \longrightarrow E \longrightarrow G(F) \longrightarrow 1.$$

The group E is the Steinberg group and it is a nonlinear covering group of $G(F)$.

1.5. Universal abstract extension. The following is a combination of results of Steinberg [**St**], Moore [**M**] and Matsumoto [**Ma**]:

THEOREM 1.1. *(i) The extension E is the universal abstract central extension, in the sense that for any abelian group A,*

$$H^2(G,A) = \mathrm{Hom}(C,A).$$

(ii) When $G \neq \mathrm{Sp}_{2n}$, C is generated by the elements

$$(s,t) = h_\alpha(st) \cdot h_\alpha(s)^{-1} \cdot h_\alpha(t)^{-1},$$

which are independent of $\alpha \in \Phi$, subject to the relations:

(i) $(s,t)\cdot(t,s) = 1$;
(ii) (s,t) *is bimultiplicative;*
(iii) $(s,t) = 1$ *if* $s+t=1$.

1.6. Milnor-Quillen K_2. The above theorem is subsequently extended to all isotropic simply-connected G by Deodhar [**D**] and Prasad-Raghunathan [**PR1, PR2**], and E is called the universal abstract covering group. Its kernel C is the Milnor-Quillen K_2-group of F:

$$K_2(F) = F^\times \otimes_{\mathbb{Z}} F^\times/(\text{relations (i) - (iii)}).$$

A map

$$F^\times \times F^\times \longrightarrow A$$

satisfying the relations (i) - (iii) is called an A-valued symbol, and the map

$$(-,-) : F^\times \times F^\times \longrightarrow K_2(F)$$

is the universal symbol. Thus, any A-valued symbol arises from a group homomorphism $K_2(F) \longrightarrow A$.

1.7. Universal Topological Extension and Hilbert Symbol. Topological coverings of $G(F)$ are obtained from the universal abstract extension by applying locally constant symbols [**M, Ma**].

THEOREM 1.2. *The universal locally constant symbol is the Hilbert symbol*

$$(-,-)_F : F^\times \times F^\times \to K_2(F) \to \mu(F)$$

where

$$\mu(F) = \{\textit{roots of unity in } F\}.$$

For every simply-connected isotropic simple algebraic group over F, we have thus seen that there is a canonical abstract extension

$$1 \longrightarrow K_2(F) \longrightarrow E \longrightarrow G(F) \longrightarrow 1$$

and a canonical topological extension:

$$1 \longrightarrow \mu(F) \longrightarrow E \longrightarrow G(F) \longrightarrow 1.$$

However, this nice picture becomes a bit murky once G is not simply-connected or isotropic:

- there is no universal cover, because of presence of automorphisms;
- there are too many coverings, with no reasonable classification;
- it is hard to describe any given cover in terms of 2-cocycles;
- there are no good functorial properties.

Thus it is reasonable to ask if there is a class of covering groups which possesses better functorial properties, admits a sensible classification and yet is sufficient large to include all interesting examples. We will discuss such a class of covering groups singled out by Brylinski-Deligne [**BD**] in the next section.

2. Brylinski-Deligne Extensions

Let us return to the natural extension constructed by Steinberg when G is simply-connected:

$$1 \longrightarrow K_2(F) \longrightarrow E \longrightarrow G(F) \longrightarrow 1$$

Brylinski-Deligne's idea is to remove the F and consider the extension

$$1 \longrightarrow K_2 \longrightarrow E \longrightarrow G \longrightarrow 1$$

in an appropriate category. More precisely, they considered the problem:

Problem: Classify all central extensions of a connected reductive linear algebraic group G over F by Quillen's K_2-sheaf in the category of sheaves of groups on the big Zariski site of F-schemes of finite type.

Such extensions are also called *multiplicative K_2-torsors on G*, and we shall refer to them as BD-extensions.

2.1. Classification of BD extensions.

Brylinski-Deligne [**BD**] gave a classification of the category (a groupoid actually) of multiplicative K_2-torsors on G. We shall describe their answer in two cases:

- $G = T$ is a torus;
- G is simply-connected absolutely simple.

The classification will be in terms of root theoretic data, with some extra enhancements.

2.2. Simply-Connected G.

When G is simply connected, one has the following theorem.

THEOREM 2.1. *The category of multiplicative K_2-torsors on G is equivalent to the rigid groupoid with objects*

$$Q : Y(T) \to \mathbb{Z},$$

which are $W(G,T) \rtimes \mathrm{Gal}(\overline{F}/F)$-invariant $\mathbb{Z}$-valued quadratic forms. Here $W(G,T)$ is the Weyl group of T in G.

