

ON SHALIKA PERIODS AND A THEOREM OF JACQUET-MARTIN

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ABSTRACT. Let π be a cuspidal automorphic representation of $GL_4(\mathbb{A})$ with central character μ^2 . It is known that π has Shalika period with respect to μ if and only if the L -function $L^S(s, \pi, \wedge^2 \otimes \mu^{-1})$ has a pole at $s = 1$. In [JM], Jacquet and Martin considered the analogous question for cuspidal representations π_D of the inner form $GL_2(D)(\mathbb{A})$, and obtained a partial result via the relative trace formula. In this paper, we provide a complete solution to this problem via the method of theta correspondence, and give necessary and sufficient conditions for the existence of Shalika period for π_D . We also resolve the analogous question in the local setting.

1. Introduction

Let F be a number field with adèle ring \mathbb{A} , and let D be a (possibly split) quaternion algebra over F . We consider the linear algebraic group $GL_2(D)$, so that if D is split, then $GL_2(D) \cong GL_4$. The group $GL_2(D)$ is thus an inner form of GL_4 . Let π_D be a cuspidal automorphic representation of $GL_2(D)(\mathbb{A})$ and assume that its central character ω_{π_D} is a square, say

$$\omega_{\pi_D} = \mu^2.$$

One may consider the Shalika period of π_D with respect to μ . More precisely, $GL_2(D)$ has a parabolic F -subgroup $P_D = M_D \cdot N_D$ with Levi factor and unipotent radical given by:

$$\begin{cases} M_D \cong D^\times \times D^\times; \\ N_D \cong D. \end{cases}$$

Let ψ_D be the nondegenerate unitary character of $N_D(F) \backslash N_D(\mathbb{A})$ defined by $\psi_D(x) = \psi(\text{Tr}_D(x))$ for $x \in N_D(\mathbb{A}) \cong D_\mathbb{A}$. Its stabilizer in P_D is the Shalika subgroup

$$\tilde{S}_D = \Delta D^\times \cdot N_D$$

and we may extend ψ_D to a character of \tilde{S}_D via:

$$\psi_D(h \cdot n) = \mu(\mathbb{N}_D(h)) \cdot \psi_D(n) \quad \text{for } h \in D^\times \text{ and } n \in N_D.$$

We shall in fact mostly be concerned with the quotient group

$$S_D = \tilde{S}_D / \Delta \mathbb{G}_m = PD^\times \cdot N_D.$$

The Shalika period of π_D is the linear form on π_D defined by:

$$\mathcal{S}_D : f \mapsto \int_{S_D(F) \backslash S_D(\mathbb{A})} f(nh) \cdot \mu(\mathbb{N}_D(h))^{-1} \cdot \psi_D(n)^{-1} dn dh.$$

We say that π_D has Shalika period with respect to μ if the linear form \mathcal{S}_D is non-zero. If μ is trivial, then we simply say that π_D has Shalika period. In the following, if D is split, we shall suppress the symbol D from the above notations. So, for example, GL_4 has a parabolic subgroup $P = M \cdot N$.

Now suppose that D is split, so that $GL_2(D) = GL_4$. There is a well-known theorem of Jacquet and Shalika [JS] that relates the existence of Shalika period on GL_{2n} to the existence of poles of a twisted exterior square L -function. For the case of GL_4 , their theorem reads as follows.

Theorem 1.1 (Jacquet-Shalika). *Let π be a cuspidal automorphic representation of $GL_4(\mathbb{A})$ whose central character is μ^2 . Then the following are equivalent:*

- (a) π has Shalika period with respect to μ .
- (b) The (incomplete) twisted exterior square L -function $L^S(s, \pi, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 1$.

Now it is natural to ask whether the same theorem holds when D is not split. In their recent paper [JM], Jacquet and Martin obtained the following result by using the relative trace formula.

Theorem 1.2 (Jacquet-Martin). *Suppose that D is a quaternion division algebra and π_D is a cuspidal representation of $GL_2(D)(\mathbb{A})$ which has a cuspidal Jacquet-Langlands lift π to GL_4 . Further assume that*

- (i) D is non-split at some archimedean place;
- (ii) π_D has trivial central character;
- (ii) π_D has at least one supercuspidal local component at a place where D splits.

Then

$$\pi_D \text{ has Shalika period} \implies \pi \text{ has Shalika period.}$$

In light of Theorem 1.1, their theorem shows that, modulo some technicalities, if π_D has Shalika period, then the exterior square L -function $L^S(s, \pi_D, \bigwedge^2)$ has a pole at $s = 1$. Then the natural question to ask is whether the converse is also true; in other words, whether the analog of Thm. 1.1 remains true for non-split D .

In this paper, we resolve this question completely by giving a characterization of the existence of Shalika period for all cuspidal representations π_D . It turns out that the converse of the Jacquet-Martin theorem does not hold and must be augmented with a certain local condition. Our main theorem is:

Theorem 1.3. *Let D be a quaternion division algebra and Σ_D the set of places at which D ramifies. Further assume that D splits at every archimedean place.*

(i) *Suppose that π_D is a cuspidal automorphic representation on $GL_2(D)(\mathbb{A})$ with central character μ^2 and whose Jacquet-Langlands lift $JL(\pi_D)$ to $GL_4(\mathbb{A})$ is cuspidal. Then the following are equivalent.*

- (A) π_D has Shalika period with respect to μ .
- (B) The (incomplete) twisted exterior square L -function $L^S(s, \pi_D, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 1$, and for all $v \in \Sigma_D$, $\pi_{D,v}$ is not of the form $\text{Ind}_{P_D,v}^{GL_2(D_v)} \delta_{P_D}^{1/2} \cdot (\tau_{D,1,v} \boxtimes \tau_{D,2,v})$ where $\tau_{D,i,v}$ are representations of D_v^\times with central character μ_v .

(ii) *Suppose that π_D is cuspidal with central character μ^2 but its Jacquet-Langlands lift $JL(\pi_D)$ to $GL_4(\mathbb{A})$ is not cuspidal. In this case, $JL(\pi_D)$ is contained in the residual spectrum and is isomorphic to the unique irreducible quotient of $\text{Ind}_P^{GL_4} \delta_P^{1/2} \cdot (\tau - |\cdot|^{1/2} \boxtimes \tau - |\cdot|^{-1/2})$ for a cuspidal representation τ of $GL_2(\mathbb{A})$. Then the following are equivalent:*

- (C) π_D has Shalika period with respect to μ .
- (D) μ is equal to the central character ω_τ of τ .
- (E) The (incomplete) twisted exterior square L -function $L^S(s, \pi_D, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 2$.

The reason for assuming that D is split at every archimedean place in the above theorem is that we make use of recent results of Badulescu [B] concerning the Jacquet-Langlands correspondence and this assumption is present in his work. In fact, for the Jacquet-Martin theorem, which is part of the implication $(A) \implies (B)$, one does not need the assumption that D be split at all archimedean places. But, for the ease of exposition, we will make this assumption throughout this paper; see, however, Remarks 2.3, 3.5 and 5.8.

Clearly the interesting point in our theorem is the local condition in (B), which is not present in the split case. Let us briefly explain the origin of this local condition. For each place v of F and a representation $\pi_{D,v}$ of $GL_2(D_v)$ with central character μ_v^2 , we say that $\pi_{D,v}$ has local Shalika period with respect to μ_v if

$$\mathrm{Hom}_{\Delta D_v^\times \cdot N_{D,v}}(\pi_{D,v}, (\mu \circ \mathbb{N}_{D,v}) \boxtimes \psi_{D_v}) \neq 0.$$

It is known that this Hom space has dimension at most 1. One may consider the problem of existence of local Shalika periods, and indeed we will show that a generic representation π_v of $GL_4(F_v)$ has a local Shalika period if and only if its Langlands parameter factors through $GSp_4(\mathbb{C})$, *i.e.* is of symplectic type. Suppose that this holds and $\pi_{D,v}$ is the local Jacquet-Langlands lift of π_v to $PGL_2(D)$. Then it is possible that $\pi_{D,v}$ does not have local Shalika period. Indeed, whether $\pi_{D,v}$ has a local Shalika period or not is an issue addressed by a special case of the local Gross-Prasad conjecture, and thus it is controlled by a local epsilon factor condition. To show the implication $(B) \implies (A)$, one needs (at least) these local epsilon factor conditions to be satisfied.

However, let us mention here that even if the local epsilon factor conditions are satisfied, it turns out that they are not sufficient for the global representation π_D to have Shalika period. In fact, we prove the following perhaps somewhat surprising result:

Theorem 1.4. *Assume that D is split at every archimedean place. There are cuspidal representations π_D of $PGL_2(D)$, with a cuspidal Jacquet-Langlands lift to PGL_4 , satisfying:*

- (i) *for all places v , $\pi_{D,v}$ has local Shalika period, and*
- (ii) *$L^S(s, \pi_D, \bigwedge^2)$ has a pole at $s = 1$, but*
- (iii) *π_D does not have global Shalika period.*

We should mention that though the study of the existence of local Shalika periods elucidates the nature of the local conditions in our theorem, the proofs of the above global theorems are largely independent of this local study. The exception is Prop. 3.4, whose proof relies on the local study of Section 8.

Our main local results, which complete some initial work of D. Prasad, are summarized as follows:

Theorem 1.5. *Let F_v be a non-archimedean local field and D_v the unique quaternion division algebra over F_v .*

(i) *Let π_v be a generic representation of $GL_4(F_v)$ with central character μ_v^2 . Then π_v has Shalika period with respect to μ_v if and only if the Langlands parameter φ_{π_v} of π_v factors through $GSp_4(\mathbb{C})$ with similitude character μ_v .*

If π_v is a discrete series representation, the above conditions are equivalent to:

$$L(s, \pi_v, \bigwedge^2 \otimes \mu_v^{-1}) \text{ has a pole at } s = 0.$$

Here, the local L -function $L(s, \pi_v, \bigwedge^2 \otimes \mu_v^{-1})$ is that defined by Shahidi, and is equal to the Artin L -function $L(s, \bigwedge^2 \varphi_{\pi_v} \otimes \mu_v^{-1})$ by a recent result of Henniart [He].

(ii) Let $\pi_{D,v}$ be a representation of $GL_2(D_v)$ with generic Jacquet-Langlands transfer $\pi_v = JL(\pi_{D,v})$ on GL_4 and central character μ_v^2 . Then $\pi_{D,v}$ has Shalika period with respect to μ_v if and only if

$$\epsilon(1/2, (\bigwedge^2 \varphi_{\pi_v} \otimes \mu_v^{-1}) \otimes S_2) = -1.$$

Here S_2 denotes the 2-dimensional representation of $SL_2(\mathbb{C})$ (which is the Langlands parameter of the Steinberg representation of GL_2) and $(\bigwedge^2 \varphi_{\pi_v} \otimes \mu_v^{-1}) \otimes S_2$ is a representation of the Weil-Deligne group $W_{F_v} \times SL_2(\mathbb{C})$ of F_v . Moreover, if the above holds, then the Jacquet-Langlands transfer π_v has Shalika period with respect to μ_v .

In fact, we also determine whether a generalized Speh representation of GL_4 or $GL_2(D)$ has Shalika period with respect to μ ; the result is contained in Thm. 8.6 and is the local analog of Theorem 1.3(ii). Since those local results are largely independent of our global results, we will take them up at the end of the paper (Section 7 and 8).

The main technique used in this paper is the theta correspondence (for similitudes). Indeed, consider the quadratic space

$$(V_D, q_D) = (D, \mathbb{N}_D) \oplus \mathbb{H}$$

where \mathbb{H} is the hyperbolic plane. Then one has

$$GSO(V_D) \cong (GL_2(D) \times GL_1) / \{(z, z^{-2}) : z \in GL_1\}.$$

To see this, note that the quadratic space V_D can also be described as the space of 2×2 -Hermitian matrices with entries in D , so that a typical element has the form

$$(a, d; x) = \begin{pmatrix} a & x \\ \bar{x} & d \end{pmatrix}, \quad a, d \in F \text{ and } x \in D,$$

equipped with the quadratic form

$$-\det : \begin{pmatrix} a & x \\ \bar{x} & d \end{pmatrix} \mapsto -ad + \mathbb{N}_D(x).$$

The action of $GL_2(D) \times GL_1$ on this space is given by

$$(g, z)(X) = z \cdot g \cdot X \cdot \bar{g}^t$$

Observe that an irreducible representation of $GSO(V_D)$ is of the form $\pi \boxtimes \mu$ where π is a representation of $GL_2(D)$ and μ is a square root of the central character of π . This is precisely the data needed to define a Shalika period.

One can thus consider the theta correspondence for the (almost) dual pair

$$GSp_4 \times GSO(V_D).$$

When D is split, this theta correspondence can be used to prove the (weak) lifting of globally generic cuspidal representations of GSp_4 to GL_4 ; this is a well-known result of Jacquet, Piatetski-Shapiro and Shalika that was announced almost thirty years ago, but whose proof was never published. However, most of the details of their proof can be found in the paper [So], where Soudry made use of this same dual pair to prove the strong multiplicity one theorem for globally generic cuspidal representations of

GSp_4 . Locally, a preliminary study of this theta correspondence has been conducted by Waldspurger as *un exercice* [W].

The understanding of this theta correspondence for both split and non-split D underlies the results of this paper. In the framework of theta correspondence, the local conditions which intervene in Theorem 1.3 can be explained as follows. If v is a non-archimedean place, a representation π_v of $GL_2(D_v)$ has Shalika period with respect to μ_v if and only if the corresponding representation $\pi_v \boxtimes \mu_v$ of $GSO(V_{D_v})$ has a big theta lift to GSp_4 which is *generic*. For most representations of $GSO(V_{D_v})$, the genericity of their big theta lifts is equivalent to the genericity of their small theta lifts; this is always the case for generic representations of $GSO(V_{D_v})$ when D_v is split. However, when D_v is non-split, the local representations which intervene in Theorem 1.3(B) are precisely those such that

- their small theta lifts to GSp_4 are not generic, even though their big theta lifts may be generic;
- their Jacquet-Langlands lifts to GL_4 are generic and have the relevant Shalika period.

This peculiarity is the reason for their appearance in Theorem 1.3.

In addition to the theory of theta correspondence, we should mention another key tool for the proof of Thm. 1.3. It is a Rankin-Selberg integral representation of the degree 5 L-function of a cuspidal representation of Sp_4 , which was first discovered by Andrianov in the classical setting and recast in the adelic setting by Piatetski-Shapiro and Rallis [PSR].

Finally, we should mention that the theorem of Jacquet and Martin has an obvious analog for D an arbitrary division algebra of degree d . Though the use of theta correspondence gives a simple proof in the case when D is quaternion, it has no hope of addressing the general case. On the other hand, one fully expects the relative trace formula approach of Jacquet and Martin to work for general d , as long as one can master the analytic difficulties. However, as we learned shortly after the completion of this paper, the analog of the Jacquet-Martin theorem (i.e. Thm. 1.2) for general D has been proven in a recent paper of Jiang-Nien-Qin [JNQ] using entirely different methods. Our Theorem 1.3 indicates that the correct converse statement in the case of general D may be quite delicate.

