Representations of Metaplectic Groups

Wee Teck Gan
University of California, San Diego
National University of Singapore

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Local Langlands Correspondence (LLC)

Let F be a local field with Weil group

$$W_F \subset \mathrm{Gal}(\overline{F}/F),$$

and Weil-Deligne group

$$WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$$

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The LLC for G gives a classification of

$$\mathrm{Irr}(G) = \{\text{isom. classes of irreducible representations of } G(F)\}$$

in terms of arithmetic data.

L-parameters

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Then LLC postulates a bijection

$$\mathcal{L} : \bigsqcup_{G'} \mathrm{Irr}(G') \longleftrightarrow \Phi(G)$$

as G' ranges over pure inner forms of G.

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The LLC is, on one hand, a generalization of local class field theory, and on the other hand, a generalization of the Cartan-Weyl theory of highest weights for representations of connected compact Lie groups.

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- GSp(4) and its inner form, Sp(4): joint with Takeda and Tantono;
- all classical groups SO(n), Sp($2n$) and U(n): pending book of Arthur, and work of Moeglin.

Thus, I will take the view that the LLC is essentially known.

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The time is perhaps ripe for a systematic extension of the Langlands program (local and global) to general covering groups.

Today, we will look at a particular family of such groups: the metaplectic groups.

Shimura Correspondence

Cuspidal Hecke eigenform
of weight $(2k + 1)/2$
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Cuspidal Hecke eigenform of weight $2k$ and level $\Gamma_0(N)$

Established using converse theorem for classical modular forms. Alternative approach via Theta Correspondence was pursued by Niwa and Shintani.

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Cuspidal Representations of $\mathrm{PGL}_2(\mathbb{A})$

Here, $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ denotes the double cover of $\mathrm{SL}_2(\mathbb{A}) = \mathrm{Sp}_2(\mathbb{A})$. The map depends on an additive character

$$\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times.$$

Waldspurger's Local Results

The above result depends on a local analog:

Irreducible Genuine Representations of $\widetilde{\mathrm{SL}}_2(F)$

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Irreducible Representations of $\mathrm{PGL}_2(F)$

Here an irreducible *genuine* representation of $\widetilde{\mathrm{SL}}_2(F)$ is one which does not factor to a representation of $\mathrm{SL}_2(F)$. The map depends on an additive character

$$\psi : F \longrightarrow \mathbb{C}^\times$$

Refinement

If D denotes the quaternion division F-algebra, then $PD^\times$ is a pure inner form of $\mathrm{PGL}_2(F)$ and the above map can be refined to a bijection:

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$$\boxed{\text{Genuine Irreps of } \widetilde{\mathrm{SL}}_2(F)}$$

$$\updownarrow \Theta_\psi$$

$$\boxed{\text{Irreps of } \mathrm{PGL}_2(F)} \quad \bigcup \quad \boxed{\text{Irreps of } PD^\times}$$

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where V^+ (resp. V^-) denotes the 3-dim split (resp. non-split) quadratic space of discriminant 1.

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where V^+ (resp. V^-) denotes the 3-dim split (resp. non-split) quadratic space of discriminant 1.

So the map Θ_ψ is a classification of the genuine irreps of the double cover of the symplectic group of rank 1 by the irreps of the two special orthogonal groups of rank 1.

(iii) Coupled with the LLC, one has:

$$\boxed{\text{Genuine Irreps of } \widetilde{\mathrm{SL}}_2(F)}$$

$$\updownarrow \mathcal{L}_\psi$$

$$\boxed{\text{pairs } (\phi, \eta) : \eta \in \mathrm{Irr}(A_\phi)}$$

where

$$\phi : WD_F \longrightarrow \mathrm{SL}_2(\mathbb{C}),$$

and

$$A_\phi = \pi_0(Z_{\mathrm{SL}_2(\mathbb{C})}(\phi)) = 1 \text{ or } \mathbb{Z}/2\mathbb{Z}.$$

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• Highlight some other recent developments, especially with regards to a theory of endoscopy. (Work of Wenwei Li)

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$$1 \longrightarrow \mu_2 \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1$$

- If $H \subset \mathrm{Sp}(W)$, then $\tilde{H}$ is the preimage of H in $\mathrm{Mp}(W)$
- If $Z \cong \mu_2 =$ the center of $\mathrm{Sp}(W)$, then $\tilde{Z}$ is the center of $\mathrm{Mp}(W)$.