For example, if G is split and absolutely simple, then such a Q is determined by $Q(\alpha^\vee)$ where $\alpha^\vee$ is a short coroot. So the set of isomorphism classes of objects is just $\mathbb{Z}$.

2.3. Tori.

Let T be a torus defined over F, with cocharacter group $Y(T)$, which is a $\mathrm{Gal}(\overline{F}/F)$-module.

THEOREM 2.2. *The category of multiplicative K_2-torsors on T is equivalent to the category with*

(i) *Objects: pairs $(Q, \mathcal{E})$ where*
   - $Q : Y(T) \to \mathbb{Z}$ *is a* $\mathrm{Gal}(\overline{F}/F)$*-invariant $\mathbb{Z}$-valued quadratic form;*

– *$\mathcal{E}$ is a* $\mathrm{Gal}(\overline{F}/F)$*-equivariant extension of abstract groups*

$$1 \longrightarrow F^{\times} \longrightarrow \tilde{Y} \longrightarrow Y(T) \longrightarrow 1$$

whose commutator is given by

$$[\tilde{y_1}, \tilde{y_2}] = (-1)^{b_Q(y_1, y_2)},$$

with $b_Q(y_1, y_2) = Q(y_1 + y_2) - Q(y_1) - Q(y_2)$.

(ii) *Morphisms:* $\mathrm{Hom}((Q, \mathcal{E}), (Q', \mathcal{E}'))$ *is nonempty only if* $Q = Q'$*, in which case a morphism is a* $\mathrm{Gal}(\overline{F}/F)$*-equivariant isomorphism of extensions:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F^{\times} & \longrightarrow & \tilde{Y} & \longrightarrow & Y(T) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \| & & \\ 1 & \longrightarrow & F^{\times} & \longrightarrow & \tilde{Y}' & \longrightarrow & Y(T) & \longrightarrow & 1 \end{array}$$

For general reductive G, the classification is an amalgam of the above two cases and is slightly more complicated to describe. However, the point to note here is that the classification of BD-extensions is largely root-theoretic in nature. As such, it is not too far from the familiar classification of linear algebraic groups by their root data.

2.4. BD covering groups. Starting with a multipicative K_2-torsor

$$1 \longrightarrow K_2 \longrightarrow E \longrightarrow G \longrightarrow 1,$$

one may take F-points to get an extension of abstract groups:

$$1 \longrightarrow K_2(F) \longrightarrow E(F) \longrightarrow G(F) \longrightarrow H^1(F, K_2) = 1.$$

Pushing out by the Hilbert symbol $K_2(F) \longrightarrow \mu(F)$, one obtains:

$$1 \longrightarrow \mu(F) \longrightarrow \tilde{G} \longrightarrow G(F) \longrightarrow 1$$

If n is such that $\mu_n(\overline{F}) \subset \mu(F)$, then one has a natural map $\mu(F) \longrightarrow \mu_n(F)$ and one can push out further to get

$$1 \longrightarrow \mu_n(F) \longrightarrow \tilde{G} \longrightarrow G(F) \longrightarrow 1$$

This gives a class of (topological) covering groups, associated to (E, n), which we will call the BD-covering groups.

Though the set of all BD-extensions does not exhaust all covering groups, there are some good reasons to believe that they are a good class of covering groups to consider:

- Algebraicity: they begin life in the world of algebraic geometry;
- Functoriality: BD data behaves reasonably with respect to base change, pullbacks and other natural operations in algebraic geometry;
- Exhaustive: BD coverings capture a large class of covering groups, e.g. all covers for simply-connected G;
- Classification: they admit an essentially root theoretic classification.

Given these advantages, we would like to ask:

Question: Do BD covering groups give a good framework for extending the Langlands program?

3. Local Langlands Correspondence (LLC)

If F is a local field and G is a linear algebraic group over F, the LLC gives a classification

$$\begin{array}{c} \mathrm{Irr}(G(F)) = \{\text{irreducible representations of } G(F)\} \\ \downarrow \quad \text{finite-to-one} \\ \Phi(G(F)) = \{L\text{-parameters for } G \text{ over } F\} \end{array}$$

We describe some of the objects appearing in this statement more precisely.