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2. Theta Correspondence for Similitudes

In this section, we give a brief introduction of the necessary background on theta correspondence for similitudes. We shall follow the reference [Ro2] closely.

Let us begin by establishing some more notations. Recall that F is a number field with ring of adèles \mathbb{A} and D is a quaternion F -algebra which is split at every archimedean place. The norm form \mathbb{N}_D equips D with the structure of a quadratic space and we fix the isomorphism

$$GSO(D, \mathbb{N}_D) \cong (D^\times \times D^\times) / \{(z, z^{-1}) : z \in GL_1\}$$

via the action of the latter on D given by

$$(\alpha, \beta) \mapsto \alpha x \bar{\beta}.$$

In particular, if D is split, then $GSO(D, \mathbb{N}_D)$ is the split orthogonal group $GSO(2, 2)$ and we have:

$$GSO(2, 2) \cong (GL_2 \times GL_2) / \{(z, z^{-1}) : z \in GL_1\}$$

In any case, an irreducible representation of $GSO(D, \mathbb{N}_D)$ is thus of the form $\tau_{D,1} \boxtimes \tau_{D,2}$ where the central characters of $\tau_{D,1}$ and $\tau_{D,2}$ are equal.

Recall from the introduction that $V_D = (D, \mathbb{N}_D) \oplus \mathbb{H}$, where \mathbb{H} is the hyperbolic plane. As we explained in the introduction, there is a natural isomorphism

$$(GL_2(D) \times GL_1) / \{(z, z^{-2}) : z \in GL_1\} \xrightarrow{\sim} GSO(V_D).$$

In particular if D is split, we simply write V for V_D and

$$(GL_4 \times GL_1) / \{(z, z^{-2}) : z \in GL_1\} \xrightarrow{\sim} GSO(V).$$

The similitude factor of $GSO(V_D)$ is:

$$\lambda_D : (g, z) \mapsto N(g) \cdot z^2,$$

where N is the reduced norm on the central simple algebra $M_2(D)$. Thus,

$$SO(V_D) = \{(g, z) \in GSO(V_D) : N(g) \cdot z^2 = 1\}.$$

Under the above isomorphism, the parabolic subgroup $P_D \times GL_1$ is identified with the stabilizer of the line in V_D spanned by $(1, 0; 0_D)$. Indeed, using matrix representation with respect to the decomposition $V_D = F \cdot (1, 0) \oplus D \oplus F \cdot (0, 1)$, we have:

$$\left(\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, z \right) \mapsto \begin{pmatrix} z \cdot \mathbb{N}(\alpha) & & \\ & z \cdot (\alpha, \beta) & \\ & & z \cdot \mathbb{N}(\beta) \end{pmatrix} \right)$$

and

$$\left(\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & Tr_D(\bar{y}-) & \mathbb{N}(y) \\ & 1 & y \\ & & 1 \end{pmatrix} \right).$$

Observe that there is an embedding

$$\iota : S_D \cong PD^\times \cdot N_D \hookrightarrow GSO(V_D).$$

The embedding of N_D is the one given above, whereas PD^\times is embedded via:

$$\iota(\alpha) = \left(\left(\begin{pmatrix} \alpha & \\ & \alpha \end{pmatrix}, \mathbb{N}\alpha^{-1} \right) \in (GL_2(D) \times GL_1) / \{(z, z^{-2})\} \right).$$

Thus, if $\pi \boxtimes \mu$ is a cuspidal representation of $GSO(V_D)$, then the Shalika period on π with respect to μ is simply the linear form on $\pi \boxtimes \mu$ given by

$$f \mapsto \int_{S_D(F) \backslash S_D(\mathbb{A})} f(\iota(n)) \cdot \overline{\psi_D(\iota(n))} dn$$

where ψ_D is extended from N_D to S_D by requiring that ψ_D be trivial on PD^\times .

Now let W be the 4-dimensional symplectic vector space and fix a Witt decomposition $W = X \oplus Y$. Let $P_Y = GL(Y) \cdot N_Y$ be the parabolic subgroup stabilizing the maximal isotropic subspace Y . Then

$$N_Y = \{b \in \text{Hom}(X, Y) : b^t = b\},$$

where $b^t \in \text{Hom}(Y^*, X^*) \cong \text{Hom}(X, Y)$.

Fix a unitary character ψ of $F \backslash \mathbb{A}$ and consider the Weil representation ω_D associated to ψ for the dual pair $Sp(W)(\mathbb{A}) \times O(V_D)(\mathbb{A})$. It can be realized on $S((X \otimes V_D)(\mathbb{A}))$ and the action of $P_Y \times O(V_D)$ is given by the usual formulas:

$$\begin{cases} \omega_D(h)\phi(x) = \phi(h^{-1}x), & \text{for } h \in O(V_D); \\ \omega_D(a)\phi(x) = |\det_Y(a)|^{\frac{1}{2} \dim V_D} \cdot \phi(a^{-1} \cdot x), & \text{for } a \in GL(Y); \\ \omega_D(b)\phi(x) = \psi(\langle bx, x \rangle) \cdot \phi(x), & \text{for } b \in N_Y, \end{cases}$$

where $\langle -, - \rangle$ is the natural symplectic form on $W \otimes V_D$. To describe the full action of $Sp(W)$, one needs to specify the action of a Weyl group element, which acts by a Fourier transform.

Now let

$$R_D = \{(g, h) \in GSp(W) \times GO(V_D) : \lambda(g) \cdot \lambda_D(h) = 1\},$$

where the λ 's refer to the similitude factor of the relevant group. Note that this differs from the normalization in [Ro2]. The Weil representation can then be extended in a natural way to the group $R_D(\mathbb{A})$, via:

$$\omega_D(g, h)\phi = |\lambda_D(h)|^{-\frac{1}{2} \dim V_D} \omega_D(g_1, 1)(\phi \circ h^{-1})$$

where

$$g_1 = g \begin{pmatrix} \lambda(g)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in Sp(W).$$

Observe that the central elements $(t, t^{-1}) \in R_D$ act trivially. We shall in fact only be interested in the action of

$$R_D^0 = \{(g, h) \in R_D : h \in GSO(V_D)\}.$$

For $\phi \in \mathcal{S}((X \otimes V_D)(\mathbb{A}))$ and $(g, h) \in R_D(\mathbb{A})$, set

$$\theta(\phi)(g, h) = \sum_{x \in (X \otimes V_D)(F)} \omega_D(g, h)\phi(x).$$

Then $\theta(\phi)$ is a function of moderate growth on $R_D(F) \backslash R_D(\mathbb{A})$. If $\pi_D \boxtimes \mu$ is a cuspidal representation of $GSO(V_D)$ and $f \in \pi_D \boxtimes \mu$, we set

$$\theta(\phi, f)(g) = \int_{SO(V_D)(F) \backslash SO(V_D)(\mathbb{A})} \theta(\phi)(g, h_1 h) \cdot \overline{f(h)} dh$$

where h_1 is any element of $GSO(V_D)$ such that $\lambda_D(h_1) = \lambda(g)$. Moreover, set

$$\Theta(\pi_D \boxtimes \mu) = \langle \theta(\phi, f) : \phi \in \omega_D, f \in \pi_D \boxtimes \mu \rangle.$$

Then $\theta(\phi, f)$ is an automorphic form (possibly zero) on $GSp(W)$ and $\Theta(\pi_D \boxtimes \mu)$ is an automorphic representation (possibly zero) of $GSp(W)(\mathbb{A})$ whose central character is equal to that of π_D . Similarly, starting from a cuspidal representation σ of $GSp(W)$, we have the automorphic representation $\Theta_D(\sigma)$ of $GSO(V_D)$.

Many questions about these similitude theta liftings can be easily reduced to the analogous questions for the isometry case. We highlight two such questions here.

The first such question is the vanishing or non-vanishing of $\Theta(\pi_D \boxtimes \mu)$. If res denotes the restriction of functions from a similitude group to the corresponding isometry group, then $res(\pi_D \boxtimes \mu)$ is a nonzero cuspidal representation of $SO(V_D)$ which is possibly reducible. From the definition of $\theta(\phi, f)$, it is immediate that $\Theta(\pi_D \boxtimes \mu)$ is nonzero iff the global theta lift of $res(\pi_D \boxtimes \mu)$ from $SO(V_D)$ to $Sp(W)$ is nonzero. Similarly, if σ is a cuspidal representation of $GSp(W)$, then $\Theta_D(\sigma)$ is nonzero iff the global theta lift of $res(\sigma)$ from $Sp(W)$ to $SO(V_D)$ is nonzero.

The second such question is the cuspidality of $\Theta(\pi_D \boxtimes \mu)$. Again, it is evident from the definition that $\Theta(\pi_D \boxtimes \mu)$ is contained in the space of cusp forms of $GSp(W)$ iff the global theta lift of $res(\pi_D \boxtimes \mu)$ is contained in the space of cusp forms of $Sp(W)$. Now in the isometry case, if one has a tower of theta liftings in the sense of Rallis, then a standard result in the theory is the so-called tower property of theta correspondence. This says that the global theta lift of a cuspidal representation of an isometry group to a particular step in the tower is contained in the space of cusp forms iff its theta lift to the previous step of the tower vanishes. Together with the above discussion, one sees immediately that the same statement applies to the theta liftings for similitude groups. Moreover, after the first nonzero lift, the theta lifts to higher steps of the tower do not vanish and are not contained in the space of cusp forms (though its intersection with the space of cusp forms may be nonzero because we are working with SO rather than O).

The following lemma will be used in this paper.

Lemma 2.1. *Let σ be a globally generic cuspidal representation of GSp_4 . Then the global theta lift $\Theta(\sigma)$ of σ to $GSO(V) = GSO(3, 3)$ is globally generic and thus is nonzero.*

Proof. This is essentially the main theorem of [GRS]. There they considered the isometry groups, but by our discussion above, it is easy to see that their theorem applies to the similitude case. \square

We now note:

Proposition 2.2. (i) *Suppose that $\pi_D \boxtimes \mu$ is a cuspidal representation of $GSO(V_D)$ such that the Jacquet-Langlands lift $JL(\pi_D)$ of π_D to GL_4 is cuspidal. If $\Theta(\pi_D \boxtimes \mu)$ is non-zero, then $\Theta(\pi_D \boxtimes \mu)$ is contained in the space of cusp forms of GSp_4 .*

(ii) *Suppose now that $JL(\pi_D)$ is non-cuspidal. If $\Theta(\pi_D \boxtimes \mu)$ is non-zero, then $\Theta(\pi_D \boxtimes \mu)$ does not contain any globally generic cuspidal representation of GSp_4 .*

Proof. (i) By the tower property of theta correspondence, if $\Theta(\pi_D \boxtimes \mu)$ is non-cuspidal, then the theta lift of $\pi_D \boxtimes \mu$ to GL_2 (which is the lower step of the tower) is nonzero cuspidal. Denote this cuspidal representation of GL_2 by Σ . Consider the theta lift of Σ to $GSO(V) = GSO(3, 3)$. Since the theta lift of Σ to $GSO(2, 2)$ is well-known to be nonzero, it follows by the tower property again that its theta lift $\Theta(\Sigma)$ to $GSO(V)$ is nonzero and not contained in the space of cusp forms. But any irreducible subquotient of $\Theta(\Sigma)$ is nearly equivalent to the cuspidal $JL(\pi_D) \boxtimes \mu$. This contradicts the generalized strong multiplicity one theorem of Jacquet-Shalika.

(ii) If $\Theta(\pi_D \boxtimes \mu)$ contains a globally generic cuspidal representation σ , then the theta lift of σ to $GSO(V)$ is nonzero by the above lemma and all its irreducible constituents are nearly equivalent to $JL(\pi_D) \boxtimes \mu$. By the generalized strong multiplicity one theorem for GL_4 , this contradicts the fact that $JL(\pi_D)$ is non-cuspidal. \square

We now consider the local situation. Over a local field, one has the analogous Weil representation $\omega_{D,v}$ for $R_D(F_v)$. If $\pi_D \otimes \mu$ and σ are irreducible representations of $GSO(V_D)(F_v)$ and $GSp(W)(F_v)$

respectively, then one says that they correspond under theta correspondence if

$$\mathrm{Hom}_{R_D^0(F_v)}(\omega_{D,v}, \sigma \boxtimes (\pi_D \boxtimes \mu)) \neq 0.$$

Necessarily, the central characters of σ and $\pi_D \boxtimes \mu$ are equal. It is perhaps easier to work with the compactly induced Weil representation

$$\Omega_{D,v} = \mathrm{ind}_{R_D^0}^{GSp(W) \times GSO(V_D)} \omega_{D,v}.$$

It follows from Frobenius reciprocity that $\pi_D \boxtimes \mu$ and σ correspond if and only if

$$\mathrm{Hom}_{GSp(W) \times GSO(V_D)}(\Omega_{D,v}, \sigma \boxtimes (\pi_D \boxtimes \mu)) \neq 0.$$

As usual, given $\pi_D \boxtimes \mu$, the maximal $\pi_D \boxtimes \mu$ -isotypic quotient of $\Omega_{D,v}$ is of the form

$$(\pi_D \boxtimes \mu) \boxtimes \Theta(\pi_D \boxtimes \mu)$$

for some smooth representation $\Theta(\pi_D \boxtimes \mu)$ of $GSp(W)$. One knows that $\Theta(\pi_D \boxtimes \mu)$ is of finite length and for lack of a better terminology, we call $\Theta(\pi_D \boxtimes \mu)$ the **big theta lift** of $\pi_D \boxtimes \mu$.

Set

$$\theta(\pi_D \boxtimes \mu) = \text{the maximal semisimple quotient of } \Theta(\pi_D \boxtimes \mu);$$

we call it the **small theta lift** of $\pi_D \boxtimes \mu$. For the case at hand, one knows the following:

- (i) When $\pi_D \boxtimes \mu$ is supercuspidal, $\Theta(\pi_D \boxtimes \mu)$ is irreducible if it is nonzero, regardless of the residual characteristic of F_v . This follows from the corresponding result of Kudla in the isometry case, as shown in [GT, Prop. 2.3 and Lemma 3.1].
- (ii) It is a result of Roberts [Ro2] that if the Howe conjecture for isometry group holds (for example, away from residual characteristic two), then $\theta(\pi_D \boxtimes \mu)$ is zero or irreducible. Actually, the results of Roberts a priori only apply if we are working with representations of $GO(V_D)$. The passage from $GO(V_D)$ to $GSO(V_D)$ is explained in [GT, Lemma 3.1].
- (iii) In fact, for the representations $\pi_D \boxtimes \mu$ of $GSO(V_D)$ considered in this paper, we show in Section 8 that $\theta(\pi_D \boxtimes \mu)$ is zero or irreducible, regardless of residual characteristic of F_v .

Similarly, starting with the representation σ of GSp_4 , one has the representations $\Theta_D(\sigma)$ and $\theta_D(\sigma)$ of $GSO(V_D)$. For the same reasons as above, one knows that the small theta lift $\theta_D(\sigma)$ is either zero or irreducible (see [GT, Prop. 2.3 and Lemma 3.1]).

