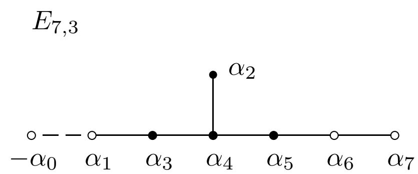
Modular Forms of Level p on the Exceptional Tube Domain

Wee Teck Gan, Hung Yean Loke

March 2, 2009

Introduction

Let G be a simply connected group defined over $\mathbb{Q}$ of type E_7 with maximal split torus of rank 3. It is known that $G(\mathbb{R}) \simeq E_{7,3}(\mathbb{R})$ but G splits over $\mathbb{Q}_p$ for all finite primes p. The Satake diagram is: (see Onishchik and Vinberg [O-V, pg 230])



where α_0 will denote the highest root. We will follow the notation of Bourbaki [Bou, Planche VI] for the root system of E_7.

The group $G(\mathbb{R})$ acts on its Hermitian symmetric space which in this case is an exceptional tube domain, to be denoted by $\mathcal{T}$, and which is a higher dimensional analogue of the upper half plane. The action factors through the adjoint group G^{ad} and G has a maximal discrete subgroup Γ. In [K], Kim has constructed a singular modular form E_4 of weight 4 on $\mathcal{T}$ with respect to Γ, and obtained its Fourier expansion.

The purpose of this paper is to obtain other modular forms on $\mathcal{T}$ by using the action of Hecke operators. This is done by first transforming Kim's form to an L^2-function $\tilde{E}_4$ on the adele group, as in the case of SL_2. It is known that $\tilde{E}_4$ generates the minimal representation of $G(\mathbb{A})$; hence E_4 is the analogue of the classical theta function, which is a modular form of weight $\frac{1}{2}$ on the upper half plane. We then obtain modular forms of level p by computing the action of the Iwahori-Hecke algebra at p. From general facts on the minimal representation, this gives an 8-dimensional space of modular forms of level p on $\mathcal{T}$.

Formally, the computation of the action of Hecke operators is similar to that on the upper half plane. However, it is complicated by the facts that $\mathcal{T}$ has higher dimension, and G is not split. This implies, for example, that the explicit information one needs about strong approximation is not readily available.

Now we give a brief summary of the layout of the paper. In the first five sections of the paper, we shall set up the basic notations and equations concerning the tube domain $\mathcal{T}$, the group G and its Lie algebra, and the action of G on $\mathcal{T}$. Then we shall present the representation theoretic point of view of Hecke operators. In Section 8, we do an explicit computation of the action of the Hecke operator T_0, and obtain the Fourier coefficients of three other modular forms of level p. These are the cases where the strong approximation is still tractable. Details about strong approximation, which is necessary in the computation of other Hecke operators will be given in the remaining sections. Finally, we discuss our initial motivations for carrying out these computations.

We would like to thank Professor B. H. Gross, and D. Pollack for many helpful discussions, as well as Professor D. Prasad for his careful reading of the manuscript and his various suggestions which improve the presentation of this paper.

1 The Exceptional Tube Domain and the Construction of G

1.1 Let $\mathbb{O}_k$ be the standard octonion algebra over a field k, which is defined as follows:

$$\mathbb{O}_k = k \oplus ke_1 \oplus ke_2 \oplus ke_3 \oplus ke_4 \oplus ke_5 \oplus ke_6 \oplus ke_7,$$

$$e_i^2 = -1, \quad (e_i e_{i+1})e_{i+3} = e_i(e_{i+1}e_{i+3}) = -1$$

where the index i is assumed to be taken modulo 7. Note that $\mathbb{O}_{\mathbb{Q}}$ has an integral structure given by the Coxeter order(see [E-G]). We shall denote this order by $\mathbb{O}_{\mathbb{Z}}$.

Let $\mathcal{J}_k$ be the Jordan algebra over k. Hence, an element X of $\mathcal{J}_k$ can be represented as :

$$X = (\gamma_1, \gamma_2, \gamma_3; c_1, c_2, c_3) := \begin{pmatrix} \gamma_1 & c_3 & \overline{c_2} \\ \overline{c_3} & \gamma_2 & c_1 \\ c_2 & \overline{c_1} & \gamma_3 \end{pmatrix} \tag{1.2}$$

where $\gamma_i \in k$ and $c_i \in \mathbb{O}_k$.

The composition in $\mathcal{J}_k$ is given by :

$$X \circ Y = \frac{1}{2}(XY + YX)$$

where XY denotes usual matrix multiplication.

We have an inner product on $\mathcal{J}_k$ given by:

$$(X, Y) = \mathrm{Tr}\,(X \circ Y)$$

where Tr denotes the usual trace of matrices. Recall also that there is a cubic form det on $\mathcal{J}_k$ which induces a trilinear form on $\mathcal{J}_k \times \mathcal{J}_k \times \mathcal{J}_k$ such that:

$$(X, X, X) = detX$$

From this, we obtain another bilinear map $\mathcal{J}_k \times \mathcal{J}_k \longrightarrow \mathcal{J}_k$, $(X, Y) \mapsto X \times Y$ by requiring that:

$$(X \times Y, Z) = 3\,(X, Y, Z)$$

Note that $X \times X$ is just the adjoint matrix of X.

The integral sructure $\mathbb{O}_{\mathbb{Z}}$ of $\mathbb{O}_{\mathbb{Q}}$ enables us to give $\mathcal{J}_{\mathbb{Q}}$ an integral sructure $\mathcal{J}_{\mathbb{Z}}$, by requiring that $\gamma_i \in \mathbb{Z}$ and $c_i \in \mathbb{O}_{\mathbb{Z}}$ in equation (1).

Finally, let R_i be the rank i matrices in $\mathcal{J}_k$, for i=0,1,2,3 (see [K]). Hence, we have:

$$\begin{aligned} R_3 &= \{X \in \mathcal{J}_k : detX \neq 0\} \\ R_2 &= \{X \in \mathcal{J}_k : detX = 0, X \times X \neq 0\} \\ R_1 &= \{X \in \mathcal{J}_k : X \neq 0, X \times X = 0\} \\ R_0 &= \{0\} \end{aligned}$$

Let $R_i^+ \subset R_i$ be the elements which are squares. Then we define the tube domain $\mathcal{T}$ to be:

$$\mathcal{T} = \{X + iY : X \in \mathcal{J}_{\mathbb{R}}, Y \in R_3^+\}$$

1.3 We shall now discuss the construction of G, following [Ba]. Let $\mathcal{J}$ denote $\mathcal{J}_{\mathbb{Q}}$, for simplicity.

Let

$$L = \{g \in GL\,(\mathcal{J}_{\mathbb{Q}}) : det\,(gX) = \lambda\,(g)\,detX, \forall X \in \mathcal{J}_{\mathbb{Q}}\}$$

Note that $\lambda : L \longrightarrow \mathbb{Q}^*$ is a group homomorphism. Its kernel is a simply connected group of type $E_{6,2}$, and $L\,(\mathbb{Q}) \simeq (E_{6,2}(\mathbb{Q}) \times \mathbb{Q}^*)/\mu_3$. The center $\mathbb{G}_m$ of L acts as scalars and $\lambda\,(a) = a^3$, for all $a \in \mathbb{G}_m$. We shall denote this representation of L on $\mathcal{J}$ by $\mathcal{J}_n$ and call it the natural representation.

For $g \in L$, set g^* to be the unique element of L such that for all $X, Y \in \mathcal{J}$:

$$(gX, g^*Y) = (X, Y)$$

The action of L on $\mathcal{J}$ via $g \mapsto g^*$ is simply the dual of the natural representation and is denoted $\overline{\mathcal{J}_n}$. Now let:

$$W = \mathcal{J} \oplus \mathbb{Q} \oplus \mathcal{J} \oplus \mathbb{Q}$$

be a 56-dimensional $\mathbb{Q}$-vector space. We define an embedding of L into $GL(W)$ as follows:

$$\rho : L \longrightarrow GL(W)$$

$$\rho(g)\,(X, \gamma, X', \gamma') = \left(gX, \lambda(g)^{-1}\gamma, g^* X', \lambda(g)\gamma'\right).$$

Here, gX denotes the natural action and we denote the image of ρ by L again. In addition we define an embedding of $\mathcal{J}$ into $GL(W)$:

$$p' : \mathcal{J} \longrightarrow GL(W)$$

$$p'_B(X, \gamma, X', \gamma') = (Y, \delta, Y', \delta')$$

where,

$$\begin{aligned} Y &= X + 2B \times X' + \gamma B \times B \\ \delta &= \gamma \\ Y' &= X' + \gamma B \\ \delta' &= \gamma' + (B, X) + (B \times B, X') + \gamma det B \end{aligned}$$

Call the image U. With these embeddings, we compute to see that :

$$\rho(g) p'_B \rho(g)^{-1} = p'_{\lambda(g) g^*(B)}$$

Hence L normalises $U \simeq \mathcal{J}$ and its adjoint action on U is the representation $\lambda \otimes \overline{\mathcal{J}_n} := \mathcal{J}_{ad}$.

Finally, we have an element $\iota \in GL(W)$ given by:

$$\iota(X, \gamma, X', \gamma') = (-X', -\gamma', X, \gamma)$$

Let $G = \langle \iota, U \rangle$. Then G is the simply connected group of type $E_{7,3}$. Moreover, $L \subset G$ and $P = L \ltimes U$ is a maximal parabolic subgroup which stabilises the line $(0, 0, 0, \gamma')$ in W.

Set $p_B = \iota^{-1} p'_B \iota$. This gives another embedding of $\mathcal{J}$ into G. Call the image $\overline{U}$. Then L normalises $\overline{U}$ as well, and its adjoint action on $\overline{U}$ is the dual of that on U.

Now let e_i be the element of $\mathcal{J}$ whose $(i+1, i+1)$ entry is 1,and whose other entries are 0 for $i = 0, 1, 2$. Set

$$\iota_{e_i} = p'_{e_i} p_{e_i} p'_{e_i}$$

Then the ι_{e_i}'s commute with one another and $\iota_{e_0} \iota_{e_1} \iota_{e_2} = \iota$. So G can also be generated by U and $\overline{U}$.

Let

$$Q := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$Q^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $Q(JQ^{-1}) = (QJ)Q^{-1}$ for all $J \in \mathcal{J}$. We define $\iota_0 \in L$ to be the element whose natural action on $\mathcal{J}$ is given by:

$$\iota_0 : J \mapsto QJQ^{-1}$$

Then via the embedding ρ, ι_0 acts on W by:

$$\iota_0(X, \gamma, X', \gamma') = (QXQ^{-1}, \gamma, QXQ^{-1}, \gamma') \tag{1.4}$$

Finally,we give G an integral structure as follows: let

$$W_{\mathbb{Z}} = \mathcal{J}_{\mathbb{Z}} \oplus \mathbb{Z} \oplus \mathcal{J}_{\mathbb{Z}} \oplus \mathbb{Z}.$$

Then let $\Gamma = G(\mathbb{Z})$ be the elements of G which stabilises the lattice $W_{\mathbb{Z}}$. $G(\mathbb{Z})$ is a maximal discrete subgroup of $G(\mathbb{R})$ [Ba].

2 Composition Algebras

2.1 In this section we shall review the theory of composition algebras. A composition k-algebra K is defined as a k-algebra with a nondegenerate quadratic form N satisfying $N(xy) = N(x)N(y)$. By a celebrated theorem of Hurwitz [Sch, Thm 3.25], the only possible structures for K are:

(1) k with $N(x) = x^2$;
(2) $D(k) = k[X]/f(x)$ where $f(x) = X^2 - X - \alpha$ is a polynomial, with $\alpha \in k$ and $4\alpha + 1 \neq 0$;
(3) A quaternion algebra $\mathbb{H}(k)$ and;
(4) a Cayley algebra or an algebra of octonians $\mathbb{O}(k)$.