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- An irred rep π of $\mathrm{SO}(V)$ has 2 extensions π^+ and π^- to $\mathrm{O}(V)$. The sign in π^{ϵ} simply encodes the central character:
  $$\epsilon = \pi^{\epsilon}(-1).$$

Theorem A

Assume that F is p-adic with $p \neq 2$. Fix an additive character $\psi : F \longrightarrow \mathbb{C}^\times$. Then there is a bijection

$$\mathrm{Irr}(\mathrm{Mp}(W))$$

$$\updownarrow \Theta_\psi$$

$$\mathrm{Irr}(\mathrm{SO}(V^+)) \quad \bigsqcup \quad \mathrm{Irr}(\mathrm{SO}(V^-))$$

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Corollary B (LLC): If the LLC for $\mathrm{SO}(V^\pm)$ holds (e.g. for $n = 1$ or 2), one has a bijection

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- for an irrep σ of $\mathrm{Mp}(W)$, the maximal σ-isotypic quotient of Ω_ψ has the form
$$\sigma \boxtimes \Theta_{V,W,\psi}(\sigma),$$
for some smooth representation $\Theta_{V,W,\psi}(\sigma)$ of $\mathrm{O}(V)$ (the *big* theta lift of σ).

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- (Kudla) $\Theta_{V,W,\psi}(\sigma)$ is zero or of finite length.

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- (Howe, Waldspurger) If $p \neq 2$, then $\theta_{V,W,\psi}(\sigma)$ is zero or irreducible.
- the above discussion holds with the roles of V and W switched.

Key Proposition

Thm A follows from:

(1) For $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$, exactly one of

$$\Theta_{V^+,W,\psi}(\sigma) \quad \text{or} \quad \Theta_{V^-,W,\psi}(\sigma)$$

is nonzero.

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(2) For $\pi \in \mathrm{Irr}(\mathsf{SO}(V))$, exactly one extension π^ϵ of π satisfies

$$\Theta_{V,W,\psi}(\pi^\epsilon) \neq 0.$$

Part (1) was also shown by C. Zorn.

Hence, if $p \neq 2$, one defines Θ_ψ by the composite:

$$\begin{array}{ccc} \mathrm{Irr}(\mathrm{Mp}(W)) & \xrightarrow{\theta_{V,W,\psi}} & \mathrm{Irr}(\mathrm{O}(V^+)) \sqcup \mathrm{Irr}(\mathrm{O}(V^-)) \\ & & \downarrow \mathit{rest} \\ & & \mathrm{Irr}(\mathrm{SO}(V^+)) \sqcup \mathrm{Irr}(\mathrm{SO}(V^-)) \end{array}$$

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Key Prop implies that the composite is bijective, proving Thm A.

Refinement of Key Prop

- for $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$, which $\Theta_{V^\epsilon,W,\psi}(\sigma)$ is nonzero?
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Theorem C (Epsilon Dichotomy):
(i) Given $\pi \in \mathrm{Irr}(\mathrm{SO}(V))$, π^ϵ has nonzero theta lift to $\mathrm{Mp}(W)$ if and only if

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(ii) Given $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$, σ has nonzero theta lift to $\mathrm{O}(V^\epsilon)$ if and only if

$$\begin{aligned} \epsilon &= z_\psi(\sigma) \cdot \epsilon(1/2, \sigma, \psi) \\ &= z_\psi(\sigma) \cdot \epsilon(1/2, \Theta_\psi(\sigma)). \end{aligned}$$

Here, the epsilon factors are the standard epsilon factors defined by the doubling zeta integral of PS-Rallis (Lapid-Rallis).

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- If σ is a Langlands quotient of an induced representation, is its L-parameter obtained from that of the inducing data in the usual way?
- If η is trivial, does (ϕ, η) correspond to a ψ-generic rep of $\mathrm{Mp}(W)$?
- Are natural invariants such as various L-factors and ϵ-factors preserved by $\mathcal{L}_\psi$?

Parabolic Subgroups of $O(V)$

The parabolic subgroups $Q = L \cdot U$ of $O(V)$ have Levi factors

$$L = GL(n_1) \times .... \times GL(n_r) \times O(V_0)$$

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$$2(n_1 + ... + n_r) + \dim V_0 = \dim V.$$

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Given reps τ_i of GL(n_i) and π_0 of O(V_0), can form principal series rep

$$I_Q(\tau_1, ..., \tau_r, \pi_0) = \mathrm{Ind}_Q^{\mathrm{O}(V)} \tau_1 \boxtimes ....\tau_r \boxtimes \pi_0.$$

If this is a standard module, its unique irred quotient is denoted by

$$J_Q(\tau_1, ..., \tau_r, \pi_0).$$

Parabolic Subgroups of $\mathrm{Mp}(W)$

The parabolic subgroups $P = M \cdot N$ of $\mathrm{Sp}(W)$ have Levi subgroups:

$$M \cong \mathrm{GL}(n_1) \times .... \times \mathrm{GL}(n_r) \times \mathrm{Sp}(W_0)$$

with

$$2n_1 + ... + 2n_r + \dim W_0 = \dim W.$$

The restriction of the metaplectic cover to N splits uniquely, so one has:

$$\tilde{P} = \tilde{M} \cdot N.$$

The double cover $\tilde{M}$ of M can be described as follows.