The above discussion can be summarized in the following ‘‘theta lifting diagram’’.

$$\begin{array}{ccccccc}
 GL_2(D) \times GL_1 & \longrightarrow & GSO(V_D) & \xleftarrow{JL} & GSO(V) & \longleftarrow & GL_4 \times GL_1 \\
 & & \searrow^{\theta} & & \swarrow^{\theta} & & \\
 & & & & & & \\
 & & \swarrow^{\theta_D} & & \searrow^{\theta_V} & & \\
 & & & & GSp_4 & & \\
 & & \swarrow^{\theta} & & \swarrow^{\theta} & & \\
 D^\times \times D^\times & \longrightarrow & GSO(D) & \xleftarrow{JL} & GSO(2, 2) & \longleftarrow & GL_2 \times GL_2
 \end{array}$$

We shall denote all the theta lifts to GSp_4 by θ and the theta lift from GSp_4 to $GSO(V_D)$ and $GSO(V)$ by θ_D and θ_V respectively. Moreover, JL indicates the Jacquet-Langlands transfer.

Remark 2.3. Let Σ_D be the set of places of F where D ramifies, and let $\Sigma_{D,\infty} \subset \Sigma_D$ be the subset of archimedean places. We have assumed in this section (and throughout this paper) that $\Sigma_{D,\infty}$ is empty. This assumption is not essential for the above discussion but a slight subtlety needs to be noted. Namely, the natural projection map

$$R_D(\mathbb{A}) \longrightarrow GSp(W)(\mathbb{A})$$

is the subgroup

$$GSp(W)(\mathbb{A})^+ = \{g \in GSp(W)(\mathbb{A}) : \lambda(g) \in \text{Im}(\lambda_D)\},$$

and λ_D is surjective onto \mathbb{A}^\times if and only if $\Sigma_{D,\infty}$ is empty, in which case $GSp(W)(\mathbb{A})^+ = GSp(W)(\mathbb{A})$. In general,

$$GSp(W)(\mathbb{A})^+ = \prod_{v \in \Sigma_{D,\infty}} GSp_4(F_v)^+ \times \prod'_{v \notin \Sigma_{D,\infty}} GSp_4(F_v),$$

where $GSp_4(F_v)^+$ is the identity component of the real Lie group $GSp_4(\mathbb{R})$ for $v \in \Sigma_{D,\infty}$. Thus, in the above discussion, we need to replace $GSp(W)(\mathbb{A})$ by $GSp(W)(\mathbb{A})^+$ where appropriate. For example, the function $\theta(\phi, f)$ is an automorphic form on $GSp(W)(\mathbb{A})^+$ and $\Theta(\pi_D \boxtimes \mu)$ is a representation of $GSp(W)(\mathbb{A})^+$. We stress however that this technical complication is not the reason for our assumption that $\Sigma_{D,\infty}$ be empty.

We conclude this section with a brief discussion on the functoriality of the above theta correspondence for spherical representations. The L-group of GSp_4 is $GSp_4(\mathbb{C})$ and so an unramified representation of GSp_4 corresponds to a semisimple class in $GSp_4(\mathbb{C})$. On the other hand, the L-group of $GSO(V)$ is the subgroup of $GL_4(\mathbb{C}) \times GL_1(\mathbb{C})$ given by

$$\{(g, z) \in GL_4(\mathbb{C}) \times GL_1(\mathbb{C}) : \det(g) = z^2\}.$$

There is a natural map

$${}^L GSp_4 \longrightarrow {}^L GSO(V)$$

given by

$$g \mapsto (g, \Lambda(g))$$

where Λ is the similitude factor of $GSp_4(\mathbb{C})$. The following proposition is shown in [GT, Cor. 12.14]:

Proposition 2.4. *If σ_v is the unramified representation of GSp_4 corresponding to the semisimple class $s \in GSp_4(\mathbb{C})$, then the small theta lift of σ_v is the unramified representation of $GSO(V)$ corresponding to the semisimple class $(s_v, \Lambda(s_v)) \in {}^L GSO(V)$.*

Moreover, corresponding to the inclusion $SO(V) \hookrightarrow GSO(V)$, one has a map of L-groups

$$\text{std} : {}^L GSO(V) \longrightarrow {}^L SO(V) = SO_6(\mathbb{C}).$$

Indeed, one has the map

$$GL_4(\mathbb{C}) \times GL_1(\mathbb{C}) \longrightarrow GSO_6(\mathbb{C})$$

given by:

$$(g, z) \mapsto z^{-1} \cdot \wedge^2 g,$$

and the map std is simply the restriction of this map to the subgroup ${}^L GSO(V)$. Thus, one may consider the (partial) standard degree 6 L-function of a cuspidal representation $\pi \boxtimes \mu$ of $GSO(V)$, which we denote by $L^S(s, \pi \boxtimes \mu, \text{std})$. If we regard $\pi \boxtimes \mu$ as a representation of $GL_4 \times GL_1$, then this L-function is nothing but the twisted exterior square L-function $L^S(s, \pi, \wedge^2 \otimes \mu^{-1})$.

Observe finally that if we consider the composite

$$GSp_4(\mathbb{C}) \longrightarrow {}^L GSO(V) \longrightarrow SO_6(\mathbb{C}),$$

then this 6-dimensional representation of $GSp_4(\mathbb{C})$ decomposes as the sum of the trivial representation and the standard 5-dimensional representation

$$GSp_4(\mathbb{C}) \longrightarrow PGSp_4(\mathbb{C}) \cong SO_5(\mathbb{C}).$$

Indeed, one has the commutative diagram:

$$\begin{array}{ccccc} GSp_4(\mathbb{C}) & \longrightarrow & {}^L GSO(V) & \longrightarrow & GL_4(\mathbb{C}) \times GL_1(\mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ SO_5(\mathbb{C}) & \longrightarrow & SO_6(\mathbb{C}) & \longrightarrow & GSO_6(\mathbb{C}) \end{array}$$

In view of this, Prop. 2.4 immediately gives:

Proposition 2.5. *If $\Theta(\pi \boxtimes \mu)$ is cuspidal and contains σ as an irreducible constituent, then*

$$L^S(s, \pi, \bigwedge^2 \otimes \mu^{-1}) = L^S(s, \pi \boxtimes \mu, std) = \zeta^S(s) \cdot L^S(s, \sigma, std),$$

where $L^S(s, \sigma, std)$ is the (partial) standard degree 5 L-function of σ .

3. The Implication (A) \implies (B)

In this section, we prove the implication (A) \implies (B) of Thm. 1.3. In particular, we give a very short proof of the Jacquet-Martin theorem without the 3 conditions present there.

For a nondegenerate character χ on the unipotent radical U of a Borel subgroup of GSp_4 , let \mathcal{W}_χ denote the global Whittaker functional on $\mathcal{A}(GSp_4)$:

$$\mathcal{W}_\chi : F \mapsto \int_{U(F) \backslash U(\mathbb{A})} F(u) \cdot \overline{\chi(u)} du$$

The following proposition is the key computation (see [So] for the same computation when D is split):

Proposition 3.1. *Let π_D be a cuspidal representation of $GL_2(D)$ whose central character is μ^2 , so that we may consider the representation $\pi_D \boxtimes \mu$ of $GSO(V_D)$. Then we have:*

$$\mathcal{W}_\chi(\theta(\phi, f)) = \int_{S_D(\mathbb{A}) \backslash SO(V_D)(\mathbb{A})} \overline{\mathcal{S}_D(h \cdot f)} \cdot \left(\int_{U_Y(\mathbb{A})} \overline{\chi(u)} \cdot \omega(u, h) \phi(\underline{x}_0) du \right) dh,$$

where S_D is embedded in to $SO(V_D)$ via ι and

$$\underline{x}_0 = (x_1^0, x_2^0) \in V_D(\mathbb{A})^2$$

with

$$\begin{cases} x_1^0 = ((1, 0); 0_D) \in \mathbb{H} \oplus D = V_D \\ x_2^0 = ((0, 0); 1_D) \in \mathbb{H} \oplus D = V_D. \end{cases}$$

Proof. Let us write $U = N_Y \rtimes U_Y$ with U_Y a maximal unipotent subgroup of $GL(Y)$ and N_Y the unipotent radical of the Siegel parabolic subgroup P_Y . Then we may restrict the character χ to N_Y and U_Y . Its restriction to N_Y is a degenerate character, whereas its restriction to U_Y is nondegenerate.

Now we have:

$$\begin{aligned}
& \mathcal{W}_\chi(\theta(\phi, f)) \\
&= \int_{U(F) \backslash U(\mathbb{A})} \overline{\chi(u)} \cdot \int_{SO(V_D)(F) \backslash SO(V_D)(\mathbb{A})} \theta(\phi)(u, h) \cdot \overline{f(h)} \, dh \, du \\
&= \int_{SO(V_D)(F) \backslash SO(V_D)(\mathbb{A})} \overline{f(h)} \cdot \int_{U_Y(F) \backslash U_Y(\mathbb{A})} \overline{\chi(u)} \cdot \int_{N_Y(F) \backslash N_Y(\mathbb{A})} \overline{\chi(n)} \cdot \sum_{\underline{x} \in (X \otimes V_D)(F)} \omega(nu, h) \phi(\underline{x}) \, dn \, du \, dh \\
&= \int_{SO(V_D)(F) \backslash SO(V_D)(\mathbb{A})} \overline{f(h)} \cdot \left(\int_{U_Y(F) \backslash U_Y(\mathbb{A})} \overline{\chi(u)} \cdot \sum_{\underline{x} \in \Omega} \omega(u, h) \phi(\underline{x}) \, du \right) \, dh
\end{aligned}$$

where

$$\Omega = \{\underline{x} = (x_1, x_2) \in V_D^2 : q_D(x_1) = 0, q_D(x_2) = 1 \text{ and } x_2 \in x_1^\perp\}.$$

Clearly, one has a decomposition

$$\Omega = \Omega_0 \bigsqcup \Omega_1$$

where Ω_0 (resp. Ω_1) is the subset of elements with $x_1 = 0$ (resp. $x_1 \neq 0$). It is easy to see that the sum over Ω_0 does not contribute and so we need only consider the sum over Ω_1 above.

Now the element \underline{x}_0 lies in Ω_1 and the group $SO(V_D)(F)$ acts transitively on Ω_1 . Indeed, if we set

$$\Xi = \{\underline{x}_t = (x_1^0, x_2^0 + tx_1^0) : t \in F\} \subset \Omega_1,$$

then

$$\Omega_1 = SO(V_D)(F) \times_{S_D(F)} \Xi.$$

Moreover, identifying Ξ with F in the obvious way, the action of $U_Y(F) \times S_D(F)$ on Ξ is given by:

$$(u, d \cdot n) : t \mapsto (u + Tr_D(n)) \cdot t.$$

Hence $U_Y(F)$ acts simply transitively on Ξ .

Thus we have:

$$\begin{aligned}
& \mathcal{W}_\chi(\theta(\phi, f)) \\
&= \int_{S_D(F) \backslash SO(V_D)(\mathbb{A})} \overline{f(h)} \cdot \int_{U_Y(F) \backslash U_Y(\mathbb{A})} \overline{\chi(u)} \cdot \left(\sum_{t \in F} \omega(u, h) \phi(\underline{x}_t) \right) \, du \, dh \\
&= \int_{S_D(F) \backslash SO(V_D)(\mathbb{A})} \overline{f(h)} \cdot \int_{U_Y(\mathbb{A})} \overline{\chi(u)} \cdot \omega(u, h) \phi(\underline{x}_0) \, du \, dh \\
&= \int_{S_D(\mathbb{A}) \backslash SO(V_D)(\mathbb{A})} \int_{S_D(F) \backslash S_D(\mathbb{A})} \overline{f(rh)} \cdot \int_{U_Y(\mathbb{A})} \overline{\chi(u)} \cdot \omega(u, rh) \phi(\underline{x}_0) \, du \, dr \, dh \\
&= \int_{S_D(\mathbb{A}) \backslash SO(V_D)(\mathbb{A})} \int_{S_D(F) \backslash S_D(\mathbb{A})} \overline{f(rh)} \cdot \int_{U_Y(\mathbb{A})} \overline{\chi(u)} \cdot \omega(h) \phi(\underline{x}_{u+Tr_D(r)}) \, du \, dr \, dh \\
&= \int_{S_D(\mathbb{A}) \backslash SO(V_D)(\mathbb{A})} \int_{S_D(F) \backslash S_D(\mathbb{A})} \overline{f(rh)} \chi(Tr_D(r)) \cdot \int_{U_Y(\mathbb{A})} \overline{\chi(u)} \omega(h) \phi(\underline{x}_u) \, du \, dr \, dh \\
&= \int_{S_D(\mathbb{A}) \backslash SO(V_D)(\mathbb{A})} \overline{S_D(h \cdot f)} \cdot \left(\int_{U_Y(\mathbb{A})} \overline{\chi(u)} \cdot \omega(u, h) \phi(\underline{x}_0) \, du \right) \, dh.
\end{aligned}$$

□

Corollary 3.2. *The automorphic representation $\Theta(\pi_D \boxtimes \mu)$ is globally generic if and only if π_D has Shalika period with respect to μ .*

Proof. This follows from the above proposition by a standard argument analogous to that in [GS, Pg. 2718-2719]. \square

Now we can prove a part of the implication $(A) \implies (B)$, which is essentially the Jacquet-Martin theorem without the three conditions present there:

Theorem 3.3. *If π_D has Shalika period with respect to μ , then the partial L-function $L^S(s, \pi_D, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 1$. Hence its Jacquet-Langlands lift $\pi = JL(\pi_D)$ (if cuspidal) has Shalika period with respect to μ .*

Proof. Suppose that $\pi_D \boxtimes \mu$ has Shalika period. Then by Cor. 3.2, $\Theta(\pi_D \boxtimes \mu)$ contains a globally generic cuspidal representation σ of $GS\!p_4$. By Prop. 2.5, we have:

$$L^S(s, \pi_D, \bigwedge^2 \otimes \mu^{-1}) = L^S(s, \pi_D \boxtimes \mu, std) = L^S(s, \sigma, std) \cdot \zeta^S(s).$$

Now because σ is globally generic, $L^S(s, \sigma, std)$ is non-zero at $s = 1$ (by Shahidi [Sh, Thm. 5.1]). Thus $L^S(s, \pi_D, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 1$. It follows from results of Jacquet-Shalika [JS] that π has Shalika period with respect to μ . \square

Finally the following proposition completes the proof of the implication $(A) \implies (B)$ of Thm. 1.3.