Each of these comes with an anti-automorphism $x \mapsto \overline{x}$ called *conjugation* and $N(x) = x\overline{x} = \overline{x}x$. We define $\mathrm{tr}(x) = x + \overline{x}$. It satisfies $\mathrm{tr}(x) = \mathrm{tr}(\overline{x})$ and $\mathrm{tr}(xy) = \mathrm{tr}(yx)$. This gives us a non-degenerate inner product on K, defined by: $(x, y) = tr(x\overline{y})$. Note that although we have used the same notation $(\cdot, \cdot)$ for the inner product on the Jordan algebra $\mathcal{J}_k$ and on K, there should be no cause for confusion.

The conjugations on $M_2(k)$ and $\mathbb{O}(k)$ extend to a conjugation action on $M_2(k) \otimes \mathbb{O}$ given by: $\overline{a \otimes b} := \overline{a} \otimes \overline{b}$.

2.2 A split quaternion algebra over k can be represented by the set of 2 by 2 matrices $M_2(k)$. We identify $\mathfrak{sl}_2(k)$ as the subspace of matrices of trace zero. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $D(k) = k + kI$ is a subalgebra of $M_2(k)$.

2.3 Let $(V, \langle\ ,\ \rangle)$ be a 3 dimensional inner product space over k with orthonormal basis $\{\nu_2, \nu_3, \nu_4\}$. A split octonion algebra $\mathbb{O}(k)$ can be represented by the set of 2 by 2 matrices of the form [J2, pg 142]:

$$\begin{pmatrix} a & v \\ v^* & d \end{pmatrix}$$

where $a, d \in k$ and $v, v^* \in V$. The product is given by

$$\begin{pmatrix} a_1 & v_1 \\ v_1^* & d_1 \end{pmatrix}\begin{pmatrix} a_2 & v_2 \\ v_2^* & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - \langle v_1, v_1^* \rangle & a_1v_2 + d_2v_1 + v_1^* \wedge v_2^* \\ a_2v_1^* + d_1v_2^* + v_1 \wedge v_2 & d_1d_2 - \langle v_2, v_2^* \rangle \end{pmatrix}.$$

The split quaternion $M_2(k)$ embeds (non-canonically) into the split octonian $\mathbb{O}(k)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b\nu_2 \\ c\nu_2 & d \end{pmatrix}.$$

2.4 First, we specialise to $k = \mathbb{F}_p$. With a slight abuse of notations we define the following elements in $\mathbb{O}(k)$:

$$\begin{aligned} \nu_1 &:= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \nu_1^* &:= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \nu_i &:= \begin{pmatrix} 0 & \nu_i \\ 0 & 0 \end{pmatrix} \qquad i = 2, 3, 4 \\ \nu_i^* &:= \begin{pmatrix} 0 & 0 \\ \nu_i & 0 \end{pmatrix} \qquad i = 2, 3, 4. \end{aligned}$$

All the above elements have norm 0. Define V_0 and V_0^* to be the span of $\{\nu_i\}_{i=1}^4$ and $\{\nu_i^*\}_{i=1}^4$ respectively. Then these are 2 complementary isotropic subspace of $\mathbb{O}(k)$ with respect to the norm $N(\cdot)$.

2.5 Now, we specialise to $k = \mathbb{Q}$. As described in Section 1, $\mathbb{O}(\mathbb{Q})$ has a standard orthonormal basis $\{1, e_1, \ldots, e_7\}$ which satisfies $e_i^2 = -1$ and $(e_ie_{i+1})e_{i+3} = e_i(e_{i+1}e_{i+3}) = -1$ for $i \geq 1$ (taken modulo 7). We denote the Coxeter order in $\mathbb{O}(\mathbb{Q})$ by $\mathbb{O}(\mathbb{Z})$. It is a maximal order of rank 8 over $\mathbb{Z}$ and it contains the maximal order of $\mathbb{H}(\mathbb{Q})$:

$$\mathbb{H}(\mathbb{Z}) = \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_4 + \mathbb{Z}\left(\frac{1 + e_1 + e_2 + e_4}{2}\right) \subset \mathbb{H}(\mathbb{Q}).$$

In standard notations we denote $i := e_1$, $j := e_2$ and $k = ij := e_4$. Also, we let $\tilde{I} \in \mathbb{H}(\mathbb{Z})$ be a lift of $I = \nu_1 - \nu_1^*$.

Let p be a prime number. It is well known that $\mathbb{O}(\mathbb{Z}_p) = \mathbb{O}(\mathbb{Z}) \otimes \mathbb{Z}_p$ and $\mathbb{O}(\mathbb{F}_p) = \mathbb{O}(\mathbb{Z}) \otimes \mathbb{F}_p$ are split octonions (cf. §2.3), and if $p \neq 2$,

$$\mathbb{H}(\mathbb{Z}) \otimes \mathbb{Z}_p \simeq M_2(\mathbb{Z}_p), \quad \mathbb{H}(\mathbb{Z}) \otimes \mathbb{F}_p \simeq M_2(\mathbb{F}_p).$$

Hence we have the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{H}(\mathbb{Z}) & \stackrel{\otimes \mathbb{Z}_p}{\longrightarrow} & M_2(\mathbb{Z}_p) & \stackrel{\bmod p}{\longrightarrow} & M_2(\mathbb{F}_p) \\ \bigcap & & \cap & & \bigcap \\ \mathbb{O}(\mathbb{Z}) & \stackrel{\otimes \mathbb{Z}_p}{\longrightarrow} & \mathbb{O}(\mathbb{Z}_p) & \stackrel{\bmod p}{\longrightarrow} & \mathbb{O}(\mathbb{F}_p) \end{array}$$

Note that the composition of any 2 horizontal arrows is a surjective homomorphism of $\mathbb{Z}$-algebras.

2.6 We may assume that $k = ij \in \mathbb{H}(\mathbb{Z})$ maps to $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{F}_p)$. Indeed the image $\overline{k}$ of k satisfies $\overline{k}^2 = -1$. Hence we define a map $\overline{k} \mapsto w$ and it is well known that such a map can be extended to an automorphism of $M_2(\mathbb{F}_p)$. Next we further extend the automorphism to an automorphism of $\mathbb{O}(\mathbb{F}_p)$. We just compose the reduction $\pmod p$ map with this automorphism.

2.7 Suppose $p \neq 2$ and let $x = \nu_i$ or $\nu_i^* \in \mathbb{O}(\mathbb{F}_p)$. Then there exists a $\tilde{x} \in \mathbb{O}(\mathbb{Z})$ such that

(1) $\tilde{x} \pmod p = x$ and

(2) $N(\tilde{x}) \equiv p^2 \pmod{p^3}$.

Indeed choose a lift of x, say $\tilde{y} = \sum_i \beta_i e_i$ where $\beta_i \in \mathbb{Z}$ and $\{e_i\}$ is the standard basis. Then $p | N(\tilde{y})$. Suppose $\beta_1 \neq 0$. Then $\tilde{x} := \tilde{y} + np^m e_1$ for appropiate integers m and n will do. This fact will be used later.

3 Exceptional Lie Algebras

3.1 Now we consider the Lie algebra of G. It is well known that the exceptional Lie algebras can be constructed using composition algebras and Jordan algebras. We will review a construction of the Lie algebra $\mathfrak{e}_7$ over a field k of characteristic not equal to 2. See Loke [L, Part 1] for details.

Our objective in this section is to relate the construction of the Lie algebras to the construction of G given in Section 1.

3.2 We denote the positive roots of D_4 by $\epsilon_i \pm \epsilon_j$ where $1 \leq j < i \leq 4$. $Spin(8)$ is defined as the norm preserving linear transformations acting via triality on 3 copies of octonion algebras, namely $W_i = (\mathbb{O}, \text{Norm})$, $i = 1, 2, 3$. We assume that

(1) (W_1, π_1) is the standard representation and it has weights $\pm\epsilon_i$;
(2) (W_2, π_2) is the half spin representation with weights

$$\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$$

where there is an *even* number of postive signs;
(3) (W_3, π_3) is the other half spin representation with weights

$$\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$$

where there is an *odd* number of postive signs.

$g \in Spin(8)$ acts on the Jordan algebra by

$$(a, b, c; x, y, z) \mapsto (a, b, c; \pi_1(g)x, \pi_2(g)y, \pi_3(g)z).$$

In the case when $k = \mathbb{Q}$, g stablises the cubic form *det* and hence $Spin(8)(\mathbb{Q}) \subset L(\mathbb{Q}) \subset G(\mathbb{Q})$.

3.3 Let $\mathfrak{d}_4 = \mathfrak{so}(\mathbb{O})$ be the Lie algebra of $Spin(8)$ defined in §3.2. Define an isomorphism of $Spin(8)$-modules $\mathbb{O} \bigwedge \mathbb{O} \longrightarrow \mathfrak{d}_4$ by

$$v \wedge w \mapsto [u \mapsto 2\langle u, w\rangle v - 2\langle u, v\rangle w]$$

where $u, v, w \in \mathbb{O}$.

3.4 Recall from §2.4 that we have $\mathbb{O}(\mathbb{F}_p) = V_0 \oplus V_0^*$ and we can write an element X of $\mathfrak{so}(\mathbb{O})$ as 8 by 8 matrices with respect to the basis $\{\nu_i, \nu_i^* : i = 1, \ldots, 4\}$. Then

$$X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$$

where $A, B, C \in M_4(\mathbb{F}_p)$ such that $B = -B^t$ and $C = -C^t$. In this way, we identify the CSA of $\mathfrak{so}(\mathbb{O})$ as the subspace of elements X where $B = C = 0$ and, A is a diagonal 4 by 4 matrix.

3.5 We associate the weight spaces of the representation $W_1 = \mathbb{O}$ (cf. §3.2) of $\mathfrak{d}_4$ as

$$\nu_i \in X_{\epsilon_i}, \quad \nu_i^* \in X_{-\epsilon_i}$$

We identify the root spaces $\mathfrak{g}_{\epsilon_i \pm \epsilon_j}$ of $\mathfrak{d}_4$:

$$\mathbb{C} \cdot \nu_i \wedge \nu_j^* = \mathfrak{g}_{\epsilon_i - \epsilon_j} \tag{3.6}$$

$$\mathbb{C} \cdot \nu_i \wedge \nu_j = \mathfrak{g}_{\epsilon_i + \epsilon_j} \tag{3.7}$$

$$\mathbb{C} \cdot \nu_i^* \wedge \nu_j^* = \mathfrak{g}_{-\epsilon_i - \epsilon_j} \tag{3.8}$$

The CSA of $\mathfrak{d}_4$ is spanned by $\{H_i := \nu_i \wedge \nu_i^* : i = 1, \ldots, 4\}$.

3.9 The Lie algebra $\mathfrak{e}_7$ can be realised as

$$\mathfrak{e}_7 = \mathfrak{d}_4 \oplus \mathfrak{sl}_2^3 \oplus (M_2(k) \otimes W_1) \oplus (M_2(k) \otimes W_2) \oplus (M_2(k) \otimes W_3). \tag{3.10}$$

An element $X \in \mathfrak{e}_7$ can be written as:

$$\begin{aligned} X &= (d; x_1, x_2, x_3; v_1, v_2, v_3) \\ &:= d \oplus (x_1, x_2, x_3) \oplus v_1 \oplus v_2 \oplus v_3 \\ &= d \oplus \begin{pmatrix} x_1 & v_3 & -\overline{v_2} \\ -\overline{v_3} & x_2 & v_1 \\ v_2 & -\overline{v_1} & x_3 \end{pmatrix} \end{aligned} \tag{3.11}$$

where $d \in \mathfrak{d}_4$, $x_i \in \mathfrak{sl}_2(k)$ and $v_i \in M_2(k) \otimes W_i$.