3.1. *L*-parameter. An L-parameter is an equivalence class of homomorphisms

$$\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G(\mathbb{C})$$

(with some properties), where

- W_F = the Weil group of F; it is almost the same as the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$.
- ${}^L G(\mathbb{C})$ = the *Langlands L-group of G over F*:
$$ {}^L G(\mathbb{C}) = G^\vee \rtimes W_F \xrightarrow{pr} W_F $$
where its identity component $G^\vee$ is the so-called Langlands dual group of G.

One requirement for ϕ is that

$$pr \circ \phi|_{W_F} = \text{the identity map on } W_F$$

and equivalence is defined by $G^\vee$-conjugacy. So an L-parameter is more or less a local Galois representation valued in the connected complex Lie group $G^\vee(\mathbb{C})$.

Here is another view on L-parameters. One has a (split) short exact sequence

$$1 \longrightarrow G^\vee \longrightarrow {}^L G \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow 1$$

Then an L-parameter is simply a $G^\vee$-orbit of splittings of this short exact sequence.

3.2. The dual group $G^\vee$. Recall that for a connected reductive G, with maximal torus T, one can associate its root datum:

$$(X(T), \Phi, Y(T), \Phi^\vee).$$

The quadruple

$$(Y(T), \Phi^\vee, X(T), \Phi)$$

can be shown to be a root datum as well. Since root data classify the connected reductive linear algebraic groups over algebraically closed fields, this root datum corresponds to a linear algebraic group $G^\vee$ over $\mathbb{C}$. This complex Lie group $G^\vee$ is the Langlands dual group of G.

Here is a table of examples of $G^\vee(\mathbb{C})$:

G	GL_n	Sp_{2n}	SO_{2n+1}	SO_{2n}	E_8
$G^\vee$	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{SO}_{2n+1}(\mathbb{C})$	$\mathrm{Sp}_{2n}(\mathbb{C})$	$\mathrm{SO}_{2n}(\mathbb{C})$	$E_8(\mathbb{C})$

4. Genuine Representations

For a covering group

$$1 \longrightarrow \mu \longrightarrow \tilde{G} \longrightarrow G(F) \longrightarrow 1,$$

fix an embedding

$$\epsilon : \mu \hookrightarrow \mathbb{C}^{\times}.$$

We are interested in the set of ϵ-genuine representations of $\tilde{G}$, namely representations π such that

$$\pi|_{\mu} = \epsilon.$$

The question we want to consider is:

Question: Is there a classification of $\mathrm{Irr}_{\epsilon}(\tilde{G})$ in the style of the LLC for linear algebraic groups?

This leads naturally to:

Question: What should be the dual group or the L-group of $\tilde{G}$?

In this section, we shall discuss some results which shed some light on these two questions.

4.1. Unramified Langlands Correspondence. One of the initial evidence for the Langlands program is Langlands' re-interpretation of Satake's work on spherical functions and unramified representations, which leads to the creation of the Langlands L-group. This re-interpretation gives a classification of unramified representations, namely a bijection:

$$\{\text{Unramified representations of } G(F)\} \longleftrightarrow \{\text{Unramifed L-parameters for } G\},$$

which is a partial realization of the LLC.

Assume that G is split henceforth for simplicity. Let $K \subset G(F)$ be a hyperspecial maximal compact subgroup. We define the notions of unramified representations and L-parameters:

- An irreducible representation (π, V) of $G(F)$ is unramified if
$$V^K \neq 0.$$
In this case, one knows that $\dim V^K = 1$.
- Let $I_F \subset W_F$ be the inertia group. An L-parameter
$$\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G = G^{\vee} \times W_F$$
is unramified if
$$\phi(w, g) = (1, w)$$
for $(w, g) \in I_F \times \mathrm{SL}_2(\mathbb{C})$.

4.2. Satake Isomorphism. It is not hard to see that the map $V \mapsto V^K$ defines a bijection of the set of irreducible unramified representations of $G(F)$ with the irreducible characters of the spherical Hecke algebra:

$$H(G(F), K) = C_c^{\infty}(K \backslash G(F)/K).$$

The following is Langlands' re-interpretation of Satake's results:

Theorem 4.1. *There is a natural algebra isomorphism (the Satake isomorphism)*

$$H(G(F),K) \longrightarrow H(T(F),K_T)^W,$$

where T is a maximal split torus with Weyl group W and $K_T = K \cap T(F)$.