Proposition 3.4. *Suppose that D is split at every archimedean place. If π_D has Shalika period with respect to μ , then for all $v \in \Sigma_D$, the local representation $\pi_{D,v}$ is not equal to $\text{Ind}_{P_{D,v}}^{GL_2(D_v)} \delta_{P_{D,v}}^{1/2} \cdot (\tau_{D,1,v} \boxtimes \tau_{D,2,v})$, where $\tau_{D,i,v}$ are representations of D_v^\times with central character μ_v .*

Proof. For this, we need one of our local results proved in Section 8. Indeed, if π_D has Shalika period with respect to μ , then it follows by Cor. 3.2 that the global theta lift $\theta(\pi_D \boxtimes \mu)$ to $GS\!p_4$ is generic. In particular, the local theta lift $\theta(\pi_{D,v} \boxtimes \mu_v)$ is generic. However, by Thm. 8.3(i), one sees that if $\pi_{D,v}$ (for $v \in \Sigma_D$) is of the “bad” type described in the proposition (which is denoted by $PS(\tau_{D,1,v}, \tau_{D,2,v})$ in Section 7 and 8), then the local theta lift is non-generic. With this contradiction, the proposition is proved. \square

Remark 3.5. *As we noted in the introduction, the assumption that $\Sigma_{D,\infty}$ be empty is not needed for Theorem 3.3. When $\Sigma_{D,\infty}$ is non-empty, we have already noted in Remark 2.3 that one needs to work with the group $GS\!p_4(\mathbb{A})^+$ in the theta correspondence. One subtlety that this creates is that there are more than one orbit of nondegenerate characters χ on the unipotent radical U of the Borel subgroup $B(\mathbb{A})^+ = B(\mathbb{A}) \cap GS\!p_4(\mathbb{A})^+$; here, we are considering orbits of nondegenerate characters under the action of $GS\!p_4(F)^+ = GS\!p_4(F) \cap GS\!p_4(\mathbb{A})^+$. These orbits are determined by sign conditions at the places in $\Sigma_{D,\infty}$, so that there are $2^{\#\Sigma_{D,\infty}}$ orbits of nondegenerate characters. However, when one computes the Whittaker functionals on $\theta(\phi, f)$ with respect to these characters χ as in Proposition 3.1, one finds that $W_\chi(\theta(\phi, f))$ is identically zero for all but one of these orbits of χ . For the remaining orbit of characters χ_0 , $W_{\chi_0}(\theta(\phi, f))$ is as given in Proposition 3.1 with the same proof. Hence, Corollary 3.2 continues to hold, provided we replace the condition that $\Theta(\pi_D \boxtimes \mu)$ be “globally generic” by “globally χ_0 -generic”. With these modifications duly noted, the proof of Theorem 3.3 remains the same.*

4. A Counterexample to the Converse of Jacquet-Martin

Before proving the other implication $(B) \implies (A)$ of our main theorem, we describe in this section a concrete counterexample to the converse of the Jacquet-Martin theorem. Namely we shall construct a cuspidal representation π of PGL_4 which has cuspidal Jacquet-Langlands lift π_D on $PGL_2(D)$ and show that π has Shalika period but π_D does not. To do so, we shall construct an irreducible cuspidal representation σ on $PGSp_4$ with the following properties:

- (i) σ is globally generic.
- (ii) at two finite places v_1 and v_2 , σ_{v_i} is supercuspidal and has a non-zero theta lift to the split $PGSO(2, 2)$, say $\sigma_{v_i} = \Theta(\tau_i \boxtimes \tau'_i)$ for supercuspidal $\tau_i \boxtimes \tau'_i$ on $PGSO(2, 2) = PGL_2 \times PGL_2$.
- (iii) the global theta lift of σ to $GSO(2, 2)(\mathbb{A})$ is zero.

Before showing how to construct such a σ , let us see why having such a σ gives a counterexample. Because σ is globally generic and cuspidal, $\pi := \Theta_V(\sigma)$ is nonzero on PGL_4 by Lemma 2.1. Moreover, π is cuspidal: this follows from the tower property of theta lifts, since the theta lift of σ to $PGSO(2, 2)$ is zero by assumption. By the strong multiplicity one theorem, we see that π is irreducible. For $i = 1$ or 2 , a simple calculation of local theta correspondence (cf. Thm. 8.4(i)) shows that

$$\pi_{v_i} = PS(\tau_i, \tau'_i) := \text{Ind}_P^{GL_4} \delta_P^{1/2} \cdot (\tau_i \boxtimes \tau'_i).$$

Moreover, we have

$$L^S(s, \pi, \bigwedge^2) = L^S(s, \sigma, std) \cdot \zeta^S(s),$$

and $L^S(s, \sigma, std)$ is nonzero at $s = 1$ by genericity of σ . Thus $L^S(s, \pi, \bigwedge^2)$ has a pole at $s = 1$, and so π has Shalika period.

Now let D be the quaternion algebra ramified precisely at v_1 and v_2 . By recent results of Badulescu [B], π has a cuspidal Jacquet-Langlands lift π_D on $PGL_2(D)$ (since π_{v_i} is compatible with $GL_2(D_{v_i})$ for $i = 1$ and 2 in the sense of [B]). We need to show that π_D has no Shalika period.

For $i = 1$ and 2 , we have:

$$\pi_{D, v_i} = PS(JL(\tau_i), JL(\tau'_i)) := \text{Ind}_{P_D}^{GL_2(D)} JL(\tau_i) \boxtimes JL(\tau'_i).$$

In [P1, Prop. 7], Prasad showed that

$$(PS(JL(\tau_i), JL(\tau'_i)))_{N_D, \psi_D} \cong JL(\tau_i) \boxtimes JL(\tau'_i)$$

as representations of $PD_{v_i}^\times$, and π_{D, v_i} does not possess Shalika period if $\tau_i = \tau_i^\vee \neq \tau'_i$; we recall his results in Prop. 7.1 below. Thus, if we assume that $\tau_i \neq \tau'_i$ for $i = 1$ and 2 , then the local components of π_D at v_1 and v_2 do not possess local Shalika periods. Hence π_D does not possess Shalika period.

In fact, we can give another argument to show that the same conclusion holds even if $\tau_i^\vee = \tau_i = \tau'_i$ (in which case π_{D, v_i} does possess local Shalika period). Suppose for the sake of contradiction that π_D has Shalika period. Then as in the proof of Thm. 3.3, its theta lift $\Theta_D(\pi_D)$ to $G = PGSp_4$ is non-zero cuspidal and globally generic. Moreover, any of its irreducible constituents is nearly equivalent to σ . By the strong multiplicity one theorem for globally generic cuspidal representations of $PGSp_4$ [J-So, Cor. 2], we have

$$\sigma \subset \Theta(\pi_D).$$

This implies that for all v , σ_v has non-zero theta lift to $PGSO(V_D)$. On the other hand, Kudla-Rallis [KR1, Thm. 3.9] showed that if a representation of $PGSp_4$ lifts to $PGSO(2, 2)$, then it does not lift

to $PGSO(V_D)$. So at the two places v_1 and v_2 , the theta lift of σ_{v_i} to $PGSO(V_D)$ is zero. This gives the desired contradiction.

It remains then to construct a cuspidal representation σ with the properties (i)-(iii) above. Such a σ can be obtained as a theta lift from a cuspidal representation of a quasi-split $PGSO(3, 1)$. Choose a quadratic extension E/F such that v_1 and v_2 split in E . Let V_E be the quadratic space $(E, \mathbb{N}_E) \oplus \mathbb{H}$. Then (cf. [Ro1, §2])

$$PGSO(V_E) \cong PGL_2(E).$$

Let Σ be a cuspidal representation of $PGL_2(E)$ satisfying the following properties:

- (a) for $i = 1$ or 2 , Σ_{v_i} is a supercuspidal representation of $PGL_2(E_{v_i}) = PGL_2(F_{v_i}) \times PGL_2(F_{v_i})$.
- (b) at some finite place v_3 which is inert in E , Σ_{v_3} is a supercuspidal representation which is not the base change lift of a representation of $GL_2(F_{v_3})$.

Such a Σ exists by Poincare series techniques, as shown in [Sh2].

Now consider the theta lift of Σ from $PGSO(V_E)$ to $PGSp_4$, which has been studied in great detail by B. Roberts [Ro1]. One knows that the theta lift of Σ to $PGSp_4$ is non-zero cuspidal (because of condition (b)) and contains a globally generic constituent. We take such a constituent to be our σ . By construction, σ satisfies the requirements (i) and (ii). To show that the global theta lift of σ to $GSO(2, 2)$ vanishes, one considers the standard degree 5 L-function of σ . Since σ is a theta lift from $PGSO(V_E)$, we have

$$L^S(s, \sigma, std) = L^S(s, \omega_{E/F}) \cdot L^S(s, \Sigma, Asai)$$

where $\omega_{E/F}$ is the quadratic character associated to E/F and the last L-function is the Asai L-function of Σ . It is well-known that $L^S(s, \omega_{E/F})$ is nonzero holomorphic at $s = 1$. Also one knows that $L^S(s, \Sigma, Asai)$ is nonzero holomorphic at $s = 1$ (see [F], especially §5.) Hence the RHS is nonzero holomorphic at $s = 1$. On the other hand, if the theta lift of σ to $GSO(2, 2)$ is nonzero, say $\Theta(\sigma) \supset \tau_1 \boxtimes \tau_2$, then

$$L^S(s, \sigma, std) = \zeta^S(s) \cdot L^S(s, \tau_1 \times \tau_2),$$

which has a pole at $s = 1$. This contradiction shows that σ satisfies the condition (iii).

This completes the construction of the counterexample. Indeed, by our construction above, we obtain:

Theorem 4.1. *Suppose that D is split at every archimedean place. Then there are cuspidal representations π_D of $PGL_2(D)$, with a cuspidal Jacquet-Langlands lift to PGL_4 , satisfying:*

- (i) *for each place v , $\pi_{D,v}$ has local Shalika period, and*
- (ii) *$L^S(s, \pi_D, \wedge^2)$ has a pole at $s = 1$, but*
- (iii) *π_D does not have global Shalika period.*

Proof. As in the construction of the counterexample above, we pick a quadratic field extension E/F such that for all $v \in \Sigma_D$, E_v is split. For all $v \in \Sigma_D$, pick supercuspidal representations $\Sigma_v = \tau_v \boxtimes \tau_v$ of $PGL_2(F_v) \times PGL_2(F_v)$. At some finite place $w \notin \Sigma_D$ such that E_w is a field, pick a supercuspidal representation Σ_w of $PGL_2(E_w)$ which is not in the image of the base change from $GL_2(F_w)$. Let Σ be a cuspidal representation of $PGSO(V_E)$ with these local components and let σ be a globally generic constituent of the theta lift of Σ to $PGSp_4$. As above, if π is the global theta lift of σ to $PGSO(V) \cong PGL_4$, then π is a cuspidal representation with Shalika period whose local components

at $v \in \Sigma_D$ are of the form $PS(\tau_v, \tau_v)$. In particular, $L^S(s, \pi, \wedge^2)$ has a pole at $s = 1$. By Badulescu [B], this π has a Jacquet-Langlands transfer π_D to $PGL_2(D)$ and for $v \in \Sigma_D$,

$$\pi_{D,v} = PS(JL(\tau_v), JL(\tau_v)),$$

which has local Shalika period by Prop. 7.1. Then, as we have seen above, all the local components of π_D have Shalika period, but π_D itself does not. \square

5. The Implication (B) \implies (A)

In this section we will prove the other implication (B) \implies (A) in Thm. 1.3. For this, we shall need some precise results about the global Jacquet-Langlands correspondence between $GL_2(D)$ and GL_4 . Such results have now been obtained by Badulescu [B] in essentially complete generality. However, for technical reasons, Badulescu [B] assumes that the quaternion algebra D is split at every archimedean place. This is the reason why we make this assumption in this paper.

We first note the following lemma:

Lemma 5.1. *Assume that D is split at every archimedean place. If the cuspidal representation $\pi = JL(\pi_D)$ of GL_4 has Shalika model with respect to μ , so that $\Theta(\pi \boxtimes \mu)$ contains an irreducible globally generic cuspidal representation σ of GSp_4 , then the following are equivalent:*

- (i) π_D has Shalika period with respect to μ ;
- (ii) the theta lift of σ to $GSO(V_D)$ is non-zero.

When these conditions hold, the theta lift of σ to $GSO(V_D)$ is equal to $\pi_D \boxtimes \mu$.

Proof. We first prove that (i) implies (ii). If π_D has Shalika period with respect to μ , then by Corollary 3.2, its theta lift to GSp_4 contains an irreducible cuspidal globally generic representation σ' . Moreover, σ' and σ are nearly equivalent and so are equal by the results of Jiang-Soudry [J-So]. The theta lift $\Theta_D(\sigma)$ of σ to $GSO(V_D)$ is thus nonzero cuspidal and all its constituents are nearly equivalent to $\pi_D \boxtimes \mu$; it is thus equal to $\pi_D \boxtimes \mu$ by the strong multiplicity one theorem for $GL_2(D)$ due to Badulescu [B, Thm. 5.1 (b) and (c)].

Conversely, if the theta lift of σ to $GSO(V_D)$ is non-zero, then all its irreducible constituents are nearly equivalent to $\pi_D \boxtimes \mu$ and thus $\Theta_D(\sigma)$ is equal to $\pi_D \boxtimes \mu$ by the strong multiplicity one theorem for $GL_2(D)$ [B, Thm. 5.1 (b) and (c)]. This shows that the theta lift of $\pi_D \boxtimes \mu$ to GSp_4 is not orthogonal to σ , and thus contains an irreducible constituent σ' isomorphic to σ . Again, the results of [J-So] implies that $\sigma' = \sigma$, so that $\Theta(\pi_D \boxtimes \mu)$ is globally generic. Corollary 3.2 then implies that $\pi_D \boxtimes \mu$ has Shalika period with respect to μ . \square

Thus, a necessary condition for π_D to have Shalika period is that the local representations σ_v has a non-zero theta lift to $GSO(V_D)(F_v)$. This is of course automatic for $v \notin \Sigma_D$, but is not automatic for $v \in \Sigma_D$ (as the counterexample shows), and hence we need the local condition as in our main theorem. Of course, even when these local obstructions to theta lifting are absent, one still has to show that the global theta lift is non-zero.

In any case, the following theorem immediately implies the implication (B) \implies (A):

Theorem 5.2. *Assume that D is split at every archimedean place of F . Suppose that π_D is a cuspidal representation of $GL_2(D)$ with central character μ^2 and a cuspidal Jacquet-Langlands lift π on GL_4 . If*

(i) the L -function $L^S(s, \pi_D, \wedge^2 \otimes \mu^{-1})$ has a pole at $s = 1$, i.e. π has Shalika period with respect to μ , and

(ii) for $v \in \Sigma_D$, π_v is not of the form $PS(\tau_{1,v} \boxtimes \tau_{2,v})$ where $\tau_{i,v}$ are representations of $GL_2(F_v)$ with central character μ ,

then π_D possesses Shalika period with respect to μ .

Proof. Let $\sigma = \Theta(\pi \boxtimes \mu)$; it is an automorphic representation of $GSp_4(\mathbb{A})$. By (i), σ is globally generic and irreducible cuspidal. By (ii), for $v \in \Sigma_D$, the local components σ_v do not participate in the local theta correspondence with $GSO(2, 2)$. In view of the previous lemma, in order to prove the theorem, it suffices to show that $\pi'_D := \Theta_D(\sigma) \neq 0$. This is a consequence of the following result:

Theorem 5.3. *Suppose that σ is an irreducible cuspidal (not necessarily generic) representation of $GSp_4(\mathbb{A})$ satisfying:*

- σ has nonzero Fourier coefficient along N_Y corresponding to the split binary quadratic space;
- $L^S(1, \sigma, \text{std})$ is finite but nonzero;
- for all $v \in \Sigma_D$, σ_v does not participate in the theta correspondence with $GO(2, 2)$.