Let $X = d \oplus A$, $X' = d' \oplus A' \in \mathfrak{e}_7$ (cf. equation (3.11)). Then

$$[X, X'] := ([d, d'] + D(A, A')) \oplus (AA' - A'A)_*$$

where $D(A, A') \in \mathfrak{d}_4$ is defined in [L] and $(A)_*$ denotes the operation of removing elements of $1 \otimes \mathbb{O}$ from the diagonal entries. Moreover

$$[(0; 0, 0, 0; 1 \otimes v, 0, 0), (0; 0, 0, 0; 1 \otimes u, 0, 0)] = v \wedge w \in \mathfrak{d}_4$$

as in §3.3.

3.12 If $\mathbb{O}(k)$ splits, we can define the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{e}_7$ as the subspace consisting of elements of the form

$$X = (d; x_1, x_2, x_3; v_1, v_2, v_3)$$

where

$$\begin{aligned} d &\in \text{Borel subalgebra of } \mathfrak{d}_4, \\ x_1, x_2, x_3 &\in \left\{ \begin{pmatrix} r & s \\ 0 & -r \end{pmatrix} \in M_2(k) \right\}, \\ v_1, v_2, v_3 &\in \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \otimes \mathbb{O}(k). \end{aligned}$$

3.13 Recall the definition of $D(k)$ from (2.2). In $\mathfrak{e}_7$, we have a Lie subalgebra $\mathfrak{l}$ of type $\mathfrak{t} \oplus \mathfrak{e}_6$,with $\mathfrak{t}$ 1-dimensional, which is given by:

$$\mathfrak{l} = \{(d; x_1, x_2, x_3; v_1, v_2, v_3) : d \in \mathfrak{d}_4, x_i \in D(k), v_i \in D(k) \otimes \mathbb{O}\}.$$

Moreover,

$$\begin{aligned} \mathfrak{e}_6 &= \{(d; x_1, x_2, x_3; v_1, v_2, v_3) \in \mathfrak{l} : x_1 + x_2 + x_3 = 0\} \\ \mathfrak{t} &= \{(0; x, x, x; 0, 0, 0) \in \mathfrak{l}\}. \end{aligned}$$

Further, we have the Lie subalgebra $\mathfrak{u}$ (respectively $\overline{\mathfrak{u}}$), whose elements are given by:

$$\begin{pmatrix} x_1 & v_3 & -\overline{v_2} \\ -\overline{v_3} & x_2 & v_1 \\ v_2 & -\overline{v_1} & x_3 \end{pmatrix}$$

where $x, y, z \in \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$ (respectively $\begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}$), and $v_1, v_2, v_3 \in \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \otimes \mathbb{O}$ (respectively $\begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \otimes \mathbb{O}$). So $\mathfrak{u}$ can be canonically identified as a vector space with $\mathcal{J}_k$.

Also, $\mathfrak{e}_7 = \overline{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u}$ as a representation of $\mathfrak{l}$ by adjoint action.

3.14 The construction given in §3.6 works equally well when we replace k by $\mathbb{Z}$, and the composition algebras by their maximal orders (cf. §2.5). For more details, see Loke [L, Chap. 2]. This gives a $\mathbb{Z}$ model of the exceptional Lie algebras which is functorial with respect to base extensions or reduction $\pmod p$.

3.15 Now we specialise to $k = \mathbb{Q}$. Then $\mathfrak{e}_7$ has $\mathbb{Q}$-rank 3,and the maximal split torus can be chosen to be:

$$\mathfrak{a} = \{(0; aI, bI, cI; 0, 0, 0) : a, b, c \in \mathbb{Q}\}.$$

Going to $\overline{\mathbb{Q}}$, we choose a maximal torus (of dimension 7) containing $\mathfrak{a}$. With this torus, the 4 compact simple roots $\alpha_2, \ldots, \alpha_5$ in the Satake diagram on page 1 spans the sub-root system of the compact Lie algebra $\mathfrak{d}_4$. There are 3 mutually perpendicular positive roots perpendicular to the 4 compact simple roots. Among these 3 roots, 2 of them are in the Satake Diagram, namely α_0 and α_7. The third one is

$$\alpha_8 := \alpha_7 + 2\alpha_6 + 2\alpha_5 + 2\alpha_4 + \alpha_3 + \alpha_2. \tag{3.16}$$

The coroot h_α corresponding to these 3 roots exactly span $\mathfrak{a}$, and the TDS (3-dimensional subalgebra) corresponding to them are exactly the 3 copies of $\mathfrak{sl}_2$ in the construction of $\mathfrak{e}_7$ given above.

By choosing a different simple system of roots if necessary, we can assume that the TDS corresponding to the highest root α_0 is $\{(0; x, 0, 0; 0, 0, 0)\}$.

3.17 Now let us return to the group G. We can define 3 embeddings of SL_2 into G as follows: Let

$$s = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and for $i =$ 0,7,8, let $\Phi_i : SL_2 \to G$ be such that

$$\begin{aligned} \Phi_0(s) &= p'_{e_0} & \Phi_0(w) &= \iota_{e_0} \\ \Phi_7(s) &= p'_{e_1} & \Phi_7(w) &= \iota_{e_1} \\ \Phi_8(s) &= p'_{e_2} & \Phi_8(w) &= \iota_{e_2}. \end{aligned}$$

One can check that Φ_i is indeed an embedding of SL_2 into G and that the Φ_i's commute with one another. Notice also that Φ_i maps $SL_2(\mathbb{Z})$ into $G(\mathbb{Z})$ and so is defined over $\mathbb{Z}$. We shall denote the image of Φ_i by Φ_i also.

Let

$$T = \langle \Phi_i \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, a \in \mathbb{Q}^*, i = 0, 7, 8 \rangle$$

be the group generated by the images of the torus in SL_2. Then T is a maximal split torus in $G(\mathbb{Q})$. We can represent an element $t \in T$ by $t(a, b, c)$ where:

$$t(a, b, c) = \Phi_0 \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \Phi_7 \begin{pmatrix} b & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \Phi_8 \begin{pmatrix} c & 0 \\ 0 & \frac{1}{c} \end{pmatrix} \tag{3.18}$$

3.19 If we consider the Lie algebra of $G_{\mathbb{Q}}$, then we find that we can make the following identifications:

$$\begin{aligned} \mathrm{Lie}(T(\mathbb{Q})) &= \mathfrak{a} \\ \mathrm{Lie}(L(\mathbb{Q})) &= \mathfrak{l} \\ \mathrm{Lie}(U(\mathbb{Q})) &= \mathfrak{u} \\ \mathrm{Lie}(\overline{U}(\mathbb{Q})) &= \overline{\mathfrak{u}} \end{aligned}$$

$$\mathrm{Lie}(\Phi_0) = \mathfrak{sl}_2(\alpha_0), \quad \mathrm{Lie}(\Phi_7) = \mathfrak{sl}_2(\alpha_7), \quad \mathrm{Lie}(\Phi_8) = \mathfrak{sl}_2(\alpha_8).$$

The last 3 equations explain the awkward indices in §3.11. Furthermore, we can choose our identifications such that for $Z \in \mathfrak{u}$ (canonically identified with $\mathcal{J}$ as we have seen), $\exp(Z) = p'_Z$ and for $t = (0; aI, bI, cI; 0, 0, 0) \in \mathfrak{a}$, we have $\exp(t) = t(exp(a), exp(b), exp(c))$. In particular, we see that Φ_0, Φ_7, Φ_8 are root subgroups. Moreover, $L(\mathbb{Q})$ acts on $\mathrm{Lie}(G(\mathbb{Q}))$ via adjoint action and as a $L(\mathbb{Q})$-representation,

$$\mathrm{Lie}(G(\mathbb{Q})) = \overline{\mathcal{J}_{ad}} \oplus \mathfrak{l} \oplus \mathcal{J}_{ad}.$$

4 Group Action on The Tube Domain $\mathcal{T}$

4.1 First, we briefly recall the action $G(\mathbb{R})$ on the tube domain $\mathcal{T}$. See Baily [Ba] for details. The maps p' and ρ that we defined in §1.2 induces under base extension to give:

$$\begin{aligned} p' &: \mathcal{J}_{\mathbb{C}} \simeq \mathrm{Lie}(U(\mathbb{C})) \to U(\mathbb{C}) \\ \rho &: L(\mathbb{C}) \to GL(W \otimes \mathbb{C}) \end{aligned}$$

If $g \in G(\mathbb{R})$ and $Z \in \mathcal{T} \subset \mathcal{J}_{\mathbb{C}} \simeq U(\mathbb{C})$, it was shown in [Ba] that we can uniquely factorize $p'_Z g$ as

$$p'_Z g = p_A k p'_{Z_1}$$

with $k \in L(\mathbb{R})$ and $Z_1 \in \mathcal{T}$.

Then we set

$$\begin{aligned} Zg &= Z_1 \\ j(Z,g) &= \lambda(k). \end{aligned}$$

For our own convenience (and hopefully that of the reader), we shall define the left-action of $G(\mathbb{R})$ on $\mathcal{T}$ by:

$$\begin{aligned} gZ &:= Zg^{-1} \\ j(g,Z) &:= j(Z,g^{-1}). \end{aligned}$$

$G(\mathbb{R})$ acts transitively on $\mathcal{T}$. Let $e = iI$. Then the isotropy subgroup of e is the maximal compact subgroup of $G(\mathbb{R})$.

In the lemma below, we investigate the action of Φ_0 on $\mathcal{T}$.

Lemma 4.2 *(a) The action of $L(\mathbb{R})$ on $\mathcal{T}$ defined above agrees with its adjoint action on* $\mathrm{Lie}(U(\mathbb{C}))$.

(b) The element $t(a,0,0) \in T(\mathbb{R}) \subset L(\mathbb{R})$ is the element which in the natural representation acts as follows:

$$t(a,0,0)X = (\frac{1}{a}x, ay, az, ac_1, c_2, c_3)$$

(c) Let $x \in \mathbb{R}$ and $a \in \mathbb{R}^\times$, then

$$\begin{aligned} &(i) \quad \Phi_0\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot Z &&= Z - xe_1 \\ &(ii) \quad \Phi_0\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \cdot Z &&= (a^2x, y, z; c_1, ac_2, ac_3); \\ &(iii) \quad j(\Phi_0\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}, Z) &&= \frac{1}{a}. \end{aligned}$$

(d)

$$\mathrm{Tr}_{\mathcal{J}}(\left(\Phi_0\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \cdot Z\right) \circ Z') = \mathrm{Tr}_{\mathcal{J}}(Z \circ \left(\Phi_0\begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \cdot Z'\right)) \quad .$$

(e)

$$\iota Z = -(\det Z)^{-1}(Z \times Z).$$

Proof. (a) By definition, if $g \in L(\mathbb{R})$, then we need to look at $p'_Z g^{-1}$. But since L normalises U, $p'_Z g^{-1} = g^{-1}(g p'_Z g^{-1})$, and the result follows.

(b) Let $g = t(a,0,0) \in T(\mathbb{R})$. Then to see how g acts in the natural representation, it suffices to compute its action on $(X,0,0,0) \in W$. Going through the computation gives the result.