The Group $\tilde{M}$

The restriction of the metaplectic cover to $\mathrm{GL}(m)$ is a double cover described by:

$$\tilde{\mathrm{GL}}(m) = \mathrm{GL}(m) \times \{\pm 1\}$$

with multiplication

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) =$$

$$(g_1 \cdot g_2,\ \epsilon_1 \cdot \epsilon_2 \cdot (\det g_1, \det g_2)).$$

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Then the preimage $\tilde{M}$ of M in $\mathrm{Mp}(W)$ is

$$\tilde{M} = \tilde{\mathrm{GL}}(n_1) \times_{\mu_2} .... \times_{\mu_2} \tilde{GL}(n_r) \times_{\mu_2} \mathrm{Mp}(W_0).$$

In particular, to form genuine principal series reps of $\mathrm{Mp}(W)$, we need genuine reps of $\tilde{\mathrm{GL}}(m)$.

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(i) The determinant map det has a lifting:

$$\begin{array}{ccc} \tilde{G}L(m) & \xrightarrow{\det} & \tilde{GL}(1) \\ \downarrow & & \downarrow \\ GL(m) & \xrightarrow{\det} & GL(1) \end{array}$$

defined by

$$\det(g, \epsilon) = (\det(g), \epsilon)$$

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(ii) Using ψ, one may define a genuine character χ_ψ of $\tilde{\mathrm{GL}}(1)$ (Weil). By composition, one obtains a genuine 1-dim character χ_ψ of $\tilde{\mathrm{GL}}(m)$.
(iii) Tensoring with χ_ψ gives

$$\mathrm{Irr}(\mathrm{GL}(m)) \to \mathrm{Irr}(\tilde{\mathrm{GL}}(m)).$$

Given reps τ_i of $\mathrm{GL}(n_i)$ and σ_0 of $\mathrm{Mp}(W_0)$, one has the induced rep

$$I_{P,\psi}(\tau_1, ..., \tau_r, \sigma_0) = \mathrm{Ind}_{\tilde{P}}^{\mathrm{Mp}(W)} \chi_\psi \tau_1 \boxtimes ... \boxtimes \chi_\psi \tau_r \boxtimes \sigma_0.$$

If this is a standard module, its unique irred quotient is denoted by

$$J_{P,\psi}(\tau_1, ..., \tau_r, \sigma_0)$$

Central Sign of $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$

Recall that the center of $\mathrm{Mp}(W)$ is the preimage $\tilde{Z}$ of the center $Z \cong \mu_2$ of $\mathrm{Sp}(W)$. So $\tilde{Z}$ is a group of size 4 and has 2 irred genuine characters.

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Every genuine irrep σ of $\mathrm{Mp}(W)$ has a central character, and we set

$$z_\psi(\sigma) = \begin{cases} +1, & \text{if its central character is } \chi_\psi; \\ -1, & \text{otherwise.} \end{cases}$$

Theorem D

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- σ is tempered iff π is tempered. Indeed, if

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  with τ_i and σ_0 unitary d.s., then

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- If $\sigma = J_{P,\psi}(\tau_1, ..., \tau_r, \sigma_0)$ is a Langlands quotient, then

  $$\pi = J_Q(\tau_1, ..., \tau_r, \Theta_\psi(\sigma_0)).$$

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(LHS: Shahidi, RHS: Szpruch)

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How are (ϕ, η) and (ϕ_c, η_c) related?
Suppose that
$$\phi = \bigoplus_i n_i \phi_i,$$
then
$$A_\phi = \bigoplus_{i:\phi_i \text{ is symplectic}} \mathbb{Z}/2\mathbb{Z} a_i.$$

Theorem E

(i) $\phi_c = \phi \otimes \chi_c$;

It follows by (i) that we have canonical identification of component groups:

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The proof of Thm. E uses the Gross-Prasad conjecture for special orthogonal groups recently proved by Wadlspurger.