Then the global theta lift of σ to $GSO(V_D)$ is nonzero.

Indeed, to deduce Theorem 5.2 from Theorem 5.3, one simply applies Theorem 5.3 to $\sigma = \Theta(\pi \boxtimes \mu)$, which is globally generic as we noted above. It is not difficult to see that a globally generic σ necessarily has nonzero Fourier coefficient along N_Y corresponding to the split binary quadratic space. Moreover, it satisfies the second condition of Theorem 5.3 by [Sh, Thm. 5.1] and the third condition holds by assumption (ii) of Theorem 5.2. Thus, it remains to prove Theorem 5.3.

Proof. As we explain in Section 2, the non-vanishing of the global theta lift of σ is equivalent to the non-vanishing of the global theta lift of $\sigma|_{Sp_4}$ to $O(V_D)$ (where we are considering the restriction of functions from GSp_4 to Sp_4). Thus, we may work with isometry groups below. We shall in fact show that the period of $\Theta_D(\sigma|_{Sp_4})$ over the subgroup $O(D) \subset O(V_D)$ is non-zero. Our argument below is largely inspired by [KRS, §7].

The decomposition $V_D = D \oplus \mathbb{H}$ gives a see-saw diagram:

$$\begin{array}{ccc} Sp_4 \times Sp_4 & & O(V_D) \\ \downarrow & \searrow & \downarrow \\ \Delta Sp_4 & & O(D) \times O(\mathbb{H}) \end{array}$$

On restriction to $\Delta Sp_4 \times (O(D) \times O(\mathbb{H}))$, the Weil representation of $Sp_4 \times O(V_D)$ decomposes as a tensor product:

$$S(X \otimes V_D) \cong S(X \otimes D) \hat{\otimes} S(X \otimes \mathbb{H}).$$

Let $\phi = \phi_D \otimes \phi_{\mathbb{H}} \in S((X \otimes V_D)(\mathbb{A}))$ and take $f \in \sigma$. Now let us compute the period of $\theta_D(\phi, f)$ over the anisotropic group $O(D)$. We get:

$$\begin{aligned}
& \int_{O(D)(F)\backslash O(D)(\mathbb{A})} \theta_D(\phi, f)(h) dh \\
&= \int_{O(D)(F)\backslash O(D)(\mathbb{A})} \int_{Sp_4(F)\backslash Sp_4(\mathbb{A})} \theta(\phi)(gh) \cdot \overline{f(g)} dg dh \\
&= \int_{O(D)(F)\backslash O(D)(\mathbb{A})} \int_{Sp_4(F)\backslash Sp_4(\mathbb{A})} \theta(\phi_D)(gh) \cdot \theta(\phi_{\mathbb{H}})(g) \cdot \overline{f(g)} dg dh \\
&= \int_{Sp_4(F)\backslash Sp_4(\mathbb{A})} \overline{f(g)} \cdot \theta(\phi_{\mathbb{H}})(g) \cdot \left(\int_{O(D)(F)\backslash O(D)(\mathbb{A})} \theta(\phi_D)(gh) dh \right) dg.
\end{aligned}$$

The inner integral is the theta lift of the trivial representation of $O(D)$ to Sp_4 . Note that since $O(D)$ is anisotropic, this theta integral is always convergent and hence there is no need for regularization. Now by the Siegel-Weil formula of Kudla-Rallis [KR2], this inner integral is equal to an Eisenstein series described as follows. There is a $O(D)$ -invariant and Sp_4 -equivariant map

$$F : S((X \otimes D)(\mathbb{A})) \longrightarrow I(1/2)$$

with

$$I(s) = \text{Ind}_{P_V}^{Sp_4} |\det|^s \quad (\text{normalized induction})$$

given by

$$F(\phi_D)(g) = \omega(g)\phi_D(0).$$

One may consider the Eisenstein series $E(F(\phi_D), s, g)$ associated to the standard section attached to $F(\phi_D)$. Then the result of Kudla-Rallis is:

$$\int_{O(D)(F)\backslash O(D)(\mathbb{A})} \theta(\phi_D)(gh) dh = c \cdot E(F(\phi_D), 1/2, g)$$

for some non-zero constant c . By adjusting the measure dh , there is no loss of generality in assuming that $c = 1$.

It should be noted that the family of Eisenstein series attached to an arbitrary standard section of $I(s)$ can have a pole of order 1 at $s = 1/2$. However, for the sections in the image of F , the associated Eisenstein series is holomorphic at $s = 1/2$. This is reflected by the fact that $I(1/2)$ is reducible. The structure of the local degenerate principal series $I_v(1/2)$ is described precisely in [KRS, Props. 1.1 and 1.2] and [LZ, Thm. 1]. We record the relevant facts:

Proposition 5.4. (i) *If v is non-archimedean, then $I_v(1/2) = \Theta_v(1_{O(2,2)})$, which has length 2. It has a unique irreducible submodule isomorphic to $\Theta_v(1_{O(B_v)})$ (where B_v is the quaternion division algebra over F_v) and a unique irreducible quotient isomorphic to $\Theta_v(1_{O(1,1)})$.*

(ii) *If v is archimedean, then $I_v(1/2) = \Theta_v(1_{O(2,2)})$. If v is real, then $I_v(1/2)$ has a unique irreducible quotient isomorphic to $\Theta_v(1_{O(1,1)})$ and its unique maximal submodule is isomorphic to $\Theta_v(1_{O(4,0)}) \oplus \Theta_v(1_{O(0,4)})$.*

Corollary 5.5. *The image of F is the submodule of $I(1/2)$ given by*

$$\left(\otimes_{v \in \Sigma_D} \Theta_v(1_{O(D_v)}) \right) \otimes \left(\otimes_{v \notin \Sigma_D} I_v(1/2) \right).$$

In view of the Siegel-Weil formula, we see that

$$\int_{O(D)(F)\backslash O(D)(\mathbb{A})} \theta_D(\phi, f)(h) dh = \int_{Sp_4(F)\backslash Sp_4(\mathbb{A})} \overline{f(g)} \cdot \theta(\phi_{\mathbb{H}})(g) \cdot E(F(\phi_D), 1/2, g) dg$$

and to prove the theorem, it suffices to show that the integral on the RHS is non-zero for some choices of $\phi = \phi_D \otimes \phi_{\mathbb{H}}$ and f .

In [PSR], Piatetski-Shapiro and Rallis have considered the Rankin-Selberg integral suggested by the RHS of the above equality:

$$(5.6) \quad Z(s, f, \Phi, \phi_{\mathbb{H}}) = \int_{Sp_4(F)\backslash Sp_4(\mathbb{A})} \overline{f(g)} \cdot \theta(\phi_{\mathbb{H}})(g) \cdot E(\Phi, s, g) dg.$$

This family of global zeta integrals is not identically zero if σ has non-vanishing Fourier coefficients along N_Y corresponding to the split binary quadratic space [PSR]. Moreover, Piatetski-Shapiro and Rallis showed that

$$(5.7) \quad Z(s, f, \Phi, \phi_{\mathbb{H}}) = L^S(s + \frac{1}{2}, \sigma, std) \cdot Z_S(s, f_S, \Phi_S, \phi_{\mathbb{H}, S})$$

where S is a finite set of places of F containing Σ_D .

Now let us examine the analytic behavior of both sides of (5.7) at $s = 1/2$. The Eisenstein series $E(\Phi, s, g)$ in (5.6) has a pole of order at most 1 at $s = 1/2$ and its residue there is contained in the regularized theta lift of the trivial representation of $O(1, 1)(\mathbb{A})$ [KRS, Thm. 4.1(iii)]. Thus, if the residue at $s = 1/2$ of the LHS of (5.7) is nonzero, we would conclude that σ has a non-zero theta lift to $GO(2, 2)$, which is a contradiction (since we know that σ_v does not lift to $GO(2, 2)$ for $v \in \Sigma_D$). Thus, the LHS of (5.7) is holomorphic at $s = 1/2$. On the other hand, we know that $L^S(s + \frac{1}{2}, \sigma, std)$ is holomorphic and nonzero at $s = 1/2$. This implies that the ramified factor $Z_S(s, f_S, \Phi_S, \phi_{\mathbb{H}, S})$ on the RHS of (5.7) is also holomorphic at $s = 1/2$. Moreover, it was shown in [PSR] that there are choices of data such that $Z_S(1/2, f_S, \Phi_S, \phi_{\mathbb{H}, S})$ is nonzero.

For our purpose, we need to show that for some Φ of the form $F(\phi_D)$, the ramified factor $Z_S(1/2, f_S, \Phi_S, \phi_{\mathbb{H}, S}) \neq 0$. Let us first fix f_S, Φ_S and $\phi_{\mathbb{H}, S}$ such that $Z_S(1/2, f_S, \Phi_S, \phi_{\mathbb{H}, S}) \neq 0$. Now, fixing the components of f_S in $S \setminus \Sigma_D$ while varying the components in Σ_D , we see that this ramified zeta factor at $s = 1/2$ gives a nonzero $Sp_4(F_{\Sigma_D})$ -equivariant map

$$\sigma_{\Sigma_D}^{\vee} \otimes I_{\Sigma_D}(1/2) \otimes S((X \otimes \mathbb{H})(F_{\Sigma_D})) \longrightarrow \mathbb{C}.$$

We need to show that it is still non-zero if we restrict the second argument to the submodule $\Theta_{\Sigma_D}(1_{O(D)})$. If not, then for some place $v \in \Sigma_D$, we would obtain a non-zero $Sp_4(F_v)$ -equivariant map

$$\sigma_v^{\vee} \otimes \Theta_v(1_{O(1,1)}) \otimes S((X \otimes \mathbb{H})(F_v)) \longrightarrow \mathbb{C}.$$

Since $\Theta_v(1_{O(1,1)})$ is a quotient of $S(X \otimes \mathbb{H})$, we would deduce that there is a non-zero $Sp_4(F_v)$ -equivariant map

$$S((X \otimes \mathbb{H}^2)(F_v)) \longrightarrow \sigma_v.$$

This contradicts the assumption that σ_v does not participate in the theta correspondence with $GO(2, 2)$.

This completes the proof of Theorem 5.3. □

As we explained above, the proof of Theorem 5.2 is complete. □

Remark 5.8. *Theorem 5.3 holds without the assumption that $\Sigma_{D,\infty}$ is empty. The only difference in the proof lies in the fact that for $v \in \Sigma_{\Sigma,\infty}$, D_v is the quaternion division algebra and the quadratic space (D_v, \mathbb{N}_{D_v}) is positive definite, so that in Corollary 5.5, the image of F has component $\Theta_v(1_{O(4,0)})$ at $v \in \Sigma_{D,\infty}$. In the ensuing local argument, we need to show that the local zeta integral*

$$\sigma_v^\vee \otimes I_v(1/2) \otimes S(X \otimes \mathbb{H})(F_v) \longrightarrow \mathbb{C}$$

is nonzero when one restricts the second argument to lie in the submodule $\Theta_v(1_{O(4,0)})$. But if this were not the case, then the above map would still be zero even when one restricts to the larger submodule $\Theta_v(1_{O(4,0)}) \oplus \Theta_v(1_{O(0,4)})$, because σ_v is a representation of $GSp_4(\mathbb{F}_v) = GSp_4(\mathbb{R})$. This then leads to the contradiction that σ_v participates in the theta correspondence with $GO(2, 2)$, as in the proof above.

As for Theorem 5.2, because one is not able to appeal to Baldulescu's results [B], one can only conclude that there is a cuspidal representation π'_D which is nearly equivalent to π_D and which possesses Shalika period with respect to μ . Without the strong multiplicity one theorem for $GL_2(D)$, one could not conclude that $\pi'_D = \pi_D$.

6. The Equivalences of (C), (D) and (E)

In this section, we show the equivalences of (C), (D) and (E) in Thm. 1.3. For convenience, we restate the result to be proved:

Theorem 6.1. *Assume that D is split at every archimedean place. Suppose that $\pi_D \boxtimes \mu$ is a cuspidal representation of $GSO(V_D)$ whose Jacquet-Langlands lift $JL(\pi_D)$ is not cuspidal, so that $JL(\pi_D)$ is contained in the residual spectrum and is isomorphic to the unique irreducible quotient of $PS(\tau| - |^{1/2}, \tau| - |^{-1/2})$ for a cuspidal representation τ of $GL_2(\mathbb{A})$. Then the following are equivalent:*

- (C) π_D has Shalika period with respect to μ .
- (D) μ is equal to the central character ω_τ of τ .
- (E) The (incomplete) twisted exterior square L -function $L^S(s, \pi_D, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 2$.

The equivalence of (D) and (E) is easy to verify. Indeed, since π_D is nearly equivalent to any irreducible constituent of $PS(\tau| - |^{1/2}, \tau| - |^{-1/2})$, we see that $\mu^2 = \omega_\tau^2$ so that $\mu = \omega_\tau \cdot \chi$ for some quadratic character χ and

$$L^S(s, \pi_D, \bigwedge^2 \otimes \mu^{-1}) = L^S(s+1, \chi) \cdot L^S(s-1, \chi) \cdot L^S(s, \tau \times \tau^\vee \cdot \chi^{-1}).$$

From this, one deduces that $L^S(s, \pi_D, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 2$ if and only if χ is trivial, i.e. that $\mu = \omega_\tau$.

Before proving the equivalence of (C) and (D), let us take note of the following consequence of Badulescu's paper [B].

Proposition 6.2. *The Jacquet-Langlands correspondence (as defined by Badulescu [B]) sets up a bijection between*

- (a) *the set of irreducible infinite dimensional constituents of the discrete spectrum of $GL_2(D)$ whose Jacquet-Langlands lift to GL_4 is non-cuspidal;*
- (b) *the irreducible infinite dimensional constituents of the residual spectrum of GL_4 .*

Moreover, for π_D as in (a), so that $JL(\pi_D)$ is the unique irreducible quotient of $PS(\tau|^{-1/2}, \tau|^{-1/2})$ for a cuspidal representation τ of $GL_2(\mathbb{A})$, π_D is cuspidal if and only if τ is not compatible with D , i.e. τ_v is not a discrete series representation for some $v \in \Sigma_D$.

Now suppose that (C) holds so that π_D has Shalika period with respect to μ . Then by Cor. 3.2 and Prop. 2.2(ii), the theta lift of $\pi_D \boxtimes \mu$ to GSp_4 is nonzero and non-cuspidal. Thus, by the tower property of theta correspondence, the theta lift of $\pi_D \boxtimes \mu$ to $GSp_2 \cong GL_2$ is nonzero and cuspidal. Let σ be an irreducible constituent of the theta lift of $\pi_D \boxtimes \mu$ to GL_2 , so that $\omega_\sigma = \mu$. Since the theta lift of σ to $GSO(2, 2)$ is nonzero, the theta lift of σ to $GSO(V)$ is also nonzero. Indeed, it is not difficult to check that the theta lift of σ to $GSO(V)$ is nearly equivalent to the irreducible constituents of $PS(\sigma|^{-1/2}, \sigma|^{-1/2}) \boxtimes \omega_\sigma$ (cf. Thm. 8.11). However, this is nearly equivalent to $JL(\pi_D) \boxtimes \mu$ and so we deduce by the generalized strong multiplicity one theorem that

$$\sigma = \tau \quad \text{and} \quad \omega_\tau = \mu.$$

This proves the implication (C) \implies (D).