(c) (i) is clear from the definition. For (ii), we know that if $g \in L$, then $gZ = \lambda(g)g^*(Z)$(where the first action is that on the tube domain and the second is the natural action of g^* on $\mathcal{J}$). Now let $g = t(a, 0, 0)$. Then from (b),

$$g^*Z = (ax, \frac{1}{a}y, \frac{1}{a}z, \frac{1}{a}c_1, c_2, c_3)$$

and $\lambda(g) = a$. So $gZ = ag^*Z = (a^2x, y, z; c_1, ac_2, ac_3)$ as required.

For (iii), note that $j(g, Z) = \lambda(g^{-1}) = \frac{1}{a}$.

(d) is clear.

(e) See [Ba, pg 526]. ∎

5 Modular forms on $\mathcal{T}$

5.1 Let us fix a prime p. Then as G is split over $\mathbb{Q}_p$, each positive root α_i gives rise to an embedding

$$\begin{aligned} \Phi_i &: SL_2 \longrightarrow G \\ d\Phi_i &: \mathfrak{sl}_2 \longrightarrow \mathrm{Lie}(G). \end{aligned}$$

(Recall that we have defined Φ_0, Φ_7 and Φ_8 in §3.11)

Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$, and $\mathbb{A}_f$ the ring of finite adeles. Define

$$K_p = I_p \prod_{q \neq p, \infty} G(\mathbb{Z}_q) \subset G(\mathbb{A}_f)$$

where $G(\mathbb{Z}_q)$ is the hyperspecial maximal compact subgroup of $G(\mathbb{Q}_q)$ and I_p is the Iwahori subgroup of $G(\mathbb{Z}_p)$:

$$I_p = \langle \Phi_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, d \in \mathbb{Z}_p, c \in p\mathbb{Z}_p, \alpha \in \Phi^+ \rangle.$$

5.2 A holomorphic function $f : \mathcal{T} \to \mathbb{C}$ is a modular form of weight k and level p if, for all $g \in G(\mathbb{Z}) \cap I_p$,

$$f|_g(Z) := f(gZ)j(g, Z)^{-k} = f(Z) \tag{5.3}$$

In his thesis [K], Kim has constructed a singular modular form E_4 of weight 4 by the analytic continuation of Eisenstein series, and obtained its Fourier expansion [K]:

$$E_4(Z) = 1 + 240 \sum_{T \in R_1^+ \cap \mathcal{J}_\mathbb{Z}} \sigma_3(\Delta(T))e(T, Z) \tag{5.4}$$

Here,

$$\begin{aligned}
\Delta(T) &= max\{n \in \mathbb{Z}_{>0} : \frac{1}{n}T \in \mathcal{J}_{\mathbb{Z}}\}, \\
\sigma(n) &= \sum_{d|n} d^3, \\
e(T,Z) &= \exp(2\pi i(T,Z)).
\end{aligned}$$

This is in fact a modular form of level 1, ie it satisfies (5.3) for all $g \in G(\mathbb{Z})$.

5.5 Given a modular form f, we can regard f as a function on $G(\mathbb{Q})\backslash G(\mathbb{A}))$ which is right K_p-invariant in the following way:

Since G is simply connected and $G(\mathbb{R})$ is not compact, we have, by strong approximation,

$$G(\mathbb{A}) = G(\mathbb{Q})\, G(\mathbb{R})\, K_p \tag{5.6}$$

Hence every $g \in G(\mathbb{A})$ can be written as

$$g = \gamma h(ik) \tag{5.7}$$

with respect to the decomposition in equation (5.6). This decomposition is unique up to a factor of $G(\mathbb{Q}) \cap K_p = G(\mathbb{Z}) \cap I_p$.

Define

$$F : G(\mathbb{Q})\backslash G(\mathbb{A})/K_p \longrightarrow \mathbb{C}$$

by

$$F(\gamma hik) = f|_h(e) := f(h \cdot e)j(h,e)^{-k}.$$

This is well defined. Indeed if $(\gamma hik) = (\gamma' h' i' k')$ then $h = g_0 h'$ for some $g_0 \in G(\mathbb{Z}) \cap I_p$. However

$$\begin{aligned}
f(h \cdot e)j(h,e)^{-k} &= f(g_0 h' \cdot e)j(g_0 h', e)^{-k} \\
&= f(g_0 \cdot (h' \cdot e))j(g_0, h' \cdot e)^{-k} j(h', e)^{-k} \\
&= f(h' \cdot e)j(h', e)^{-k}
\end{aligned}$$

because f is a modular form of level p. Clearly we can easily recover f from F; so we will also call F a modular form of level p.

We denote the function on $G(\mathbb{A})$ corresponding to $E_4(Z)$ by $\tilde{E}_4$.

6 The Minimal Representation of G

6.1 We shall now describe the Hecke operators from the representation theoretic point of view. First, we review some facts about the Iwahori-Hecke algebra $\mathcal{H}_p$

of $G(\mathbb{Q}_p)$. By definition, $\mathcal{H}_p$ is the algebra of locally constant, compactly supported, I_p-bi-invariant functions on $G(\mathbb{Q}_p)$, with convolution as the composition. $\mathcal{H}_p$ is generated by $T_i, i = 0, 1, ..., 7$, with the following relations:

$$\begin{aligned} T_iT_j &= T_jT_i \qquad \text{if } \langle \beta_i, \beta_j \rangle = 0 \\ T_iT_jT_i &= T_jT_iT_j \qquad \text{if } \langle \beta_i, \beta_j \rangle = -1 \end{aligned}$$

$$T_i^2 = (p-1)T_i + p$$

where $\beta_i = \alpha_i$ for $i = 1, .., 7$,and $\beta_0 = -\alpha_0$.

$\mathcal{H}_p$ has a 8-dimensional "geometric representation" $U_p = \sum_{i=0}^{7} \mathbb{C}e_i$ defined as follows:

$$T_ie_j = \begin{cases} -e_j & \text{if } i = j; \\ pe_j + p^{\frac{1}{2}}e_i & \text{if } \langle \beta_i, \beta_j \rangle = -1; \\ pe_j & \text{if } \langle \beta_i, \beta_j \rangle = 0. \end{cases}$$

6.2 By a well-known result of Borel, there is an equivalence of categories between the admissible representations of $G(\mathbb{Q}_p)$ generated by their I_p-fixed vectors, and admissible representations of $\mathcal{H}_p$. This is given by the functor:

$$V \mapsto V^{I_p}$$

Let $\mathbb{V}_p$ be the **minimal representation** of $G(\mathbb{Q}_p)$ [Sa]. Then $\mathbb{V}_p$ is characterised by the fact that its space of Iwahori-fixed vectors $\mathbb{V}_p^{I_p}$ is isomorphic as a $\mathcal{H}_p$-module to U_p. In [Sa], Savin has determined the K-types of $\mathbb{V}_p$, and found that there is a unique $G(\mathbb{Z}_p)$-fixed vector, which we denote by φ_p. As a vector in the $\mathcal{H}_p$-module U_p, it is fixed by $T_1, ..., T_7$. Moreover, if $K_{p,1}$ is the first principal congruence subgroup of $G(\mathbb{Z}_p)$, then $\mathbb{V}_p^{K_{p,1}}$ is isomorphic, as a representation of $G(\mathbb{F}_p)$, to the direct sum of the trivial representation and the reflection representation of $G(\mathbb{F}_p)$. Similarly, let $\mathbb{V}_\infty$ be the minimal representation of $G(\mathbb{R})$ [G-Sa], and let φ_∞ be its minimal K-type.

We shall call $\mathbb{V}_\infty \bigotimes (\otimes_q \mathbb{V}_q) = \mathbb{V}_{min}$ the **minimal representation** of $G(\mathbb{A})$. It is known that $\mathbb{V}_{min}$ can be realised on the space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. In other words, there is an embedding of $G(\mathbb{A})$-module: $\mathbb{V}_{min} \longrightarrow L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Then the image of $\varphi_0 = \varphi_\infty \bigotimes (\otimes_q \varphi_q)$ is just the function $\tilde{E_4}$ [G-Sa].

6.3 Now, the functions in the 8-dimensional space $\mathbb{V}_p^{I_p} \bigotimes \varphi_\infty \bigotimes (\otimes_{q \neq p} \varphi_q)$ can be regarded as modular forms of level p on $\mathcal{T}$. Since U_p is irreducible as a $\mathcal{H}_p$-module, we should be able to obtain a basis of this 8-dimensional space by the action of Hecke operators. As φ_0 is fixed by $T_1, \ldots, T_7$, we can only begin by using T_0.

6.4 Using the definition of U_p above, simple linear algebra tells us that there is a unique one dimensional subspace $U_{p,i} \subset U_p$ on which T_j acts by multiplication by p for all $j \neq i$. Hence we have:

Proposition 6.5 *There is a canonical basis $\{f_i\}$ of the modular forms of level p such that f_i is right invariant under the maximal parahoric subgroup P_i of $G(\mathbb{Q}_p)$ corresponding to the vertex α_i in the Dynkin diagram.*

Note that Kim's form is just the one stable under the hyperspecial maximal compact subgroup. In the next few sections, we shall obtain the other 7 vectors in the canonical basis. The implications of this proposition will be discussed in the last section.

7 Hecke Operators

7.1 In this section, we shall describe the action of the generators $T_0, ..., T_7$ of the Iwahori-Hecke algebra $\mathcal{H}_p$ on the modular forms of level p in U_p. First, we shall identify the T_i's as compactly-supported, I_p-bi-invariant functions on $G(\mathbb{Q}_p)$. Recall that we have the homomorphisms (see section (5.1)):

$$\Phi_j : SL_2(\mathbb{Q}_p) \longrightarrow G(\mathbb{Q}_p), \qquad j = 0, \ldots, 7.$$

Set:

$$\begin{aligned} w_j &= \Phi_j\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right), \qquad j = 1, \ldots, 7 \\ w_0 &= \Phi_0\left(\begin{pmatrix} 0 & -\frac{1}{p} \\ p & 0 \end{pmatrix}\right). \end{aligned}$$

Note that since Φ_0 and Φ_7 are defined over $\mathbb{Z}$ (see §3.17), we have $w_0, w_7 \in G(\mathbb{Z}[\frac{1}{p}])$. Now the Hecke operator T_j is the characteristic function of the double coset $I_p w_j I_p$. Its action on modular forms of level p will be described below.

7.2 If

$$F : G(\mathbb{Q})\backslash G(\mathbb{A})/K_p \longrightarrow \mathbb{C}$$

is a modular form of level p and weight k, and di is the Haar measure on $G(\mathbb{Q}_p)$ giving I_p volume 1, then the action of T_j on F is given by:

$$T_j F(g) = \int_{I_p w_j I_p} F(gi)di \tag{7.3}$$

This integral is actually a finite sum. Indeed, by a result of Iwahori and Matsumoto [I-M], the double coset $I_p w_j I_p$ can be decomposed as:

$$I_p w_j I_p = \coprod_{s \in \mathbb{F}_p} \phi_s^j w_j I_p = \coprod_{s \in \mathbb{F}_p} I_p w_j \phi_s^j$$

where,

$$\begin{aligned}\phi_s^j &= \Phi_j\left(\begin{pmatrix}1 & s\\ 0 & 1\end{pmatrix}\right), \qquad j=1,...,7\\ \phi_s^0 &= \Phi_0\left(\begin{pmatrix}1 & 0\\ sp & 1\end{pmatrix}\right).\end{aligned}$$

Then if we have $g=\gamma hk$ as in equation (5.7) (with $i=1$), we have:

$$\begin{aligned}T_jF(g) &= \int_{I_p w_j I_p} F(gi)di\\ &= \sum_{s\in\mathbb{F}_p}\int_{\phi_s^j w_j I_p} F(gi)di\\ &= \sum_{s\in\mathbb{F}_p}\int_{I_p} F(g\phi_s^j w_j i)di\\ &= \sum_{s\in\mathbb{F}_p} F((\gamma hk)\phi_s^j w_j)\\ &= \sum_{s\in\mathbb{F}_p} F(h\phi_s^j w_j)\end{aligned}$$

7.4 Now by strong approximation, there exists $g_s\in G(\mathbb{Q})$ such that:

$$g_s^{-1}\phi_s^j w_j \in G(\mathbb{R})\cdot K_p \tag{7.5}$$

Then necessarily, $g_s\in G(\mathbb{Z}_q)$ for all $q\neq p$. For $j\neq 0$, $\phi_s^j w_j\in G(\mathbb{Z}_p)$; hence we have $g_s\in G(\mathbb{Z}_p)$ as well, so that g_s will necessarily lie in $G(\mathbb{Z})$. Hence, we have:

$$T_jF(g)=\sum_{s\in\mathbb{F}_p}F(g_s^{-1}h\phi_s^j w_j)=\sum_{s\in\mathbb{F}_p}F((g_s^{-1})_\infty h) \tag{7.6}$$

where $(g_s^{-1})_\infty$ denotes g_s^{-1} regarded as an element of $G(\mathbb{R})$.