Corollary 4.2.

$$H(G(F),K) \cong \mathbb{C}[Y]^W \cong R[G^\vee],$$

where $R[G^\vee]$ denotes the Grothendieck ring of $G^\vee$. In particular, there is a bijection

$$\{\text{irreducible unramified representations of } G(F)\} \longleftrightarrow \{\text{semisimple conjugacy classes in } G^\vee\}.$$

4.3. Unramified Representations for BD coverings. Suppose now that $\tilde{G}$ is a BD-covering of $G(F)$ which is split over K. Fix such a splitting

$$s : K \hookrightarrow \tilde{G}.$$

Then a genuine representation π of $\tilde{G}$ is unramified (with respect to s) if

$$\pi^{s(K)} \neq 0.$$

As before, the set of such irreducible unramified representations is in natural bijection with the irreducible characters of a genuine spherical Hecke algebra:

$$H_\epsilon(\tilde{G},K,s) = C^\infty_{c,\epsilon}(s(K)\backslash\tilde{G}/s(K)).$$

The first systematic study of such genuine unramified representations was undertaken by Savin in [**S2**].

4.4. Satake Isomorphism for BD coverings. Recently, the Satake isomorphism has been established in the setting of BD coverings in works of P. McNamara [**Mc**], Wenwei Li [**L3**] and M. Weissman [**W4**]:

Theorem 4.3. *One has a natural algebra isomorphism*

$$H_\epsilon(\tilde{G},K,s) \to H_\epsilon(\tilde{T},s(K_T))^W.$$

Could we interpret the RHS using root theoretic data or as the Grothendieck ring of some complex Lie group?

4.5. Metaplectic modification of root datum. Recall that G has a root datum $(X,\Phi,Y,\Phi^\vee)$. Suppose $\tilde{G}$ is a BD-covering with data (E,n), with $n = |\mu|$. In the Brylinski-Deligne classification of the multiplicative K_2-torsor E, one of the ingredients is a quadratic form Q. We are going to modify the root datum of G using (Q,n).

- Set
$$Y_{Q,n} = \{y \in Y : b_Q(y,z) \in n\mathbb{Z} \text{ for all } z \in Y\}.$$
- For $\alpha^\vee \in \Phi^\vee$, set
$$n_{\alpha^\vee} = \frac{n}{(n, Q(\alpha^\vee))},$$
and
$$\tilde{\alpha}^\vee = n_{\alpha^\vee} \cdot \alpha^\vee$$
and
$$\tilde{\Phi}^\vee = \{\tilde{\alpha}^\vee : \alpha^\vee \in \Phi^\vee\}.$$

- Likewise, set:
$$X_{Q,n} = \mathrm{Hom}(Y_{Q,n}, \mathbb{Z}).$$
and
$$\tilde{\Phi} = \{n_{\alpha^\vee}^{-1} \cdot \alpha : \alpha \in \Phi\}.$$

Then we have:

Theorem 4.4. *The quadruple*
$$(Y_{Q,n}, \tilde{\Phi}^\vee, X_{Q,n}, \tilde{\Phi}\}$$
is a root datum. If $\tilde{G}^\vee$ is the associated complex Lie group, then the Satake isomorphism givens an isomorphism
$$H_\epsilon(\tilde{G}, K, s) \cong \mathbb{C}_\epsilon[\tilde{Y}_{Q,n}]^W.$$
where we have an extension induced by $\tilde{T}$ and s:
$$1 \longrightarrow \mu \longrightarrow \tilde{Y}_{Q,n} \longrightarrow Y_{Q,n} \longrightarrow 1.$$
The latter algebra $\mathbb{C}_\epsilon[\tilde{Y}_{Q,n}]^W$ is non-canonically isomorphic to $R[\tilde{G}^\vee]$.

4.6. Dual group of a BD covering. The group $\tilde{G}^\vee$ has also arised in the geometric Langlands program, through the work of Finkelberg-Lysenko [**FL**] and Riech [**Ri**]. In any case, the Satake isomorphism suggests that the dual group of a BD covering $\tilde{G}$ should be the group $\tilde{G}^\vee$ defined above.

Here are some examples:

- if $\tilde{G} = \mathrm{Mp}_{2n}$, then
$$\tilde{G}^\vee = \mathrm{Sp}_{2n}(\mathbb{C}).$$
- if G is split simple simply connected and simply-laced, let $Z(G)_n$ be the n-torsion in the centre of G. Then
$$\tilde{G}^\vee = (G/Z(G)_n)^\vee.$$

However, because of the non-canonicity at the end of the last theorem, one cannot quite associate a semisimple conjugacy class of $\tilde{G}^\vee$ to an unramified representation of $\tilde{G}$. We shall see how this can be resolved next.