Suppose now that (D) holds. Then we consider the theta lift $\Theta_D(\tau)$ of τ from GL_2 to $GSO(V_D)$. In the proposition below, we shall show that $\Theta_D(\tau)$ has nonzero Shalika period. This is sufficient to show the implication (D) \implies (C). Indeed, by Prop. 6.2, the fact that π_D is cuspidal implies that τ is not compatible with D , so that the theta lift of τ to $GSO(D)$ is zero. This shows that $\Theta_D(\tau)$ is cuspidal and all its constituents are nearly equivalent to $PS(\tau|^{-1/2}, \tau|^{-1/2}) \boxtimes \omega_\tau$ and thus to $\pi_D \boxtimes \omega_\tau$. By the strong multiplicity one theorem for $GL_2(D)$, one concludes that $\pi_D \boxtimes \omega_\tau = \Theta_D(\tau)$ and so π_D has Shalika period with respect to μ . This proves (C).

It remains then to show:

Proposition 6.3. *Let τ be a cuspidal representation of GL_2 and let $\Theta_D(\tau)$ denote the theta lift of τ to $GSO(V_D)$. Then $\Theta_D(\tau)$ has nonzero Shalika period.*

Proof. This follows by a direct computation. We begin by setting up some notations. Let $W' = F \cdot e \oplus F \cdot f$ be a rank 2 symplectic space so that $GSp(W') \cong GL_2$. Then $Sp(W')$ acts transitively on the nonzero elements of W and the stabilizer of e is the unipotent radical U of the Borel subgroup stabilizing the line $F \cdot e$.

Recall that we have a decomposition

$$V_D = F \cdot (1, 0) \oplus D \oplus F \cdot (0, 1)$$

and let us set $v_0 = (1, 0)$ and $v_0^* = (0, 1)$. The Weil representation ω'_D of $O(V_D) \times Sp(W')$ has a mixed model relative to the above decomposition of V_D . This is realized on the space of Schwarz functions on $(v_0^* \otimes W) \oplus (f \otimes D)$, i.e. on $S(v_0^* \otimes W) \otimes S(f \otimes D)$.

For $\phi \in S(v_0^* \otimes W) \otimes S(f \otimes D)$, let $\theta(\phi)$ be the associated theta function and for $f \in \tau$, we have the theta lift $\theta(\phi, f)$. Now we compute:

$$\begin{aligned}
& \mathcal{S}_D(\theta(\phi, f)) \\
&= \int_{S_D(F) \backslash S_D(\mathbb{A})} \overline{\psi_D(s)} \cdot \left(\int_{S_p(W')(F) \backslash S_p(W')(\mathbb{A})} \theta(\phi)(sg) \cdot \overline{f(g)} dg \right) ds \\
&= \int_{S_D(F) \backslash S_D(\mathbb{A})} \overline{\psi_D(s)} \cdot \left(\int_{S_p(W')(F) \backslash S_p(W')(\mathbb{A})} \sum_{w \in W} \sum_{d \in D} (\omega'_D(sg)\phi)(w, d) \cdot \overline{f(g)} dg \right) ds \\
&= \int_{S_D(F) \backslash S_D(\mathbb{A})} \overline{\psi_D(s)} \cdot \left(\int_{S_p(W')(F) \backslash S_p(W')(\mathbb{A})} \sum_{\gamma \in U(F) \backslash S_p(W')(F)} \sum_{d \in D} (\omega'_D(s\gamma g)\phi)(e, d) \cdot \overline{f(\gamma g)} dg \right) ds \\
&= \int_{S_D(F) \backslash S_D(\mathbb{A})} \overline{\psi_D(s)} \cdot \left(\int_{U(F) \backslash S_p(W')(\mathbb{A})} \sum_{d \in D} (\omega'_D(sg)\phi)(e, d) \cdot \overline{f(g)} dg \right) ds \\
&= \int_{U(F) \backslash S_p(W')(\mathbb{A})} \overline{f(g)} \cdot \int_{PD_F^\times \backslash PD_{\mathbb{A}}^\times} \sum_{d \in D} \omega'_D(hg)\phi(e, d) \cdot \left(\int_{N_D(F) \backslash N_D(\mathbb{A})} \overline{\psi_D(n)} \cdot \psi(\text{Tr}(n\bar{d})) dn \right) dh dg \\
&= \int_{U(F) \backslash S_p(W')(\mathbb{A})} \overline{f(g)} \cdot \int_{PD_F^\times \backslash PD_{\mathbb{A}}^\times} \omega'_D(hg)\phi(e, 1_D) dh dg \\
&= \int_{U(\mathbb{A}) \backslash S_p(W')(\mathbb{A})} \left(\int_{U(F) \backslash U(\mathbb{A})} \psi(u) \cdot \overline{f(ug)} du \right) \cdot \omega'_D(g)\phi(e, 1_D) dg \\
&= \int_{U(\mathbb{A}) \backslash S_p(W')(\mathbb{A})} \overline{\mathcal{W}_\psi(g \cdot f)} \cdot \omega'_D(g)\phi(e, 1_D) dg,
\end{aligned}$$

where \mathcal{W}_ψ is the global Whittaker functional on τ and we have normalized measures so that

$$\int_{N_D(F) \backslash N_D(\mathbb{A})} dn = 1 \quad \text{and} \quad \int_{PD_F^\times \backslash PD_{\mathbb{A}}^\times} dh = 1.$$

Since $\mathcal{W}_\psi(f)$ is nonzero for some f , it follows by a standard argument as in [GS, Pg. 2718-2719] that $\mathcal{S}_D(\theta(\phi, f)) \neq 0$ for some ϕ and f . The proposition is proved. \square

The cuspidal representations π_D whose Jacquet-Langlands lifts are not cuspidal are precisely the CAP representations of $GL_2(D)$. Here, the notion of CAP is as given in [G, §3.9]. Props. 6.2 and 6.3 essentially show that all CAP representations (as well as the residual spectrum) of $GL_2(D)$ can be obtained as theta lifts from GL_2 . More precisely, for any such π_D , there is a unique cuspidal representation τ of GL_2 whose theta lift to $GSO(V_D)$ is equal to $\pi_D \boxtimes \omega_\tau$.

7. The Local Problem

In this section, we shall study the local analog of Theorems 3.3 and 5.2 so as to clarify the nature of the local obstructions there. In particular, we shall relate it to the local Gross-Prasad conjecture. Thus, in this section, we shall let F denote a *non-archimedean* local field and D the unique quaternion division algebra over F .

Let π_D be an irreducible representation of $GL_2(D)$ with central character μ^2 and let π be its Jacquet-Langlands lift on $GL_4(F)$. Recall that π_D (and similarly π) has Shalika period with respect

to μ if

$$\mathrm{Hom}_{\Delta D^\times \cdot N_D}(\pi_D, (\mu \circ \mathbb{N}_D) \boxtimes \psi_D) \neq 0,$$

or equivalently, regarding $\pi_D \boxtimes \mu$ as a representation of $GSO(V_D)$,

$$\mathrm{Hom}_{S_D}(\pi_D \boxtimes \mu, \psi_D) \neq 0.$$

It is known that the dimension of this Hom space is at most 1.

In the papers [P1] and [P2], D. Prasad has studied the question of existence of local Shalika periods, especially for irreducible principal series representations. Let us recall his results briefly. Recall that if τ_1 and τ_2 are two infinite-dimensional representations of $GL_2(F)$, then $PS(\tau_1, \tau_2)$ denotes the representation of $GL_4(F)$ unitarily induced from the representation $\tau_1 \boxtimes \tau_2$ of P . Similarly, if $\tau_{D,1}$ and $\tau_{D,2}$ are two representations of D^\times , then $PS(\tau_{D,1}, \tau_{D,2})$ is the analogous principal series representation induced from P_D . The following proposition is due to D. Prasad [P1, Prop. 7] and [P2, Thm. 2]:

Proposition 7.1. (i) *We have a short exact sequence of GL_2 -modules:*

$$0 \longrightarrow \tau_1 \otimes \tau_2 \longrightarrow PS(\tau_1, \tau_2)_{N, \psi} \longrightarrow \pi(\omega_1 | - |^{1/2}, \omega_2 | - |^{-1/2}) \longrightarrow 0,$$

where ω_i is the central character of τ_i . Thus, if $PS(\tau_1, \tau_2)$ is irreducible, then it possesses local Shalika period with respect to μ if and only if one of the following holds:

- $\omega_1 = \omega_2 = \mu$;
- $\tau_1^\vee = \tau_2 \otimes \mu^{-1}$.

(ii) *We have an isomorphism of D^\times -modules:*

$$PS(\tau_{D,1}, \tau_{D,2})_{N_D, \psi_D} \cong \tau_{D,1} \otimes \tau_{D,2}.$$

Thus, if $PS(\tau_{D,1}, \tau_{D,2})$ is irreducible, it possesses local Shalika period with respect to μ if and only if $\tau_{D,1}^\vee \cong \tau_{D,2} \otimes \mu^{-1}$.

Proof. The reader may notice that the statement of (i) is different from that in [P2, Thm. 2]; namely, in the 3rd term of the short exact sequence, [P2, Thm. 2] has $\pi(\omega_1, \omega_2)$ instead of $\pi(\omega_1 | - |^{1/2}, \omega_2 | - |^{-1/2})$. Prasad has informed us that there is a normalization error in [P2, Thm. 2] and since the proof given there is somewhat sketchy, he has kindly provided us with a detailed proof, which we reproduce here.

(i) We have:

$$GL_4 = P \cup Pw_{23}P \cup PwP$$

where $w_{23} = (23) \in S_4$ (the Weyl group of GL_4) and $w = (13)(24)$. By Mackey theory, the restriction of $PS(\tau_1, \tau_2)$ to P has a filtration with successive quotients

$$\begin{cases} A = \delta_P^{1/2} \cdot (\tau_1 \boxtimes \tau_2) \\ B = \mathrm{ind}_{P \cap w_{23}Pw_{23}}^P \delta_P^{1/2} \cdot (\tau_1 \boxtimes \tau_2) \\ C = \mathrm{ind}_{P \cap wPw}^P \delta_P^{1/2} \cdot (\tau_1 \boxtimes \tau_2). \end{cases}$$

We shall be interested in the restriction of these representations to the Shalika subgroup \tilde{S} . Now the quotient A does not contribute to the twisted Jacquet module since N acts trivially. As for C , $P \cap wPw = M = GL_2 \times GL_2$ and one sees that

$$C_{N, \psi} \cong \tau_1 \otimes \tau_2$$

as representations of ΔGL_2 . Thus it remains to show that

$$B_{N, \psi} \cong \pi(\omega_1 | - |^{1/2}, \omega_2 | - |^{-1/2}).$$

The stabilizer group $P \cap w_{23}Pw_{23}$ is equal to $(B \times B) \cdot N_0$, where

$$N_0 = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset N.$$

Now consider the restriction of B to the Shalika subgroup $\tilde{S} = \Delta GL_2 \cdot N$. The double coset space

$$\tilde{S} \backslash P / (P \cap PwP) = \Delta GL_2 \backslash (GL_2 \times GL_2) / (B \times B)$$

has size 2. By Mackey theory, we need to consider each of these double cosets in turn.

Let us first consider the non-trivial double coset which is represented by the Weyl group element $w_{12} = (12)$. We first compute:

$$\Delta T \cdot N^0$$

where

$$N^0 = \left\{ \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \right\} \subset N.$$

So the representation under consideration is

$$Ind_{\Delta T \cdot N^0}^{\Delta GL_2 \cdot N} \tau_1 \otimes \tau_2,$$

where the action of an element in $T \cdot N^0$ is via the sequence of maps

$$t \cdot n \mapsto w_{12}(t \cdot n)w_{12} \mapsto w_{23}w_{12}(t \cdot n)w_{12}w_{23} \in P$$

followed by the action of P on $\tau_1 \boxtimes \tau_2$. Explicitly,

$$\begin{pmatrix} t_1 & 0 & n_3 \\ & t_2 & n_1 & n_2 \\ & & t_1 & \\ & & & t_2 \end{pmatrix} \mapsto \begin{pmatrix} t_2 & n_1 & 0 & n_2 \\ & t_1 & 0 & 0 \\ & & t_1 & n_3 \\ & & & t_2 \end{pmatrix}.$$

Since the character ψ of N is non-trivial on the 1-parameter subgroup with coordinate n_2 , but the restriction of $\tau_1 \boxtimes \tau_2$ to the image of this 1-parameter subgroup under the above map is trivial, we see that the non-trivial double coset does not contribute to the twisted Jacquet module.

It remains to consider the trivial double coset which gives rise to the representation

$$Ind_{\Delta B \cdot N_0}^{\Delta GL_2 \cdot N} \delta_P^{1/2} \cdot (\tau_1 \otimes \tau_2).$$

We note the following general lemma:

Lemma 7.2. *For a representation π of $\Delta B \cdot N$,*

$$\left(Ind_{\Delta B \cdot N}^{\Delta GL_2 \cdot N} \pi \right)_{N, \psi} \cong Ind_{\Delta B}^{\Delta GL_2} \pi_{N, \psi}$$

Therefore, we need to calculate

$$\left(ind_{\Delta B \cdot N_0}^{\Delta B \cdot N} \tau_1 \otimes \tau_2 \right)_{N, \psi}$$

as a ΔB -module. For this, note the following:

Lemma 7.3. *Let $\psi_0 = \psi|_{N_0}$. Then for a representation π of $\Delta B \cdot N_0$,*

$$\left(ind_{\Delta B \cdot N_0}^{\Delta B \cdot N} \pi \right)_{N, \psi} = \delta_B^{-1} \cdot \pi_{N_0, \psi_0}$$

where δ_B is the modulus character of ΔB (which is the inverse of the character of the action of ΔB on N/N_0).

Applying this lemma, and using the fact that the action of $\Delta B \cdot N_0$ on $\delta_P^{1/2} \cdot (\tau_1 \otimes \tau_2)$ is via the map

$$\begin{pmatrix} a & b & n_1 & n_2 \\ & c & 0 & n_3 \\ & & a & b \\ & & & c \end{pmatrix} \mapsto \begin{pmatrix} a & n_1 & b & n_2 \\ & a & 0 & b \\ & & c & n_3 \\ & & & c \end{pmatrix},$$

we conclude that

$$(\delta_P^{1/2} \cdot (\tau_1 \otimes \tau_2))_{N_0, \psi_0} = \delta_B \cdot (\omega_1 \boxtimes \omega_2)$$

as representations of ΔB . Putting everything together completes the proof of (i).