7.7 If f is the function on $\mathcal{T}$ corresponding to F, so that:

$$f(h\cdot e)=j(h,e)^kF(g)$$

then we set:

$$(T_jf)(h\cdot e)=j(h,e)^kT_jF(g) \tag{7.8}$$

If $Z=h\cdot e$, and f has Fourier expansion given by:

$$f(Z)=\sum_T a_Te(T,Z)$$

then it follows from equations (7.6) and (7.8) that:

$$T_j f(Z) = \sum_{s\in\mathbb{F}_p} j(g_s^{-1}, Z)^k \sum_T a_T e(T, g_s^{-1}\cdot Z) \tag{7.9}$$

So far, everything has been formal. It is clear that the computation of the Hecke operators will depend on the determination of the g_s's. In the next section, we will do an example of such an explicit computation.

8 Action of Hecke Operator T_0

8.1 We shall now compute the action of the Hecke operator T_0 on the function $\tilde{E}_4$. Recall that T_0 is the characteristic function of the double coset $I_p w_0 I_p$ with $w_0 \in G(\mathbb{Q}_p)$ given by

$$w_0 = \Phi_0\left(\begin{pmatrix} 0 & -\frac{1}{p} \\ p & 0 \end{pmatrix}\right).$$

Note that w_0 is a lift of the element $\overline{w}_0$ in the affine Weyl group $\tilde{W}$ of $G(\mathbb{Q}_p)$, where $\overline{w}_0$ is the reflection with respect to the affine hyperplane $\{h : \alpha_0(h) = -1\}$.

Also, we shall denote:

$$\phi_s = \Phi_0\left(\begin{pmatrix} 1 & 0 \\ sp & 1 \end{pmatrix}\right).$$

so that:

$$\phi_s \cdot w_0 = \Phi_0\left(\begin{pmatrix} 0 & -\frac{1}{p} \\ p & -s \end{pmatrix}\right).$$

8.2 In this case, since Φ_0 is defined over $\mathbb{Z}$, the strong approximation is very easy: we can simply take

$$g_s^{-1} = (\phi_s \cdot w_0)^{-1} = \Phi_0\begin{pmatrix} -s & \frac{1}{p} \\ -p & 0 \end{pmatrix} \in G(\mathbb{Q}) \bigcap \left(\bigcap_{q\neq p} G(\mathbb{Z}_p)\right).$$

8.3 We shall now compute $T_0\tilde{E}_4$. Let $u \in U(\mathbb{R})$ and $Z = u\cdot e = -u + e \in \mathcal{T}$. Then, by equation (7.6),

$$T_0\tilde{E}_4(Z) = \sum_{s=0}^{p-1} \tilde{E}_4((g_s^{-1})_\infty u) = \sum_{s=0}^{p-1} E_4|_{g_s^{-1}u}(e) = \sum_{s=0}^{p-1} E_4|_{g_s^{-1}}(Z).$$

Case 1: $s \neq 0$. In this case we can find $a, b \in \mathbb{Z}$ such that

$$g_s' := \Phi_0\begin{pmatrix} a & b \\ p & -s \end{pmatrix} \in \Phi_0(SL_2(\mathbb{Z})).$$

Note that $a \equiv s^{-1} \pmod p$ and

$$g_s' g_s^{-1} = \Phi_0 \begin{pmatrix} 1 & -\frac{a}{p} \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} S_s(Z) &:= E_4|_{g_s^{-1}}(Z) \\ &= E_4|_{g_s' g_s^{-1}}(Z) \text{ (as it is invariant under } G(\mathbb{Z})) \\ &= E_4|_{\Phi_0 \begin{pmatrix} 1 & -\frac{a}{p} \\ 0 & 1 \end{pmatrix}}(Z) \\ &= E_4(\Phi_0 \begin{pmatrix} 1 & -\frac{a}{p} \\ 0 & 1 \end{pmatrix} Z) \end{aligned}$$

since

$$u' := \Phi_0 \begin{pmatrix} 1 & -\frac{a}{p} \\ 0 & 1 \end{pmatrix} \in U(\mathbb{R})$$

and $j(u', Z) = 1$.

Applying this to equation (5.4), we get

$$\begin{aligned} S_s(Z) &= 1 + \sum_{T \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}} a(T) e(T, \Phi_0 \begin{pmatrix} 1 & -\frac{a}{p} \\ 0 & 1 \end{pmatrix} Z) \\ &= 1 + \sum_{T \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}} a(T) e(T, Z + \frac{a}{p} e_1) \quad \text{(cf. Lemma 4.2(b)(ii))} \end{aligned}$$

where $e_1 = (1,0,0;0,0,0) \in \mathcal{T}$.

Case 2: $s = 0$.

$$\begin{aligned} S_0(Z) &:= E_4|_{\Phi_0 \begin{pmatrix} 0 & -\frac{1}{p} \\ p & 0 \end{pmatrix}}(Z) \\ &= E_4|_{\Phi_0 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Phi_0 \begin{pmatrix} 0 & -\frac{1}{p} \\ p & 0 \end{pmatrix}}(Z) \\ &\qquad \text{(as it is invariant under } G(\mathbb{Z})) \\ &= E_4|_{\Phi_0 \begin{pmatrix} p & 0 \\ 0 & -\frac{1}{p} \end{pmatrix}}(Z) \\ &= E_4(\Phi_0 \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix} Z) j(\Phi_0 \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix}, Z)^{-4} \\ &= E_4(\Phi_0 \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix} Z) p^4 \end{aligned}$$

$$= p^4 \left(1 + \sum_{T \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}} a(T) e(T, \Phi_0 \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix} Z) \right)$$

$$= p^4 \left(1 + \sum_{T \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}} a(T) e(\Phi_0 \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix} T, Z) \right) .\text{(cf. Lemma 4.2(c))}$$

Combining Cases 1 and 2, we have

$$\begin{aligned} T_0 E_4(Z) &= p^4 \left(1 + \sum_{T \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}} a(T) e(\Phi_0 \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix} T, Z) \right) + \\ &\quad + (p-1) + \sum_{T \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}} a(T) e(T, Z) (\sum_{a=1}^{p-1} e(T, \frac{a}{p} e_1)). \end{aligned}$$

Since

$$\sum_{a=0}^{p-1} e(T, \frac{a}{p} e_1) = \begin{cases} p & \text{if } p \,|\, t_1 \\ 0 & \text{otherwise} \end{cases}$$

where $T = (t_1, t_2, t_3; t_{23}, t_{31}, t_{12})$, we get

Proposition 8.4

$$\begin{aligned} T_0 E_4(Z) + E_4(Z) &= p^4 \left(1 + 240 \sum_{T \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}} \sigma_3(\Delta(T)) e(\Phi_0 \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix} T, Z) \right) \\ &\quad + p \left(1 + 240 \sum_{T \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}, p | t_1} \sigma_3(\Delta(T)) e(T, Z) \right). \ \blacksquare \end{aligned}$$

Alternatively

Proposition 8.4' *Let* $T = (t_1, t_2, t_3; t_{23}, t_{31}, t_{12}) \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}$ *and* $c(T)$ *be the coefficients of the Fourier expansion*

$$T_0 E_4(Z) + E_4(Z) = (p + p^4) + \sum_{T \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}} c(T) e(T, Z).$$

Then $c(T) \neq 0$ *only if* $p \,|\, t_1$ *and in this case*

$$c(T) = 240\, p\, \sigma_3(\Delta(T)) + \begin{cases} 240\, p^4\, \sigma_3(\Delta((\frac{t_1}{p^2}, t_2, t_3; \frac{t_{23}}{p}, \frac{t_{31}}{p}, t_{12})) \\ \quad \textit{if } p^2 \,|t_1 \textit{ and } p \,|\, t_{23}, t_{31} \\ 0 \quad \textit{otherwise.} \end{cases} \quad \blacksquare$$

Note that we can divide by p and still have integral coefficients.

8.5 Recall the definition of f_i and P_i in Proposition 6.5. First note that

$$f_1 = T_0 E_4 + E_4.$$

as a function on the adele group.

Since the minimal representation factors through the adjoint group $G^{ad}(\mathbb{A})$ of $G(\mathbb{A})$, we may assume that f_i are functions on $G^{ad}(\mathbb{Q})\backslash G^{ad}(\mathbb{A})$.

Let $\rho : G \to G^{ad}$ denote the canonical quotient map. Iwahori and Matsumoto [I-M] showed that over $G^{ad}(\mathbb{Q}_p)$, there is an automorphism of the affine root system given by conjugation by

$$\overline{\tau} = \overline{t}_7^{-1} w_L w_G$$

in the affine Weyl group of G^{ad}. Here, w_L and w_G are the longest elements in the finite Weyl groups of L and G respectively, and $\overline{t}_7$ is the translation in the direction of the fundamental co-weight ϖ_7 corresponding to α_7.

Now,

$$\varpi_7 = \frac{1}{2}(\alpha_0 + \alpha_7 + \alpha_8) \in X_\bullet(T^{ad})$$

where α_8 was defined in equation (3.16). So ϖ_7 is in fact defined over $\mathbb{Q}$, and so we have:

$$\varpi_7 : \mathbb{G}_m/\mathbb{Q} \longrightarrow G^{ad}/\mathbb{Q}$$

Hence the element $t_7 = \varpi_7(p) \in G^{ad}(\mathbb{Q})$ is a lifting of $\overline{t}_7$ in $G^{ad}(\mathbb{Q}_p)$. Note that if $V/\mathbb{Q}$ is any rational representation of G^{ad}, and V_ω is the weight space of V corresponding to the weight $\omega \in X^\bullet(T^{ad})$, then, for all $v \in V_\omega$,

$$t_7 v = p^{\langle \varpi_7, \omega \rangle} v.$$

It follows from this that t_7 is in fact an element of $G^{ad}(\mathbb{Z}[\frac{1}{p}])$. So $t_7 \in G^{ad}(\mathbb{Z}_q)$ for all $q \neq p$.