4.7. L-group of a BD Covering. What about the L-group of $\tilde{G}$? If G is split, then it is natural to want to define the L-group as a direct product:
$${}^L\tilde{G} = \tilde{G}^\vee \times W_F.$$
But experience with tori, Mp_{2n} and the Satake isomorphism given in the previous theorem suggests that it is better to consider the L-group as an extension
$$1 \longrightarrow \tilde{G}^\vee \longrightarrow {}^L\tilde{G} \longrightarrow W_F \longrightarrow 1 \tag{4.1}$$
which is split, but with no canonical splitting. In other words, ${}^L\tilde{G}$ is isomorphic to $\tilde{G}^\vee \times W_F$, but not canonically so. This subtle shift in the point of view is very important, as we shall see in a moment.

In a recent paper of M. Weissman [**W3**], such a construction of ${}^L\tilde{G}$ has been given, but it seems overly complicated. Recently, Weissman has given a much simplified version of his construction, following suggestions of Deligne; see [**W4**].

4.8. Hope of an LLC.. Having

$$1 \longrightarrow \tilde{G}^\vee \longrightarrow {}^L\tilde{G} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow 1,$$

one can define an L-parameter for $\tilde{G}$ to be a $\tilde{G}^\vee$-orbit of splittings of the above short exact sequence. It is then natural to hope for a map:

$$\mathrm{Irr}_\epsilon(\tilde{G}) \xrightarrow{\text{finite-to-1}} \{L\text{-parameters for } \tilde{G}\}.$$

This would be an LLC for $\tilde{G}$, which I learn from M. Weissman.

Note that there is no canonical splitting of the above short exact sequence and hence no distinguished L-parameter. This reflects the fact that there is no distinguished irreducible genuine representation. This is unlike the case of linear group $G(F)$, where one has the trivial representation of $G(F)$ with corresponding distinguished L-parameter given by the "cocharacter" ρ (half sum of positive roots), which is the ρ-shift of the trivial L-parameter.

We shall now discuss some supporting evidence for the above hope.

4.9. Covering tori. The following is a theorem of M. Weissman [**W1**]:

THEOREM 4.5. *If T is a split torus, and $\tilde{T}$ a BD-covering, then there is a natural map*

$$\mathrm{Irr}_\epsilon(\tilde{T}) \longrightarrow \{L\text{-parameters for } \tilde{T}\}$$

whose fiber has size 0 *or* 1*. Moreover, one can describe the image of the above map precisely.*

Weissman has results for BD-covers of more general tori, for example unramified tori,but we will not discuss these here.

4.10. Unramified representations. As shown in [**W4**], the splitting $s : K \to \tilde{G}$ gives rise to a splitting of the short exact sequence (4.1), and one say that a splitting of (4.1) is an unramified L-parameter (relative to s). Then the Satake isomorphism gives a canonical bijection

$$\{\text{irreducible unramified genuine representations of } \tilde{G}\}$$

$$\updownarrow$$

$$\{\text{unramified } L\text{-parameters for } \tilde{G}\}.$$

4.11. Iwahori-Hecke algebras. Suppose G split simply-connected, simply-laced, and $\tilde{G}$ is its degree n cover. Let

$$I \subset K \subset G(F)$$

be an Iwahori subgroup. If $(n,p) = 1$, then the covering splits uniquely over K:

$$s : K \hookrightarrow \tilde{G}.$$

Savin [**S1**] has shown:

THEOREM 4.6. *There s a natural algebra isomorphism*

$$H_\epsilon(\tilde{G}, I, s) \cong H((G/Z(G)_n)(F), I').$$

This relates the representations of $\tilde{G}$ with $s(I)$-fixed vectors with the representations of $(G/Z(G)_n)(F)$ with I'-fixed vectors, and is in accordance with the fact that

$$\tilde{G}^\vee = (G/Z(G)_n)^\vee.$$

4.12. Mp_{2n}**.** The following was show by Savin and myself [**GS**]:

THEOREM 4.7. *Fix a nontrivial additive character ψ of F. There is a bijection, depending on ψ,*

$$\mathrm{Irr}_\epsilon(\mathrm{Mp}_{2n}(F)) \longleftrightarrow \mathrm{Irr}(\mathrm{SO}_{2n+1}(F)) \sqcup \mathrm{Irr}(\mathrm{SO}'_{2n+1}(F)),$$

Assuming the LLC for SO_{2n+1}*, one has a map*

$$\mathrm{Irr}_\epsilon(\mathrm{Mp}_{2n}(F)) \longrightarrow \{\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2n}(\mathbb{C})\}$$

which depends on ψ.