(ii) The proof is similar and in fact easier; we refer the reader to [P1, Prop. 7]. \square

From this proposition, one sees that it is possible that a representation π of GL_4 has local Shalika period, but its Jacquet-Langlands lift π_D does not. This was exploited in our construction of the counterexample to Thm. 3.3. Note however that our local condition in Thm. 5.2 rules out more representations π than those for which π_D has no Shalika period. For example, if $\tau_1 = \tau_2$ has central character μ , then by the Proposition, both $PS(\tau_1, \tau_2)$ and its Jacquet-Laglands transfer to $GL_2(D)$ admit Shalika period with respect to μ . However, such representations are ruled out by the local condition in Thm. 5.2. This is explained by Thm. 7.8.

To study the local Shalika period of more general representations, such as the discrete series, we note the following local analog of Proposition 3.1.

Proposition 7.4. (i) As representations of $GSO(V_D)$,

$$(\Omega_D)_{U, \chi} = \text{ind}_{S_D}^{GSO(V_D)} \psi_D.$$

(ii) Suppose that D is split and U_D is a maximal unipotent subgroup of $GSO(V_D)$ with nondegenerate character χ_D . Then as representations of GSp_4 ,

$$(\Omega_D)_{U_D, \chi_D} \cong \text{ind}_U^{GSp_4} \chi.$$

Proof. The statement (i) is proved in a similar way as Prop. 3.1. The statement (ii) is essentially [MS, Prop. 4.1]; see also [GRS, Prop. 2.4 and Cor. 2.5]. \square

Corollary 7.5. (i) Let π_D be an irreducible representation of $GSO(V_D)$ with central character μ^2 . Then π_D has Shalika period with respect to μ if and only if $\dim \Theta(\pi_D \boxtimes \mu)_{U, \chi} = 1$.

(ii) Suppose that D is split and π_D is generic. Then π_D has Shalika period with respect to μ if and only if $\theta(\pi_D \boxtimes \mu)$ is generic. Moreover, if σ is an irreducible representation of GSp_4 , then σ is generic if and only if $\Theta_D(\sigma)$ is generic, in which case $\dim \Theta_D(\sigma)_{U_D, \chi_D} = 1$.

Theorem 7.6. Let π be a generic representation of GL_4 with central character μ^2 . The following are equivalent:

- (i) π has Shalika period with respect to μ ;
- (ii) the small theta lift $\theta(\pi \boxtimes \mu)$ of $\pi \boxtimes \mu$ to GSp_4 is generic;
- (iii) the small theta lift $\theta(\pi \boxtimes \mu)$ of $\pi \boxtimes \mu$ to GSp_4 is non-zero;
- (iv) the Langlands parameter φ_π of π factors through $GSp_4(\mathbb{C})$ and its similitude character $\Lambda \circ \varphi_\pi$ is equal to μ .

If π is a discrete series representation, then the above conditions are equivalent to:

(v) The L-factor $L(s, \pi, \bigwedge^2 \otimes \mu^{-1})$ defined by Shahidi has a pole at $s = 0$.

Proof. The equivalence of (i), (ii) and (iii) is the previous corollary. For the equivalence of (iv) and (v) in the case of discrete series representations, note that if

$$\varphi_\pi : W'_F \longrightarrow GL_4(\mathbb{C})$$

is the Langlands parameter of π , then a recent result of Henniart [He] shows that the local Langlands correspondence for GL_n respects twisted exterior square L-functions, so that

$$L(s, \pi, \bigwedge^2 \otimes \mu^{-1}) = L(s, \bigwedge^2 \varphi_\pi \otimes \mu^{-1}).$$

Moreover, $L(s, \bigwedge^2 \varphi_\pi \otimes \mu^{-1})$ has a pole at $s = 0$ if and only if $(\bigwedge^2 \varphi_\pi) \otimes \mu^{-1}$ contains the trivial representation as a summand. In other words, the action of W'_F via φ_π preserves a non-zero symplectic form up to scaling by the character μ (thought of as a character of W'_F by local class field theory). This symplectic form is necessarily nondegenerate, so that φ_π factors through $GSp_4(\mathbb{C})$, for otherwise, its kernel is a non-trivial W'_F -submodule, which contradicts the irreducibility of φ_π .

The main assertion of the theorem is thus the equivalence of (i)-(iii) and (iv). In fact, the key case of discrete series representations is a special case of a beautiful theorem of Muic-Savin [MS, Thm. 2.2], which shows that the theta lift of a discrete series representation π to GSp_4 is non-zero iff (v) holds.

With the discrete series case taken care of, a non-discrete series generic representation π is of the form $Ind_Q^{GL_4} \tau$ with τ a twist of a discrete series representation on the Levi factor of some parabolic Q . If Q is not the $(1, 3)$ - or $(3, 1)$ - parabolic, then by induction-in-stages, we may assume that Q is the parabolic P , in which case τ is generic but not necessarily a discrete series. The equivalence of (i) and (iv) then follows readily from Prop. 7.1.

It remains to consider the case when $\pi = Ind_Q^{GL_4} \tau$ with Q the $(3, 1)$ -parabolic and τ is the a twist of a discrete series representation. In this case, the Langlands parameter of π is not of symplectic type and so we need to show that π does not have local Shalika period with respect to μ . This can be checked by a Mackey theory argument analogous to the proof of Prop. 7.1. \square

Remarks: In the recent preprint of Jiang-Nien-Qin [JNQ], the case of a general division algebra D of degree ≥ 2 is considered and the equivalence of (i), (iv) and (v) for supercuspidal π and trivial μ is shown.

Since it is not much of a trouble and for the convenience of reference, we list the different types of symplectic parameters $\varphi_\pi : W'_F \longrightarrow GSp_4(\mathbb{C})$ with similitude character μ which could give rise to generic representations of GL_4 :

- (1) φ_π is irreducible. In this case, π is a discrete series representation.
- (2) $\varphi_\pi = \phi_1 \oplus \phi_2$ where each ϕ_i is an irreducible 2-dim representation satisfying one of the following:
 - (a) $\det \phi_i = \mu$;
 - (b) $\phi_1^\vee = \phi_2 \otimes \mu^{-1}$.

In this case, π is the representation $PS(\tau_1, \tau_2)$ where τ_i is the discrete series representation of GL_2 associated to ϕ_i .

(3) $\varphi_\pi = \phi \oplus \chi_1 \oplus \chi_2$, where ϕ is irreducible of dimension 2 and $\det \phi = \mu$, whereas the χ_i 's are 1-dimensional with $\chi_1 \chi_2 = \mu$. In this case, $\pi = PS(\tau, \pi(\chi_1, \chi_2))$ where τ is the discrete series representation associated to ϕ .

(4) $\varphi_\pi = \chi_1 \oplus \mu \chi_1^{-1} \oplus \chi_2 \oplus \mu \chi_2^{-1}$. In this case, $\pi = PS(\pi(\chi_1, \mu \chi_1^{-1}), \pi(\chi_2, \mu \chi_2^{-1}))$.

Of these 4 classes, only (1) and (2) are relevant parameters for $GL_2(D)$. Moreover, note that the cases (2a) and (2b) are not disjoint: their intersection consists of those parameters with $\phi_1 = \phi_2$ and $\det \phi_i = \mu$.

Now we examine the analogous problem for π_D . This may be a good time to bring in the local Gross-Prasad conjecture [GP]. We have been considering Shalika periods with respect to 1-dimensional representations of the Shalika group, by using the character μ on D^\times or GL_2 . More generally, one may consider the generalized Shalika period with respect to any irreducible representation τ of D^\times or GL_2 . For example, on GL_4 , one may consider the generalized Shalika period with respect to the twisted Steinberg representation $St \otimes \mu$. The local Gross-Prasad conjecture predicts that given tempered representations π_D and $\pi = JL(\pi_D)$,

$$\dim \text{Hom}_{\tilde{S}_D}(\pi_D, \mu \boxtimes \psi_D) + \dim \text{Hom}_{\tilde{S}}(\pi, (St \otimes \mu) \boxtimes \psi) = 1,$$

and

$$\dim \text{Hom}_{\tilde{S}_D}(\pi_D, \mu \boxtimes \psi_D) = 1 \iff \epsilon((\bigwedge^2 \varphi_\pi \otimes \mu^{-1}) \otimes S_2) = -1,$$

where S_2 denotes the 2-dimensional representation of W'_F trivial on W_F (so essentially it is a representation of $SL_2(\mathbb{C})$) and is the Langlands parameter of the Steinberg representation St . This last equivalence is part of Theorem 7.8 below. At this point, we note the following lemma which is explained to us by D. Prasad:

Lemma 7.7. *Suppose that π has central character μ^2 . Then the following are equivalent:*

- (i) $\epsilon((\bigwedge^2 \varphi_\pi \otimes \mu^{-1}) \otimes S_2) = -1$.
- (ii) $\dim((\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^{W'_F})$ is odd.
- (iii) φ_π is of type (1) or (2b).

In particular, if $\epsilon((\bigwedge^2 \varphi_\pi \otimes \mu^{-1}) \otimes S_2) = -1$, then $L(s, \bigwedge^2 \varphi_\pi \otimes \mu^{-1})$ has a pole at $s = 0$.

Proof. The representation $\bigwedge^2 \varphi_\pi \otimes \mu^{-1}$ is a map

$$W'_F \longrightarrow SO_6(\mathbb{C}).$$

In particular, it is a self dual representation with trivial determinant. We shall calculate the ϵ -factor by considering different cases. The reader can find similar computations in [P2]. The following well-known identity, for which we refer to [P2, §5], will be very useful:

$$\epsilon(\rho \otimes S_n, \psi) = \epsilon(\rho, \psi)^n \cdot \det(-Frob|\rho^I)^{n-1},$$

where ρ is a representation of the Weil-Deligne group W'_F trivial on $SL_2(\mathbb{C})$ and S_n is the n -dimensional irreducible representation of $SL_2(\mathbb{C})$ regarded as a representation of W'_F . In addition, we shall use the fact that

$$\epsilon(\rho, \psi) \cdot \epsilon(\rho^*, \psi) = \det \rho(-1).$$

In particular, if $\rho^* = \rho$, then $\epsilon(\rho, \psi)^4 = 1$ and $\epsilon(\rho, \psi)^2 = 1$ if $\det \rho$ is trivial. Now we may consider the different cases, according to how φ_π decomposes as a representation of $SL_2(\mathbb{C})$.

Case 1: φ_π is a representation of the Weil group W_F

Applying the above identities for the epsilon factor, we see that

$$\epsilon(S_2 \otimes (\bigwedge^2 \varphi_\pi \otimes \mu^{-1})) = \det(-Frob | (\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^I).$$

Here, if $(\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^I$ is zero, then the determinant in question is interpreted to be 1. Now the space $(\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^I$ is precisely the submodule spanned by the unramified characters occurring in $\bigwedge^2 \varphi_\pi \otimes \mu^{-1}$. By self-duality, if an unramified character χ occurs, then so must its inverse χ^{-1} . If $\chi^{-1} \neq \chi$, then the determinant of $-Frob$ on this 2-dimensional submodule is 1. On the other hand, if $\chi^{-1} = \chi$, then χ is either the trivial character or the unique unramified quadratic character. The action of $-Frob$ on χ is then -1 and 1 respectively. So we see that:

$$\det(-Frob | (\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^I) = (-1)^{\dim(\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^{W'_F}}.$$

This shows the equivalence of (i) and (ii) in this case. Moreover, if φ_π is irreducible, then

$$\epsilon(S_2 \otimes (\bigwedge^2 \varphi_\pi \otimes \mu^{-1})) = -1$$

iff φ_π is of symplectic type with similitude character μ .

Case 2: $\varphi_\pi = \rho \oplus (\chi \cdot S_2)$

In this case, $\det(\rho) \cdot \chi^2 = \mu^2$ and

$$\bigwedge^2 \varphi_\pi \otimes \mu^{-1} = \mu^{-1} \cdot \det(\rho) \oplus \mu^{-1} \cdot \chi^2 \oplus \mu^{-1} \cdot \chi \cdot (\rho \otimes S_2)$$

and so

$$S_2 \otimes \bigwedge^2 \varphi_\pi \otimes \mu^{-1} = ((\mu^{-1} \det(\rho) \oplus \mu^{-1} \cdot \chi^2) \otimes S_2) \oplus \mu^{-1} \chi \cdot \rho \oplus \mu^{-1} \chi \cdot (\rho \otimes S_3).$$

Observe that $(\mu^{-1} \cdot \det(\rho) \oplus \mu^{-1} \cdot \chi^2)$ and $\mu^{-1} \chi \cdot \rho$ are both self-dual with determinant 1. Thus, we see that

$$\epsilon(S_2 \otimes (\bigwedge^2 \varphi_\pi \otimes \mu^{-1})) = \det(-Frob | (\mu^{-1} \det(\rho))^I) \cdot \det(-Frob | (\mu^{-1} \chi^2)^I) = (-1)^{\dim(\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^{W'_F}},$$

which shows the equivalence of (i) and (ii) in this case. Note that for each of these determinants to be -1 , we need $\det(\rho) = \mu$ and $\chi^2 = \mu$ respectively. But since $\det(\rho) \cdot \chi^2 = \mu^2$, if one of these holds, so does the other. Thus, we see that the ϵ -factor is always 1.

Case 3: $\varphi_\pi = \rho \otimes S_2$

In this case, $\det(\rho)^2 = \mu^2$, so that $\det(\rho) \cdot \mu^{-1}$ is a quadratic character. We have:

$$\bigwedge^2 \varphi_\pi \otimes \mu^{-1} = \mu^{-1} \cdot \text{Sym}^2 \rho \oplus \mu^{-1} \cdot \det(\rho) \otimes S_3$$

and

$$S_2 \otimes \bigwedge^2 \varphi_\pi \otimes \mu^{-1} = (\mu^{-1} \cdot \text{Sym}^2 \rho \otimes S_2) \oplus (\mu^{-1} \det(\rho)) \otimes S_2 \oplus \mu^{-1} \det(\rho) \otimes S_4.$$

Now observe that $\mu^{-1} \cdot \text{Sym}^2(\rho)$ and $\mu^{-1} \det(\rho)$ are self-dual with determinant $\mu^{-1} \det(\rho)$. Thus, a short calculation gives:

$$\epsilon(S_2 \otimes (\bigwedge^2 \varphi_\pi \otimes \mu^{-1})) = \det(-Frob | (\mu^{-1} \cdot \text{Sym}^2 \rho)^I) = (-1)^{\dim(\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^{W'_F}}.$$

This shows the equivalence of (i) and (ii) in this case. Moreover, if ρ is irreducible, then $\epsilon(S_2 \otimes (\bigwedge^2 \varphi_\pi \otimes \mu^{-1})) = -1$ iff ρ is induced from a character of W_K with K/F a quadratic extension of F and $\mu^{-1} \cdot \det \rho = \omega_{K/F}$ (the quadratic character associated to K/F by local class field theory).

Case 4: $\varphi_\pi = \chi_1 \oplus (\chi_2 \otimes S_3)$

We have $\chi_1 \cdot \chi_2^3 = \mu^2$. Now

$$\bigwedge^2 \varphi_\pi \otimes \mu^{-1} = (\chi_1 \chi_2 \mu^{-1} \otimes S_3) \oplus (\chi_2^2 \mu^{-1} \otimes S_3),$$

which contains no trivial representation, and a short calculation shows that $\epsilon(S_2 \otimes (\bigwedge^2 \varphi_\pi \otimes \mu^{-1}))$ is always equal to 1.