In §1.3 we defined ι and $\iota_0 \in G(\mathbb{Q})$. Both elements are defined over $\mathbb{Z}$. ι (resp. ι_0) is a lift of the longest element in the Weyl group of the real root systems of $G(\mathbb{Q})$ (resp. $L(\mathbb{Q})$) which is of type C_3 (resp. A_2). (Also see §3.17)

Let w_D be a lift of the longest element -1 in the Weyl group of $Spin(8) \subset G^{ad}(\mathbb{Q}_p)$. We can even choose $w_D \in G_2(\mathbb{Q}_p)$. Then w_D commutes with $\rho(\iota)$ and $\rho(\iota)w_D$ is a lift of w_G in $G^{ad}(\mathbb{Q}_p)$. Similarly w_D commutes with $\rho(\iota_0)$ and $\rho(\iota_0)w_D^{-1}$ is a lift of w_L. Indeed it suffices to check their actions on the root spaces as described in §3. Hence

$$\tau = t_7^{-1}\rho(\iota_0\iota_1) \in G^{ad}(\mathbb{Q}_p)$$

is a lift of $\overline{\tau}$. Indeed, $\tau \in G^{ad}(\mathbb{Z}[\frac{1}{p}]) \subset G^{ad}(\mathbb{Q})$. Then

$$\tau P_0 \tau^{-1} = P_7$$

$$\tau P_1 \tau^{-1} = P_6$$

Hence f_7 and f_6 are right translates of $\tilde{E}_4$ and f_1 by τ respectively.

Proposition 8.6 *(a) $f_7(Z) = E_4(pZ)$ (up to scalar multiple) as a function on $\mathcal{T}$.*
(b) f_6 has Fourier expansion

$$f_6(Z) = (p + p^4) + \sum_{T \in R_1^+ \cap \mathcal{J}_{\mathbb{Z}}} c(\iota_0 \cdot T) e(pT, Z).$$

where the coefficient $c(T)$ is as defined in Proposition 8.4'.

PROOF. (a) $\tilde{E}_4$ is right invariant under $G^{ad}(\mathbb{Z})$ and since $\iota_0 \iota$ is defined over $\mathbb{Z}$, f_7 is the just the right translate of $\tilde{E}_4$ by t_7^{-1}. Define (cf. equation (3.18))

$$t' := t(\sqrt{p}, \sqrt{p}, \sqrt{p}) \in L(\mathbb{R}).$$

and let $K_p^{ad} = I_p \prod_{q \neq p} G^{ad}(\mathbb{Z}_p)$. Then $\rho(t') = t_7 \in G^{ad}(\mathbb{R})$ and $t_7^{-1} \in G^{ad}(\mathbb{Q}_p)$ lies in the same double coset of

$$G^{ad}(\mathbb{Q}) \backslash G^{ad}(\mathbb{A}) / K_p^{ad}.$$

Let $Z = u \cdot e$ where $u \in U(\mathbb{R})$. Then (cf. §5.5)

$$f_7(Z) = \tilde{E}_4((t'u)_\infty) = p^6 E_4(t' \cdot Z).$$

By Lemma 4.2(c)(ii), $t' \cdot Z = pZ$. This proves (a).

(b) f_1 is right invariant under P_1 and since $\iota \in P_1$, f_6 is the just the right translate of f_1 by $t_7^{-1}\iota_0$. The result can then be proven using the same type of argument as in part (a) together with the fact that

$$e(T, \iota_0^{-1} t' \cdot Z) = e(p \iota_0^{-1} \cdot T, Z). \quad \blacksquare$$

9 Restriction to Upper Half Plane

9.1 As a check for our computation in the previous section, we can restrict the modular forms we obtained to a copy of the upper half plane $\mathfrak{h}$ in $\mathcal{T}$, and see what modular forms of SL_2 we get. For example, the map

$$\tau \mapsto \tau I$$

where $I = \mathrm{diag}(1, 1, 1) \in \mathcal{T}$, defines an embedding of $\mathfrak{h}$ into $\mathcal{T}$. If we embed SL_2 into G via $\Phi = \Phi_0 \Phi_7 \Phi_8$, then $\Phi(SL_2)$ acts on $\mathfrak{h}$ in the usual way.

9.2 For any function F on $\mathcal{T}$, and $\tau \in \mathfrak{h}$, we define:

$$f(\tau) = F(\tau I).$$

By Lemma 4.2, we have the following lemma:

Lemma 9.3 *For $g \in SL_2(\mathbb{Z})$, we have:*

$$f(g\tau) = F(\Phi(g)(\tau I))$$
$$j(\Phi(g), \tau I) = (c\tau + d)^3.$$

From the lemma, we deduce the following proposition:

Proposition 9.4 *If F is a modular form of weight k with respect to $I_p \cap G(\mathbb{Z})$, then f is a modular form of weight 3k with respect to $\Gamma_0(p)$.* ■

Hence $E_4(\tau I)$ and $T_0 E_4(\tau I)$ are modular forms of weight 12 with respect to $SL_2(\mathbb{Z})$ and $\Gamma_0(p)$ respectively.

9.5 Examples:

(i) $E_4(\tau I) = 1 + 240(3q + 747q^2 + ...) = E_4(\tau)^3$ where $q = exp(2\pi i\tau)$ and $E_4(\tau)$ is the usual Eisenstein series on the upper half plane.

(ii) Let $p = 2$ and consider $M_{12}(2)$, the modular forms of weight 12 with respect to $\Gamma_0(2)$. Let $M^0_{12}(2) \subset M_{12}(2)$ be the subspace of forms which vanish at the cusp at ∞. Then

$$M^0_{12}(2) = \langle E_4^3(\tau) - E_4^3(2\tau), \quad \Delta(\tau), \Delta(2\tau)\rangle.$$

Let

$$G = \frac{E_4(\tau I) + T_0 E_4(\tau I)}{2\sigma_3(2)}.$$

Then

$$G = 1 + 240(2q + 259q^2 + 4856q^3 + 248555q^4 + ...).$$

Moreover, if we consider $H = \frac{1}{240}(E_4^3 - G)$, then

$$\begin{aligned} H &= q + 488q^2 + 65788q^3 + 1405504q^4 + 18092766q^5 + ... \\ &= a\frac{1}{240}(E_4^3(\tau) - E_4^3(2\tau)) + b\Delta(\tau) + c\Delta(2\tau) \end{aligned}$$

Comparing the coefficients of q, q^2, q^3, we find that:

$$a = \frac{2^8}{3.7.13}, \quad b = \tfrac{-165}{7.13}, \quad c = \frac{-2^9.3^2.5}{7.13}$$

Computing the coefficient of q^4, q^5 in this linear combination, we get the same answer as those in H. This serves as a check for the Fourier coefficients obtained in Proposition 8.4.

10 Strong Approximation

From now on, we assume that $p \neq 2$ and, given an element $g \in G(\mathbb{Z}_p)$, we will denote its image mod p *by* $\overline{g} \in G(\mathbb{F}_p)$

10.1 As we have mentioned, to compute the action of T_j, one needs to determine the elements g_s in equation (7.5) explicitly. To find the remaining modular forms in U_p, we only need to compute the action of T_j, for $j = 1, ..., 6$ (T_7 can be computed as for T_0 since Φ_7 is also defined over $\mathbb{Z}$). For these, $w_j \in E_6(\mathbb{Z}_p) \subset L(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$. Since the form of E_6 is simply connected, it is possible to find $g_s \in E_6(\mathbb{Z}_p)$.

10.2 From the following commutative diagram:

$$\begin{array}{ccc} g_s \in E_6(\mathbb{Z}_p) \supset & I_p \\ \downarrow & \downarrow \\ \overline{g}_s \in E_6(\mathbb{F}_p) \supset & B_p \end{array}$$

where B_p is the Borel subgroup of $E_6(\mathbb{F}_p)$, we see that we need to find $g_s \in E_6(\mathbb{Z})$ such that:

$$\overline{g_s} B_p = \overline{\Phi}_j \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \overline{w_j} B_p = \overline{\Phi}_j \begin{pmatrix} 1 & s+1 \\ 0 & 1 \end{pmatrix} \overline{\Phi}_j \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} B_p. \qquad (10.3)$$

Here, $\overline{\Phi}_j$ is the corresponding embedding of the root subgroup corresponding to α_j over $\mathbb{F}_p$.

By equation (10.3), it suffices to find $g'_s, g''_s \in E_6(\mathbb{Z})$ such that

$$\overline{g'_s} = \overline{\Phi}_j \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \overline{g''_s} = \overline{\Phi}_j \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \in E_6(\mathbb{F}_p). \qquad (10.4)$$

Then we can set: $g_s = g'_{s+1} g''_{-1}$.

10.5 Since $g_s \in E_6(\mathbb{Z})$, we have $j(g_s, Z) = 1$, and $e(T, g_s^{-1} \cdot Z) = e(g_s^* \cdot T, Z)$. Hence, equation (7.9) becomes:

$$T_j f(Z) = \sum_s \sum_T a_T e(g_s^* \cdot T, Z) \qquad (10.6)$$

10.7 The remaining sections will be concerned with the determination of g_s for the Hecke operators $T_1, ..., T_6$, ie making the strong approximation explicit.

11 $E_6(k)$

11.1 To do strong approximation, we need more information about the structure of E_6. Recall that over a field k, $E_6(k)$ is defined as the Lie group acting on the Jordan algebra $\mathcal{J}(k)$ preseving a cubic form det. Thus it induces an action of its Lie algebra $\mathfrak{e}_6$ on $\mathcal{J}$ and we have shown in Section 3 and Lemma 4.2 that the representation $\mathcal{J}$ is indeed the adjoint action on $\overline{\mathfrak{u}} = \mathrm{Lie}(\overline{U})$. Moreover we have very explicit formulas about the Lie algebra action.

11.2 We are able to produce a lot of elements of $E_6(k)$ in the following way:

Denote ρ (or ρ_k) and $\overline{\rho}$ as the adjoint action of the $\mathfrak{e}_6(k)$ on $\mathfrak{u}(k)$ and $\overline{\mathfrak{u}}(k)$ respectively. Let $X \in \mathfrak{e}_6(k)$ and suppose $\rho(X)^3 = 0$. For any $s \in k$, define

$$g_s := 1 + s\rho(X) + \frac{s^2}{2}\rho(X)^2 \in E_6(k).$$

Indeed g preseves det so it lies in $E_6(k)$. The same works for $\overline{\rho}$ on $\overline{\mathfrak{u}}$.

11.3 Let us illustrate §11.2 with an example:

Let

$$X = \begin{pmatrix} x_1 & v_3 & -\overline{v_2} \\ -\overline{v_3} & x_2 & v_1 \\ v_2 & -\overline{v_1} & x_3 \end{pmatrix} \in \mathfrak{e}_6$$

where

$$x_i \in \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} : t \in k \right\}, \quad x_1 + x_2 + x_3 = 0, \quad v_i \in \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathbb{O}(k).$$

Further assume that

(1) $\{v_i\}$ generates a commutative quadratic subalgebra of $\mathbb{O}(k)$ and
(2) $X^2 = 0$ (usual matrix multiplication)

Recall a Thereom of Artin which states that a subalgebra of $\mathbb{O}(k)$ generated by 2 elements is associative. Hence

$$(XJ)X = X(JX) \quad \text{(usual matrix multiplication)}$$

where $J \in \mathfrak{u} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathcal{J}(k)$. Then

$$g_s \cdot J = \left(1 + s\rho(X) + \frac{s^2}{2}\rho(X)^2\right) J = 1 + sXJ - sJX - s^2(XJ)X.$$

11.4 In the next 2 sections, we will identify $\Phi_i(SL_2(k)) \subset E_6(k)$. We assume that $\mathbb{O}(k)$ splits so that $E_6(k)$ splits. Since $\mathfrak{u}$ (or $\overline{\mathfrak{u}}$) is a faithful representation of $E_6(k)$, we only need to find the adjoint action of $\Phi_i(SL_2(k))$ on it. By §11.2, we only have to find the adjoint action on $\mathfrak{u}$ of the following 2 elements:

$$\mathrm{d}\Phi_i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \in \quad \mathfrak{n}(E_6)$$

$$\mathrm{d}\Phi_i \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \in \quad \overline{\mathfrak{n}}(E_6)$$

where $\mathfrak{n}(E_6)$ and $\overline{\mathfrak{n}}(E_6)$ are opposite nilpotent subalgebras of $\mathfrak{e}_6$:

$$\begin{aligned}
\mathfrak{n}(E_6) &= \left\{(d;0,0,0;v_1,v_2,v_3) : d \in \mathfrak{n}(\mathfrak{d}_4), v_i \in \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{O}\right\} \\
\overline{\mathfrak{n}}(E_6) &= \left\{(d;0,0,0;v_1,v_2,v_3) : d \in \overline{\mathfrak{n}}(\mathfrak{d}_4), v_i \in \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \otimes \mathbb{O}\right\}.
\end{aligned}$$

12 Actions of T_1 and T_6

12.1 Let $k = \mathbb{F}_p, p \neq 2$. We will define Φ_1 and Φ_6 but first we have to determine $\mathfrak{g}_{\alpha_1}$ and $\mathfrak{g}_{\alpha_6}$.