If one works with the L-group of Mp_{2n} rather than of SO_{2n+1}, then one should be able to refine this to a map independent of ψ:

$$\mathrm{Irr}_\epsilon(\mathrm{Mp}_{2n}(F)) \longrightarrow \{\text{L-parameters for } \mathrm{Mp}_{2n}(F)\}.$$

5. Global Langlands Correspondence

Now suppose F is a global field with ring of adeles $\mathbb{A}$. An automorphic form on G is a function

$$f : G(F)\backslash G(\mathbb{A}) \longrightarrow \mathbb{C}$$

with some smoothness and finiteness properties. One considers the $G(\mathbb{A})$-representation

$$L^2(G(F)\backslash G(\mathbb{A})) = L^2_{disc}[G] \oplus L^2_{cont}[G].$$

Then

$$L^2_{disc}[G] = \bigoplus_\pi m(\pi) \cdot \pi$$

The global Langlands correspondence classifies the set

$$\{\text{irreducible representations } \pi \text{ of } G(\mathbb{A}) \text{ with } m(\pi) > 0\}$$

of square-integrable automorphic representations.

For BD-coverings, we would like to consider the analogous problem.

5.1. Global BD coverings. Given a multiplicative K_2-torsor

$$1 \longrightarrow K_2 \longrightarrow E \longrightarrow G \longrightarrow 1$$

one inherits:

$$1 \longrightarrow \mu(F) \longrightarrow \tilde{G}_{\mathbb{A}} \longrightarrow G(\mathbb{A}) \longrightarrow 1$$

equipped with a natural splitting

$$i : G(F) \hookrightarrow \tilde{G}_{\mathbb{A}}.$$

Using this, one may identify $G(F)$ with a subgroup of $\tilde{G}_{\mathbb{A}}$.

5.2. Genuine Automorphic Forms. For a fixed

$$\epsilon : \mu \longrightarrow \mathbb{C}^\times$$

one can consider the unitary representation

$$L^2_\epsilon(G(F)\backslash\tilde{G})$$

on the space of ϵ-genuine automoprhic forms. We are interested in the decomposition of this unitary representation.

5.3. Harmonic analysis and trace formula. In fact, many familiar harmonic analytic results for linear groups extend to the case of covering groups with little modification.

- Locally, many results of Harish-Chandra and others on local harmonic analysis has been extended with great efficiency to $\tilde{G}$ by Wen-Wei Li [**L4**]: Plancherel theorem, local integrality of character distributions, local character expansion, local trace formula,......

- Globally, the theory of Eisenstein series and spectral decomposition of $L^2_\epsilon(G(F)\backslash\tilde{G})$ has been established by Moeglin-Waldspurger [**MW**]; this reduces one to the study of the automorphic discrete spectrum.

- In a series of recent papers [**L3, L4, L5**], Wen-Wei Li has established the Invariant Trace Formula for $\tilde{G}$.

5.4. A theory of endoscopy? The natural next question: is there a theory of endoscopy for covering groups?

- for Mp_{2n}, Wen-Wei Li [**L1**] has developed a theory of endoscopy, so that the elliptic endoscopic groups of Mp_{2n} are:

$$\mathrm{SO}_{2a+1} \times \mathrm{SO}_{2b+1}$$

  where one considers all unordered pairs of non-negative integers (a, b) with $a + b = n$. There are thus $n + 1$ elliptic endoscopic groups in all. This extends earlier work of Renard [**R1, R2**] in the real case.

- for arbitrary cover of SL_2, Hiraga and Ikeda [**HI**] have developed such a theory of endoscopy.

5.5. Work of Wenwei Li. We describe Wen-Wei Li's results [**L1, L2**] for Mp_{2n} in greater detail here:

- For each pair (a, b) with $a + b = n$, Li defined transfer factors

$$\Delta_{a,b} : \mathrm{Mp}_{2n} \times (\mathrm{SO}_{2a+1} \times \mathrm{SO}_{2b+1}) \to \mathbb{C}.$$

- He showed existence of transfer mappings

$$\begin{array}{c} C_c^\infty(\mathrm{Mp}_{2n}) \\ {\scriptstyle t_{a,b}}\big\downarrow \\ C_c^\infty(\mathrm{SO}_{2a+1} \times \mathrm{SO}_{2b+1}), \end{array}$$

giving rise to a transfer

$$\begin{array}{c}\mathcal{D}_{st}(\mathrm{SO}_{2a+1}\times \mathrm{SO}_{2b+1}) \\ t^*_{a,b}\big\downarrow \\ \mathcal{D}_{gen}(\mathrm{Mp}_{2n}),\end{array}$$

between spaces of distributions.