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In his thesis, Szpruch has extended this theory to the case of $\mathrm{Mp}(W)$. For ψ-generic representations σ of $\mathrm{Mp}(W)$, and irreducible representations ρ of $\mathrm{GL}_r(F)$, he defines

$$L(s, \sigma \times \rho, \psi) \quad \text{and} \quad \epsilon(s, \sigma \times \rho, \psi).$$

Relation with Character Theory

One hallmark of the LLC is its relation with character theory; in particular the notion of stability and the theory of endoscopy.

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Basic notions in "..." needs to be developed.

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- For each pair (a, b) with $a + b = n$, he defined transfer factors

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$$\Delta_{a,b} : \mathrm{Mp}(W) \times (\mathrm{SO}(2a+1) \times \mathrm{SO}(2b+1)) \to \mathbb{C}.$$

- showed existence of transfer mappings

$$\begin{array}{c} C_c^\infty(\mathrm{Mp}(W)) \\ {\scriptstyle t_{a,b}} \big\downarrow \\ C_c^\infty(\mathrm{SO}(2a+1) \times \mathrm{SO}(2b+1)), \end{array}$$

giving rise to a map

$$\begin{array}{c} \mathcal{D}_{st}(\mathrm{SO}(2a+1) \times \mathrm{SO}(2b+1)) \\ \Big\downarrow t^*_{a,b} \\ \mathcal{D}_{gen}(\mathrm{Mp}(W)), \end{array}$$

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between spaces of distributions.

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In other words, the elliptic endoscopic groups of $\mathrm{Mp}(W)$ are the groups

$$\mathrm{SO}(2a+1)\times \mathrm{SO}(2b+1),$$

with $a+b=n$.

Automorphic Forms of Mp(W)

Now we come to the global story.
We will consider the discrete spectrum

$$L^2_{disc} = L^2_{gen,disc}(\mathrm{Sp}(W)(F)\backslash \mathrm{Mp}(W)(\mathbb{A}))$$

of genuine (square-integrable) automorphic forms on Mp(W).

Problem: describe its decomposition into irreducible reps, in the spirit of Arthur's conjecture.

The Tempered Part

Let's focus on the tempered part $L^2_{temp,disc}$. For a fixed additive automorphic character ψ, one expects that

$$L^2_{temp,disc} = \bigoplus_{\phi} L^2_{\phi,\psi}$$

with

$$\phi = \oplus_{i=1}^{r} \phi_i : L_F \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C}),$$

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Following Arthur, can reformulate the notion ϕ without recourse to L_F. One can take ϕ to be a collection $\{\pi_1, ..., \pi_r\}$ of cuspidal representations of $\mathrm{GL}(n_i)$ such that

- $n_1 + ... + n_r = n$;
- $L^2(s, \pi_i, \wedge^2)$ has a pole at $s = 1$.

Global Packet associated to ϕ

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- a global component group

$$A_\phi = \mathbb{Z}/2\mathbb{Z}a_1 \times ... \times \mathbb{Z}/2\mathbb{Z}a_r$$

and a diagonal map

$$\Delta : A_\phi \rightarrow \prod_v A_{\phi_v}$$

Langlands-Arthur's Multiplicity Formula

One conjectures that

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This conjecture amounts to a description of $L^2_{disc,temp}$ in terms of the tempered discrete spectrum of $SO(2n+1)$.
When $n = 1$, this is the main result of Waldspurger's work on the Shimura correspondence.

Towards a stable trace formula for Mp(W)

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Towards a stable trace formula for $\mathrm{Mp}(W)$

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- He has begun the study of the trace formula for general covering groups.
- For $\mathrm{Mp}(W)$, he has begun the stabilization of the trace formula.

The stable trace formula should look like:

$$\mathrm{TF}_{\mathrm{Mp}(W)}(f) =$$

$$\sum_{a+b=n} c_{a,b} \cdot \mathrm{STF}_{\mathrm{SO}(2a+1)\times\mathrm{SO}(2b+1)}(t_{a,b}(f))$$

where $c_{a,b} = 1/4$ if $ab \neq 0$ and $1/2$ otherwise.

Using a elliptic stable trace formula, together with the global theta correspondence and some extra work, one should obtain

- a proof of stability properties of local L-packets and character identities for $\mathrm{Mp}(W)$;
- the multiplicity formula for cuspidal representations of $\mathrm{Mp}(W)$ with sufficiently many supercuspidal local components

This is joint work with W. W. Li.

Conclusion

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• Several recent progresses indicate that the time is ripe for a systematic development of such a theory. A systematic study of the trace formula of covering groups will be the key, but further explicit examples would be very welcome.