From the table in [Bou, Planche VI] the roots are

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(\epsilon_8 + \epsilon_1) - \frac{1}{2}(\epsilon_2 + \ldots + \epsilon_7) \\
\alpha_6 &= \epsilon_5 - \epsilon_4.
\end{aligned}$$

and they are not roots of $\mathfrak{d}_4$.

We claim that

$$\mathfrak{g}_{\alpha_6} \subset (0;0,0,0;*,0,0) \bigcap \mathfrak{n}(E_6).$$

Indeed restricting to the CSA of $\mathfrak{d}_4$, $\mathfrak{g}_{\alpha_6}$ is the weight space X_{ϵ_4} and the claim follows from §3.2. Futhermore, let

$$\begin{aligned}
X_6 &:= (0;0,0,0; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \nu_4^*, 0, 0) \\
Y_6 &:= (0;0,0,0; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \nu_4, 0, 0),
\end{aligned}$$

then X_6 and Y_6 span $\mathfrak{g}_{\alpha_6}$ and $\mathfrak{g}_{-\alpha_6}$ respectively.

In a similar fashion, or by triality, define

$$\begin{aligned}
X_1 &:= (0;0,0,0;0,0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \nu_4^*) \\
Y_1 &:= (0;0,0,0;0,0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \nu_4).
\end{aligned}$$

Then X_1 and Y_1 span $\mathfrak{g}_{\alpha_1}$ and $\mathfrak{g}_{-\alpha_1}$ respectively.

For $i = 1, 6$, define the following elements of $E_6(\mathbb{F}_p)$:

$$\overline{\Phi}_i \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} := 1 + s\rho(X_i) + \frac{s^2}{2}\rho(X_i)^2 \tag{12.2}$$

$$\overline{\Phi}_i \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} := 1 + s\rho(Y_i) + \frac{s^2}{2}\rho(Y_i)^2. \tag{12.3}$$

This extends to a homomorphism $\overline{\Phi}_i : SL_2(\mathbb{F}_p) \to G(\mathbb{F}_p)$.

12.4 We will compute the action of T_1 on $f_1 = T_0 E_4 + E_4$. By §10.2 it suffices to find $g'_s, g''_s \in E_6(\mathbb{F}_p)$ satisfying equation (10.4). To achieve this, we rely on §11.3:

First, we find $u, v \in \mathbb{O}(\mathbb{Z})$ such that $u \mapsto \nu_4^*$ and $v \mapsto \nu_4$. Define

$$\tilde{X} := (0;0,0,0;0,0,\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes u)$$
$$\tilde{Y} := (0;0,0,0;0,0,\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes v)$$

Define

$$\begin{array}{l} g'_s := 1 + s\rho(\tilde{X}) + \frac{s^2}{2}\rho(\tilde{X})^2 \\ \\ g''_s := 1 + s\rho(\tilde{Y}) + \frac{s^2}{2}\rho(\tilde{Y})^2 \end{array} \in E_{6,2}(\mathbb{Z}).$$

Clearly by equations (12.2) and (12.3), g'_s and g''_s satisfy equation (10.4).

Let $g_{s-1} = g'_s g''_{-1}$ and $T = (a,b,c;x,y,z) \in \mathcal{J}$. Then

$$\begin{array}{rcl} g'^*_s(T) & = & T + (0, as^2 N(u) - s\,(u,z), 0; -s\overline{yu}, 0, -sau) \\ g''^*_{-1}(T) & = & T + (bN(v) + (v,z), 0, 0; 0, -vx, -bv) \\ g^*_{s-1}(T) & = & T + (bN(v) - s(v,z), s^2 N(u)(a + bN(v) - s(v,z)) - s(u,z), \\ & & 0; s\overline{(vx)}u - s\overline{yu}, -vx, -bv - su(a + bN(v) - s(v,z)). \end{array}$$

Here, $(\cdot,\cdot)$ is the inner product on $\mathbb{O}$ defined in §2.1.

With these, we can write down the Fourier coefficients of $T_1 f_1$ using equation (10.6).

12.5 We can do the same for Φ_6 except that we replace $\tilde{X}$ and $\tilde{Y}$ in §12.4 by

$$\tilde{X} := (0;0,0,0;\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes u, 0, 0)$$
$$\tilde{Y} := (0;0,0,0;\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes v, 0, 0)$$

13 Actions of $T_2, \ldots, T_5$

13.1 In this section, we devise a method of computing the actions of $T_2, \ldots, T_5$. Most of the proofs are lenghty and tedious but not difficult and we leave them to the reader. In addition we will not write down the actual computations of the Hecke operators.

13.2 Suppose G splits over a field k (for example $k = \mathbb{F}_p$) and let α be a positive root of $Spin(8) \subset G$ and $\mathfrak{g}_\alpha$ be its root space (cf §3.5). Recall from equations

(3.6), (3.7) and (3.8) that we may find $x, y, x', y' \in \{1\otimes\nu_i, 1\otimes\nu_i^* : i = 1, \dots, 4\} \subset W_1$ such that

$$X = X_\alpha := [x, y] \in \mathfrak{g}_\alpha, \quad Y = Y_\alpha := [x', y'] \in \mathfrak{g}_{-\alpha}.$$

Since $g'_s = \begin{pmatrix} 1 & 2s \\ 0 & 1 \end{pmatrix}$ and $g''_s = \begin{pmatrix} 1 & 0 \\ 2s & 1 \end{pmatrix}$ generates $SL_2(k)$, define $\Phi_\alpha : SL_2 \to Spin(8) \subset G$ by

$$\begin{aligned} \Phi_\alpha(g'_s) &:= 1 + s \cdot \rho(X), \\ \Phi_\alpha(g''_s) &:= 1 + s \cdot \rho(Y). \end{aligned}$$

Indeed by §11.2 $\Phi_\alpha(g'_s)$ and $\Phi_\alpha(g''_s)$ lies in $Spin(8)$ since $\rho(X)^2 = \rho(Y)^2 = 0$. One checks that this is a group homomorphism.

Similarly, consider $\nu \in \{1 \otimes \nu_i, 1 \otimes \nu_i^*\}$ and define

$$X_\nu = (0; 0, 0, 0; \nu, 0, 0) \in \mathfrak{e}_6(k)$$

and since $\rho(X_\nu)^3 = 0$, we define (cf. §11.3)

$$e(\nu) := 1 + \rho X_\nu + \frac{1}{2}\rho X_\nu^2 \in E_6(k).$$

Lemma 13.3

$$\begin{aligned} \Phi_\alpha(g'_s) &= e(sx)e(y)e(-sx)e(-y), \\ \Phi_\alpha(g''_s) &= e(sx')e(y')e(-sx')e(-y'). \end{aligned}$$

Proof. These are well known formulae for Chevalley groups. ■

13.4 We specialise to $k = \mathbb{F}_p$ and we will compute the actions of T_i where $i = 2, 3, 4, 5$.

By §10.2, we have to find g'_s and g''_s satisfying equation (10.4) which is rather difficult as $Spin_8(\mathbb{Z})$ is finite. Thanks to Lemma 13.3, it is sufficient to approximate $e(x), e(y), e(x')$ and $e(y')$. We will only do it for $e(x)$ and the rest is similar. To achieve this it is enough to find $\tilde{X} \in \mathfrak{e}_6(\mathbb{Z})$ such that

(1) $\tilde{X} \overset{\bmod p}{\mapsto} X = (0; 0, 0, 0; 1 \otimes x, 0, 0) \in \mathfrak{e}_6(\mathbb{F}_p)$.

(2) $\rho_{\mathbb{Q}}(\tilde{X})^3 = 0$.

so that

$$\gamma_s := 1 + s\rho(\tilde{X}) + \frac{s^2}{2}\rho(\tilde{X})^2 \in E_6(\mathbb{Z})$$

and $e(x) = \gamma_s \pmod p$.

Recall from §2.7 that there exists a lift $\tilde{x} \in \mathbb{O}(\mathbb{Z})$ of x such that $N(x) = (1 + np)p^2$ for some integer n. Define

$$\tilde{X} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & (1+np)pI \otimes 1 & z \otimes \tilde{x} \\ 0 & -\overline{z} \otimes \overline{\tilde{x}} & -(1+np)pI \otimes 1 \end{pmatrix} \in \mathfrak{e}_6(\mathbb{Z})$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$z = \begin{pmatrix} 1+np & 0 \\ 0 & 1 \end{pmatrix}, \quad \overline{z} = \begin{pmatrix} 1 & 0 \\ 0 & 1+np \end{pmatrix}.$$

It is routine to check that the above conditions (1) and (2) for $\tilde{X}$ are met.

13.5 In §13.4 we gave a rather complicated method for calculating the Hecke operators T_i. In the remaining section we will show an easier method in calculating T_3 which can also be applied to T_2 and T_5. There will be some tedious but straightforward computations and again we omit them.

We assume that the modular form f is invariant under $\Phi_2(SL_2(\mathbb{Z}_p))$ (for example f_3 in §6.3), so that $T_2 f = pf$. In this case we could weaken the argument in §10.2 by replacing the Iwahori subgroup I_p with the parahoric subgroup generated by I_p and Φ_2. Then this allows us to replace g_s in §10.2 by $h'_s = g_s t_s$ where $t_s \in \Phi_2(SL_2(\mathbb{Z}))$. In addition note that $\Phi_2(SL_2)$ and $\Phi_3(SL_2)$ commutes thus we conclude that (cf. §10.5)

$$T_3 f(Z) = \sum_{s \in \mathbb{F}_p} \sum_T a_T e(h_s^* T, Z) \tag{13.6}$$

where $h_s \in E_6(\mathbb{Z})$ satisfies

$$\overline{h_s} B_p = \delta_1 \delta_2 B_p$$

and

$$\delta_1 := \overline{\Phi}_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \overline{\Phi}_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\delta_2 := \overline{\Phi}_3 \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \overline{\Phi}_2 \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}.$$

For the rest of this section, we will find $\tilde{\delta_1}$ and $\tilde{\delta_2}$ in $E_6(\mathbb{Z})$ such that taking $\bmod p$ we get

(1) $\overline{\tilde{\delta_1}} = \delta_1$

(2) $\overline{\tilde{\delta_2}} \in \delta_2 B_p$.

We may take $h_s = \tilde{\delta_1}\tilde{\delta_2}$.