- Finally, he established the ordinary and weighted fundamental lemmas for the transfer $t_{a,b}$.

Now one is in a position to consider the stabilisation of the invariant trace formula for Mp_{2n}. In a recent preprint [**L6**], Li has completed the stabilisation of the elliptic part of the race formula for Mp_{2n}.

5.6. Arthur's conjecture for Mp_{2n}**.** We describe the analog of Arthur's conjecture for Mp_{2n}. For a fixed additive automorphic character ψ, one expects that

$$L^2_{disc} = \bigoplus_{\Psi} L^2_{\Psi,\psi}$$

where

$$\Psi = \bigoplus_i \Psi_i = \bigoplus_i \Pi_i \boxtimes S_{r_i}$$

is a global discrete A-parameter for Mp_{2n}; it is also an A-parameter for SO_{2n+1}. Here, S_{r_i} is the r_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ and Π_i is a cuspidal representation of GL_{n_i} such that

$$\begin{cases} L(s,\Pi_i,\wedge^2) \text{ has a pole at } s=1, \text{ if } r_i \text{ is odd;} \\ L(s,\Pi_i,\mathrm{Sym}^2) \text{ has a pole at } s=1, \text{ if } r_i \text{ is even.}\end{cases}$$

Moreover, we have $\sum_i n_i r_i = 2n$ and the summands Ψ_i are mutually distinct.

For a given Ψ, one obtains the following extra data:

- for each v, one inherits a local A-parameter

$$\Psi_v = \bigoplus_i \Psi_{i,v} = \bigoplus_i \Pi_{i,v} \boxtimes S_{r_i}.$$

  By the local Langlands correspondence for GL_N, we may regard each $\Pi_{i,v}$ as an n_i-dimensional representation of the Weil-Deligne group WD_{F_v}. Hence, we may regard Ψ_v as a $2n$-dimensional representation of $WD_F \times \mathrm{SL}_2(\mathbb{C})$.
- one has a global component group

$$A_\Psi = \bigoplus \mathbb{Z}/2\mathbb{Z}\cdot a_i$$

  which is a $\mathbb{Z}/2\mathbb{Z}$-vector space equipped with a distinguished basis indexed by Ψ_i. Similarly, for each v, we have the local component group A_{Ψ_v} which is defined as the component group of the centralizer of the image of Ψ_v, thought of as a representation of $WD_F \times \mathrm{SL}_2(\mathbb{C})$. Observe that

$$A_{\Psi_v} = \prod_i A_{\Psi_{i,v}}.$$

- For each v, there is a natural map
$$\Delta_v : A_\Psi \longrightarrow A_{\Psi_v}$$
by
$$\Delta_v(a_i) = (0, \cdots, -Id, \cdots 0)$$
where the only nonzero component is the i-th one, and the i-th component is the image of the element $-Id$ in $A_{\Psi_{i,v}}$. Thus, there is a natural diagonal map
$$\Delta = \prod_v \Delta_v : A_\Psi \longrightarrow \prod_v A_{\Psi_v}.$$
- For each v, one has a local A-packet associated to Ψ_v and the additive character ψ_v:
$$\Pi_{\Psi_v,\psi_v} = \{\sigma_{\eta_v} : \eta_v \in \mathrm{Irr}(A_{\Psi_v})\},$$
consisting of unitary representations (possibly zero, possibly reducible) of $\mathrm{Mp}_{2n}(F_v)$ indexed by the set of irreducible characters of A_{Ψ_v}. On taking tensor products of these local A-packets, we obtain a global A-packet
$$A_{\Psi,\psi} = \{\sigma_\eta : \eta = \otimes_v \eta_v \in \mathrm{Irr}(\prod_v A_{\Psi_v})\}$$
consisting of abstract unitary representations
$$\sigma_\eta = \otimes_v \sigma_{\eta_v}$$
of $\mathrm{Mp}_{2n}(\mathbb{A})$ indexed by the irreducible characters $\eta = \otimes_v \eta_v$ of $\prod_v A_{\Psi_v}$.
- Arthur has attached to Ψ a quadratic character (possibly trivial) ϵ_Ψ of A_Ψ, This character plays an important role in the multiplicity formula for the automorphic discrete spectrum of SO_{2n+1}. For Mp_{2n}, we need to define a modification of ϵ_Ψ.