13.7 First we deal with δ_1. Recall §2.6 that

$$k := ij \overset{\bmod p}{\mapsto} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{F}_p).$$

Define

$$\tilde{\delta_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -k \otimes 1 & 0 \\ 0 & 0 & k \otimes 1 \end{pmatrix}$$

which acts on $J = (a, b, c; x, y, z) \in \mathfrak{u}(\mathbb{Q})$ by

$$J \mapsto (\tilde{\delta}_1 J)\tilde{\delta}_1^{-1} = \tilde{\delta}_1(J\tilde{\delta}_1^{-1}) = (a, b, c; kxk, ky, zk). \tag{13.8}$$

Lemma 13.9 *(a)* $\tilde{\delta}_1 \in Spin(\mathbb{O}(\mathbb{Z})) \in E_6(\mathbb{Z})$

(b) $\tilde{\delta}_1 \overset{\bmod p}{\mapsto} \delta_1$.

Proof. (a) By definition, $\tilde{\delta}_1 \in Spin(\mathbb{O}(\mathbb{Z}))$ if and only if for all $x, y, z \in \mathbb{O}$ [L, Chap. 1]

$$\begin{aligned} \operatorname{tr}((xy)z) &= \operatorname{tr}(((\tilde{\delta}_1 x)(\tilde{\delta}_1 y))(\tilde{\delta}_1 z)) \\ &= \operatorname{tr}(((kxk)(ky))(zk)) \\ &= \operatorname{tr}((k((kxk)(ky)))z) \end{aligned}$$

and by the non-degeneracy of the trace, this is equivalent to

$$k((kxk)(ky))) = xy.$$

To prove the last equality we just have to check $x = e_i$, $y = e_j$ for all i, j and we leave this as an amusing exercise for the reader.

(b) This is due to the fact that the Lie subalgebra

$$\{\operatorname{diag}(x_1 \otimes 1, x_2 \otimes 1, x_3 \otimes 1) : x_i \in \mathbb{H}, \operatorname{tr}(x_i) = 0\} \subset \mathfrak{e}_6$$

corresponds to the embedding of the $\mathfrak{su}_2$'s corresponding to the roots α_2, α_3 and α_5.

Since $k \mapsto w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{F}_p)$ after taking $\bmod p$, $\overline{\tilde{\delta}_1}$ should act on $\overline{J} \in \mathfrak{u}(\mathbb{F}_p)$ as according to equation (13.8) with w in place of k. Thus $\overline{\tilde{\delta}_1}$ has the same action on $\mathfrak{u}(\mathbb{F}_p)$ as δ_1. ∎

13.10 Next we deal with δ_2. Choose $x \in \mathbb{O}(\mathbb{Z})$ such that $x \overset{\bmod p}{\mapsto} \nu_2^*$. Therefore $N(x) = rp$ for some nonzero integer r. Consider

$$X := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 \otimes x & z \otimes 1 \\ 0 & -\overline{z} \otimes 1 & 1 \otimes x \end{pmatrix} \in \mathfrak{e}_6(\mathbb{Z})$$

where

$$z = \begin{pmatrix} -1 & 0 \\ 0 & rp \end{pmatrix}, \quad \overline{z} = \begin{pmatrix} rp & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\rho_{\mathbb{Q}}(X)^2 = 0$ and we can apply §11.2 so that

$$\tilde{\delta}_2 := 1 + s\rho(X) \in E_6(\mathbb{Z}).$$

Define

$$Y := (d; 0, -1 \otimes \nu_2^*, 1 \otimes \nu_2^*; 0, 0, 0) \in \mathfrak{e}_6(\mathbb{F}_p).$$

and computations yield

Lemma 13.11 *(a)* $\delta_2 = 1 + s\rho(Y) \in E_6(\mathbb{F}_p)$

(b) $(\overline{\tilde{\delta}_2})^{-1}\delta_2 \in B_p$. ■

With this we complete the discussion on the computation of $T_2 f$.

13.12 By triality

$$(d; x_1, x_2, x_3; v_1, v_2, v_3) \mapsto (\pi_{12}(d); x_2, x_1, x_3; -\overline{v_2}, -\overline{v_1}, -\overline{v_3})$$

the same proceedure for computing $T_3 f_i$ where f_i is invariant under Φ_5 instead of Φ_2.

Again by triality, it is clear how the argument can be modified to compute T_2 or T_5.

14 Possible Applications to Dual Pair Correspondence

14.1 In the previous sections, we have given the full details needed to compute the action of the Hecke operators T_i. This allows us, in principle, to determine the Fourier coefficients of the modular forms of level p in the minimal representation. In this final section, we would like to indicate our initial motivations for doing these computations. Essentially it concerns global Θ-correspondences.

14.2 There are various dual reductive pairs in G [G-Sa]. These are: $F_4^{cpt} \times SL_2$, $D_4^{cpt} \times SL_2^3$ and $G_2^{cpt} \times Sp_6$, where H^{cpt} denotes a compact or anisotropic form of H. Let us denote a typical dual pair by $H_1^{cpt} \times H_2$.

Let $\alpha : H_1(\mathbb{Q})\backslash H_1(\mathbb{A}) \longrightarrow \mathbb{C}$ be a square-integrable automorphic form on H_1. Then its Θ-lift to H_2 is defined to be:

$$\beta(g') = \int_{H_1(\mathbb{Q})\backslash H_1(\mathbb{A})} \theta(g'g)\alpha(g)dg$$

for $g \in H_1(\mathbb{A})$, $g' \in H_2(\mathbb{A})$, and some $\theta \in \mathbb{V}_{min}$. It is an interesting question to see if β is non-zero or not. Suppose that α has level p, and trivial weight in the sense of [G-Sa, Chapter 1, Section 2]. This means that α is right-invariant under the open compact subgroup $H_1(\mathbb{R}) \cdot K_p$, where $K_p = I_p \cdot \prod_{q \neq p} H_1(\mathbb{Z}_q)$. Here, I_p is equal to an Iwahori subgroup of $H_1(\mathbb{Q}_p)$, as usual. Now, let us choose $\theta \in \mathbb{V}_{min}$ to be right-invariant under $H_1(\mathbb{R}) \cdot K_p$ also; for example θ could be one of the modular forms of level p that we computed before. Then putting $S = H_1(\mathbb{Q})\backslash H_1(\mathbb{A})/H_1(\mathbb{R}) \cdot K_p$, we have the finite sum:

$$\beta(g') = \sum_{s \in S} \alpha(s)\theta(sg')$$

Hence, if we regard β as a modular form on the hermitian symmetric domain $\mathfrak{h}$ corresponding to H_2, then we can see if β vanishes or not by looking at its Fourier expansion. The above equation shows that the Fourier coefficients of β is simply a linear combinations of the Fourier coefficients of θ restricted to various different embeddings of $\mathfrak{h}$ in $\mathcal{T}$.

14.3 The most interesting dual pair in G, in view of Langlands functoriality, is $G_2^{cpt} \times Sp_6 \subset E_{7,3}$ which was studied in [G-Sa]. There, they explicitly constructed 2 automorphic forms on G_2 and showed that both lift non-trivially to Sp_6. Let $\alpha : G_2(\mathbb{Q})\backslash G_2(\mathbb{A}) \to \mathbb{C}$ be any square-integrable automorphic form on G_2, and let its Θ-lift be β.

In [G-Sa, Corollary 4.9], it was shown that β is cuspidal if the automorphic representation corresponding to α admits a Whittaker functional at some finite local component. Let us assume that α is unramified at all primes not equal to p, and generates the Steinberg representation at p so that the above condition is satisfied. Further, if β is non-zero, then it will generate an automorphic representation which is unramified outside of p, and is the Steinberg representation at p. This follows from the results of local theta correspondence in [G-Sa] and [M-Sa].

In order to obtain a non-zero lift, we have to choose a θ which, when restricted to G_2, is not invariant under any parahoric subgroup (at p) other than the Iwahori subgroup of G_2. This is because, for the Steinberg representation, the space of vectors fixed by any parahoric subgroup which is strictly larger than the Iwahori subgroup is zero. Hence if θ is invariant under a parahoric subgroup other than the Iwahori subgroup, then the restriction of θ (and its translates by elements of Sp_6) to G_2 will be perpendicular to α. This implies that the integral defining the theta lift vanishes.

In view of Proposition 6.5, the above discussion implies that the eight dimensional space of Iwahori-fixed vectors in $\mathbb{V}_{min}$ will give the zero theta lift, a fact that we have overlooked at the beginning. Indeed, the intersection of any (standard) maximal parahoric subgroup of $E_7(\mathbb{Q}_p)$ with $G_2(\mathbb{Q}_p)$ contains a parahoric subgroup which is strictly larger than the Iwahori subgroup of G_2.

However, the reflection representation of $E_7(\mathbb{F}_p)$, when restricted to $G_2(\mathbb{F}_p) \times Sp_6(\mathbb{F}_p)$, contains a (unique) summand $St_{G_2} \otimes St_{Sp_6}$, where St_{G_2} and St_{Sp_6} are the Steinberg representations of G_2 and Sp_6 respectively. Hence, one should really choose θ to be $\varphi_\infty \bigotimes (\otimes_{q \neq p} \varphi_q) \bigotimes \varphi_{St_p}$, where φ_{St_p} corresponds to the unique Borel-fixed vector in $St_{G_2} \otimes St_{Sp_6}$.

Unfortunately, we do not know how to pinpoint this particular automorphic form in the space of the global minimal representation. In particular, we do not know how to obtain it from Kim's form. Moreover, it is clear that, even if one has the explicit Fourier expansion of this form, it would still take a tremendous amount of computation to settle the question of non-vanishing of global theta lifts. However, we hope that the modular forms of level p that we have computed in this paper will find some use in some other situations.

References

[Ba] Walter L. Baily, Jr *An Exceptional Arithmetic Group and its Eisenstein Series.* Ann. of Math. **91** pg 512-549 (1970).

[Bo] A. Borel *Admissible Representation of a Semi-simple Group over a Local Field with Vectors Fixed under an Iwahori Subgroup* Invent. Math. **35** pg 233-259, (1976).

[Bou] N. Bourbaki *Groupes et algebres de Lie*, Chapitres 4, 5 et 6. Hermann, Paris (1968).

[E-G] N. Elkies and B. Gross, *The Exceptional Cone and the Leech Lattice.* Preprint.

[G-Sa] B. Gross and G. Savin, *Motives with Galois Group of Type G_2: an Exceptional Theta Correspondence.* Preprint.

[I-M] N. Iwahori and H. Matsumoto, *On some Bruhat Decomposition and the Structure of the Hecke Rings of a p-adic Chevalley Groups.* I.H.E.S. **25** pg 5-48 (1965).

[J1] N. Jacobson *Exceptional Lie Algebras.* Marcel Dekker, New York (1971).

[J2] N. Jacobson *Lie Algebras.* Dover Publications Inc, New York. (1962).

[L] H. Y. Loke *Exceptional Lie Algebras and Lie Groups*, Harvard Thesis. (1997).

[M-Sa] K. Magaard and G. Savin, *Exceptional Theta Correspondences.* To appear in Compositio Math.

[O-V] A. L. Onishchik and E. B. Vinberg *Lie Groups and Lie Algebras III: Structure of Lie Groups and Lie Algebras.* Encyclopaedia of Mathematical Sciences, Springer-Verlag.

[K] Henry Kim *Exceptional Modular form of Weight 4 on an Exceptional Domain contained in* $\mathbb{C}^{27}$. Revista Matematica Iberoamericana **9**, No. 1, (1993).

[Sa] Gordon Savin *K-types of Minimal Representations (p-adic case).* Preprint.

[Sch] R. Schafer *An Introduction to Nonassociative Algebras.* Academic Press (1966).