  More precisely, consider the L-parameter Φ_Ψ associated to Ψ:
$$\Phi_\Psi = \bigoplus_i \Phi_{\Psi_i}$$
with
$$\Phi_{\Psi_i} = \bigoplus_{k=0}^{r_i-1} \Pi_i \cdot |-|^{(r_i-1-2k)/2}.$$
Then define $\eta_\Psi \in \mathrm{Irr} A_\Psi$ by
$$\eta_\Psi(a_i) = \epsilon(1/2, \Phi_{\Psi_i}) = \begin{cases} \epsilon(1/2, \Pi_i) \text{ if } L(s, \Pi_i, \wedge^2) \text{ has a pole at } s = 1; \\ 1, \text{ if } L(s, \Pi_i, \mathrm{Sym}^2) \text{ has a pole at } s = 1. \end{cases}$$
The modified quadratic character of A_Ψ in the metaplectic case is
$$\tilde{\epsilon}_\Psi = \epsilon_{\Psi_i} \cdot \eta_\Psi.$$

With the above data introduced, we can now state the conjecture.

Arthur Conjecture

The automorphic discrete spectrum $\mathcal{A}_2(\mathrm{Mp}_{2n})$ has a decomposition

$$L^2_{disc}(\mathrm{Mp}_{2n}) = \bigoplus_{\Psi} L^2_{\Psi,\psi}$$

where the sum runs over equivalence classes of discrete A-parameters of Mp_{2n} and for each such Ψ,

$$L^2_{\Psi,\psi} \cong \bigoplus_{\eta \in \mathrm{Irr}(\prod_v A_{\Psi_v}) : \Delta^*(\eta) = \tilde{\epsilon}_\Psi} \sigma_\eta$$

This conjecture amounts to a description of $L^2_{disc}(\mathrm{Mp}_{2n})$ in terms of the discrete spectrum of SO_{2n+1}. When $n = 1$, this is the main result of Waldspurger's work on the Shimura correspondence. In ongoing work with F. Gao, we are working towards establishing this Arthur conjecture for Mp_{2n}, at least for the tempered part of the spectrum. The stable trace formula for Mp_{2n} should establish this as well.

6. A Sample of Problems

We end this paper with a list of sample problems which one may want to pursue further:

- Structure theory: much remains to be understood about the map

  $$\{\text{multipicative } K_2\text{-torsors}\} \longrightarrow \{\text{topological covering groups}\}.$$

  Namely, one would like to know more about its image and its fibers.
- Bruhat-Tits theory: the paper [**W2**] studies the problem of understanding the covering of a parahoric subgroup. It gives some partial results about determining the "special fiber" of such a covering.
- Real groups: for coverings of real groups, Harish-Chandra's classification of discrete series representations has been known for decades. Could one formulate this classification result in the spirit of the LLC? J. Adams and his collaborators, and others, have thought much about this question: see [**A1, A2**], [**ABPTV**], [**AH**], [**R1, R2**] among others.
- p-adic groups: could one extend the Moy-Prasad theory of depth of a representation to the case of covering groups? The construction of depth zero supercuspidal representations was carried out in [**HW**], but could one group these into "L-packets"? Does Yu's construction of supercuspidal representations by compact induction [**Y**] extend readily to covering groups? Does J.L. Kim's result [**K**] on the exhaustion of Yu's construction (when the residue characteristic p is large) extend readily to covering groups? Together with JuLee Kim, we are working towards understanding some of these questions.
- A theory of endoscopy: as we mentioned earlier, Wen-Wei Li's work on the invariant trace formula for covering groups has led directly to the question of whether the theory of endoscopy has a natural extension to covering groups.
- Eisenstein series and L-functions: Automorphic L-functions arises naturally when one considers the constant term of Eisenstein series. For covering groups, can one have a similar interpretation of the constant terms

of Eisenstein series? What L-functions arise in this way? Addressing this question is part of the thesis work of my student Fan Gao.

- Discrete spectrum and Arthur's multiplicity formula: Can Arthur's conjecture be extended to the case of covering groups? What would the multiplicity formula look like? The case of Mp_{2n} offers some clues, but is perhaps too special a case. The ongoing work of Hiraga-Ikeda on the discrete spectrum of covering tori should shed some light on this question.

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