Case 5: $\varphi_\pi = \chi \otimes S_4$

We have $\chi^4 = \mu^2$ and

$$\bigwedge^2 \varphi_\pi \otimes \mu^{-1} = \chi^2 \mu^{-1} \oplus (\chi^2 \mu^{-1} \otimes S_5)$$

and a short computation gives

$$\epsilon(S_2 \otimes (\bigwedge^2 \varphi_\pi \otimes \mu^{-1})) = \det(-\text{Frob} | (\chi^2 \mu^{-1})^I) = (-1)^{\dim(\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^{W'_F}},$$

which shows the equivalence of (i) and (ii). Indeed, the ϵ -factor is -1 iff $\chi^2 = \mu$.

We have thus shown the equivalence of (i) and (ii) in general. From this, it follows that if the epsilon factor is -1 , then the trivial representation occurs in $\bigwedge^2 \varphi_\pi \otimes \mu^{-1}$ so that φ_π is of symplectic type and $L(s, \pi, \bigwedge^2 \otimes \mu^{-1})$ has a pole at $s = 0$. A short computation now gives the following table:

Type of φ_π	(1)	(2b) and (2a)	(2b) but not (2a)	(2a) but not (2b)
$\dim(\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^{W'_F}$	1	3	1	2

From this, we see the equivalence of (ii) and (iii). The lemma is proved. \square

Our main local theorem is:

Theorem 7.8. *Suppose that π_D is a representation of $GL_2(D)$ with central character μ^2 such that its Jacquet-Langlands lift π to GL_4 is generic. The following are equivalent:*

- (i) π_D has Shalika period with respect to μ ;
- (ii) the big theta lift $\Theta(\pi_D \boxtimes \mu)$ of $\pi_D \boxtimes \mu$ to GSp_4 is generic (and thus non-zero).
- (iii) $\epsilon((\bigwedge^2 \varphi_\pi \otimes \mu^{-1}) \otimes S_2) = -1$.

Moreover, when these conditions hold, the small theta lift $\theta(\pi_D \boxtimes \mu)$ is non-generic precisely when $\pi_D = PS(\tau_{D,1}, \tau_{D,2})$ where $\tau_{D,1} = \tau_{D,2}$ has central character μ , i.e. when φ_π is of both type (2a) and (2b), or equivalently when $\dim(\bigwedge^2 \varphi_\pi \otimes \mu^{-1})^{W'_F} > 1$.

The rest of the section is devoted to the proof of this theorem. The equivalence of (i) and (ii) is Cor. 7.5. The content of the theorem is thus the equivalence of (i)+(ii) and (iii). If $\pi_D =$

$PS(\tau_{D,1}, \tau_{D,2})$, then the equivalence of these follows immediately from Prop. 7.1. Moreover, from the explicit determination of local theta correspondence given in the next section (specifically Thm. 8.3(i)), one sees that if $\pi_D = PS(\tau_{D,1}, \tau_{D,2})$ has local Shalika period with respect to μ , then $\theta(\pi_D \boxtimes \mu)$ is non-generic iff $\tau_{D,1} = \tau_{D,2}$ has central character μ .

Thus it remains to consider the case of discrete series representations. We note first that if π_D is supercuspidal and has local Shalika period with respect to μ , then $\theta(\pi_D \boxtimes \mu) = \Theta(\pi_D \boxtimes \mu)$ is generic. In fact, the same result holds if π_D is a discrete series representation, by general results of Muic [Mu, Thm. 6.2]. Moreover, by the previous lemma, it is clear that $\epsilon((\bigwedge^2 \varphi_\pi \otimes \mu^{-1}) \otimes S_2) = -1$ if and only if φ_π is of symplectic type with similitude character μ . Hence, to complete the proof of Thm. 7.8, it remains to show:

Theorem 7.9. *Let π_D be a discrete series representation of $GL_2(D)$ with central character μ^2 and Jacquet-Langlands lift π on GL_4 . Then π_D has Shalika period with respect to μ if and only if π does.*

It is clear that the theorem follows from the following proposition, which addresses the very natural question about the compatibility of the theta correspondence and the Jacquet-Langlands transfer of discrete series representations.

Proposition 7.10. *Suppose that π_D is a discrete series representation with Jacquet-Langlands lift π on GL_4 .*

(i) *If π_D has local Shalika period with respect to μ , so that $\sigma_D = \theta(\pi_D \boxtimes \mu)$ is generic, then the small theta lift of σ_D to $GSO(V)$ is isomorphic to $\pi \boxtimes \mu$.*

(ii) *Conversely, if π has local Shalika period with respect to μ , so that $\sigma = \theta(\pi \boxtimes \mu)$ is generic, then the small theta lift of σ to $GSO(V_D)$ is isomorphic to π_D .*

Proof. Our proof of the proposition is going to involve global arguments. Let us consider the statement (ii), so that we are starting with a discrete series $\pi \boxtimes \mu$ on $GSO(V)$ with Shalika period. By Theorem 7.6, we see that in the terminology of the proof of Lemma 7.7, π is one of the following:

- π is supercuspidal with symplectic parameter and similitude character μ ;
- π is a generalized Steinberg representation attached to a dihedral supercuspidal representation π_ρ , i.e. π has parameter of the form $\rho \boxtimes S_2$ (as in Case 3 in the proof of Lemma 7.7) with ρ irreducible and monomial with respect to a quadratic extension K/F and $\det \rho = \mu \cdot \omega_{K/F}$;
- π is a twisted Steinberg St_χ with $\chi^2 = \mu$ (as in Case 5 in the proof of Lemma 7.7).

The main technical tool we need is:

Lemma 7.11. *Let $\pi \boxtimes \mu$ be as in (ii) of the proposition. Let \mathbb{F} be a number field such that $\mathbb{F}_v = F$ for some place v of \mathbb{F} . Then π can be globalized to a cuspidal representation Π of GL_4 over \mathbb{F} such that Π has global Shalika period with respect to some Υ such that $\Upsilon_v = \mu$.*

Proof. If π is supercuspidal, this is a consequence of a general result of Prasad and Schulze-Pillot [PSP, Thm. 3.1] (proved using a simple form of the relative trace formula!). So suppose that π is not supercuspidal, so that it is of the other two types described above. If $\pi = St_\chi$ with $\chi^2 = \mu$, then note that St_χ is the Langlands lift of the twisted Steinberg representation st_χ of $GL_2(F)$ under the adjoint cube lifting $Sym^3 \otimes \det^{-1}$. Let Ω be a cuspidal representation of GL_2 over \mathbb{F} such that $\Omega_v = st_\chi$ and the central character of Ω is Υ . By the results of Kim-Shahidi [KS], we may consider the adjoint cube lifting of Ω to get a cuspidal representation Π of GL_4 so that the central character of Π is Υ^2 and $\Omega_v = St_\chi$. Moreover, by [KS, Pg. 877], the partial twisted exterior square L-function $L^S(s, \Pi, \bigwedge^2 \otimes \Upsilon^{-1})$ has a pole at $s = 1$. This Π is the desired cuspidal representation.

Finally, suppose π has parameter of the type $\rho \boxtimes S_2$. Then observe that π is the Langlands lift of the representation $\pi_\rho \boxtimes st$ of $GL_2 \times GL_2$ under the Rankin-Selberg lifting $GL_2 \boxtimes GL_2 \rightarrow GL_4$. We may find a dihedral cuspidal representation Ω of GL_2 associated to a quadratic extension \mathbb{K}/\mathbb{F} such that $\Omega_v = \pi_\rho$ and the central character of Ω is $\Upsilon \cdot \omega_{\mathbb{K}/\mathbb{F}}$. Similarly, let Ω' be a cuspidal representation of PGL_2 such that $\Omega'_v = st$. Then by results of Ramakrishnan [R], the cuspidal representation $\Omega \boxtimes \Omega'$ has a functorial lifting to a cuspidal Π on GL_4 whose local component at v is π and such that $L^S(s, \Pi, \wedge^2 \otimes \Upsilon^{-1})$ has a pole at $s = 1$. This Π is what we are looking for. \square

As is evident from its proof, the lemma applies to any finite set of finite places, though we have stated it only for a singleton set. To apply the lemma, let \mathbb{F} be a number field such that for two places v_1 and v_2 , we have $\mathbb{F}_{v_i} \cong F$ for $i = 1$ and 2 . Let \mathbb{D} be a global quaternion algebra over \mathbb{F} ramified precisely at v_1 and v_2 . By the lemma, one can find a cuspidal Π on GL_4 and a character Υ such that

- $\Pi_{v_i} \cong \pi$ for $i = 1$ and 2 ;
- $\Upsilon_{v_i} = \mu$ for $i = 1$ and 2 ;
- Π has global Shalika period with respect to Υ .

Then $\Pi \boxtimes \Upsilon$ has non-zero globally generic cuspidal theta lift Σ on GSp_4 . Moreover, by Theorem 5.2 (or rather the remark following its proof), we deduce that Σ has non-zero cuspidal theta lift $\Theta_{\mathbb{D}}(\Sigma)$ to $GSO(V_{\mathbb{D}})$. Now since $\Theta_{\mathbb{D}}(\Sigma)$ and the Jacquet-Langlands transfer of $\Pi \boxtimes \Upsilon$ are nearly equivalent, it follows by the strong multiplicity one result of Badulescu [B, Thm. 5.1(c)] that they are in fact isomorphic. Extracting the component at v_1 proves (ii).

For (i), we start with π_D on $GL_2(D)$ and get $\sigma_D = \theta(\pi_D \otimes \mu)$ on $GSp_4(F)$. Let $\pi' \boxtimes \mu$ be the theta lift of σ to $GSO(V)$ so that π' has Shalika period with respect to μ . By (ii), we conclude that π' and π_D are related by the Jacquet-Langlands correspondence. This proves (i). \square

This completes the proof of Thm. 7.8. It seems to us that the proof of Prop. 7.10 for non-supercuspidal representations is an overkill, in the sense that it makes use of too much global machinery. There is in fact a purely local proof of the proposition in the non-supercuspidal case, by the explicit determination of the local theta correspondence between GSp_4 and $GSO(V_D)$ (for D both split and non-split). We discuss this in the next section.

8. Explicit Local Theta Correspondence

In this section, we describe the explicit determination of local theta correspondences for the (almost) dual pairs

$$GSp_4 \times GSO(V) \quad \text{and} \quad GSp_4 \times GSO(V_D),$$

as much as is needed for the applications of this paper. Specifically, we shall be interested in the theta lift of discrete series representations in both cases, and the theta lift of irreducible principal series in the non-split case. As we mentioned in the introduction, Waldspurger [W] has already determined part of the correspondence in the split case and our results here are a refinement of his.

To state the results, we introduce some notations. Recall from Section 2 that we have a Witt decomposition $W = X \oplus Y$. Suppose that $X = F \cdot e_1 \oplus F \cdot e_2$ and $Y = F \cdot f_1 \oplus F \cdot f_2$ and consider the decomposition $W = Fe_1 \oplus W' \oplus Ff_1$, where $W' = \langle e_2, f_2 \rangle$. Let $Q(Z) = L(Z) \cdot U(Z)$ be the maximal parabolic stabilizing the line $Z = F \cdot f_1$, so that

$$L(Z) = GL(Z) \times GSp(W')$$

and $U(Z)$ is a Heisenberg group:

$$1 \longrightarrow \text{Sym}^2 Z \longrightarrow U(Z) \longrightarrow W' \otimes Z \longrightarrow 1.$$

A representation of $L(Z)$ is thus of the form $\chi \boxtimes \tau$ where τ is a representation of $GS\!p(W') \cong GL_2$. We let $I_{Q(Z)}(\chi, \tau)$ be the corresponding parabolically induced representation (i.e. via normalized induction). The module structure of this induced representation is known. In particular, we note the following lemma (cf. [W, Prop. 5.1] and [ST]):

Lemma 8.1. (a) *Let τ be a supercuspidal representation of GL_2 . The induced representation $I_{Q(Z)}(\chi, \tau)$ is reducible iff one of the following holds:*

(i) $\chi = 1$;

(ii) $\chi = \chi_0 | - |^{\pm 1}$ and χ_0 is a non-trivial quadratic character such that $\tau \otimes \chi_0 \cong \tau$.

In case (i), the representation $I_{Q(Z)}(1, \tau)$ is the direct sum of two irreducible representations, exactly one of which is generic. In case (ii), assuming without loss of generality that $\chi = \chi_0 \cdot | - |$, one has a (non-split) short exact sequence:

$$0 \longrightarrow St(\chi_0, \tau_0) \longrightarrow I_{Q(Z)}(\chi_0 \cdot | - |, \tau_0 \cdot | - |^{-1/2}) \longrightarrow Sp(\chi_0, \tau_0) \longrightarrow 0$$

where $St(\chi_0, \tau_0)$ is a (generic) discrete series representation and the Langlands quotient $Sp(\chi_0, \tau_0)$ is non-generic.

(b) *If τ is the twisted Steinberg representation of GL_2 , then $I_{Q(Z)}(\chi, \tau)$ is reducible iff one of the following holds:*

(i) $\chi = 1$;

(ii) $\chi = | - |^{\pm 2}$.

In case (i), $I_{Q(Z)}(1, st_\chi)$ is the sum of two irreducible representations, exactly one of which is generic. In case (ii), $I_{Q(Z)}(| - |^2, st_\chi \cdot | - |^{-1})$ has the twisted Steinberg representation $St_{PGSp_4} \otimes \chi$ as a unique irreducible submodule.

(c) *For general τ , there is a standard intertwining operator*

$$I_{Q(Z)}(\chi^{-1}, \tau \otimes \chi) \longrightarrow I_{Q(Z)}(\chi, \tau).$$

Now consider the group $GSO(V_D)$ where D is possibly split. We may identify $GSO(V_D)$ as a quotient of $GL_2(D) \times GL_1$ as in Section 2. We have:

$$V_D = F \cdot (1, 0) \oplus D \oplus F \cdot (0, 1)$$

and the stabilizer $P(J)$ of $J = F \cdot (1, 0)$ is the image of the parabolic $P_D \times GL_1 \in GL_2(D) \times GL_1$. A representation of its Levi subgroup is thus of the form $(\tau_1 \boxtimes \tau_2) \boxtimes \chi$ with $\omega_{\tau_1} \cdot \omega_{\tau_2} = \chi^2$, and we denote the associated induced representation of $GL_2(D) \times GL_1$ by $PS(\tau_1, \tau_2) \boxtimes \chi$. Now we note the following lemma (cf. [T]):

Lemma 8.2. *Let τ be a representation of D^\times (where D is possibly split) and let $JL(\tau)$ denote its Jacquet-Langlands lift to GL_2 .*

(i) *Suppose that $JL(\tau)$ is supercuspidal. Then one has a short exact sequence of representations of $GL_2(D)$:*

$$0 \longrightarrow St(\tau) \longrightarrow PS(\tau | - |^{1/2}, \tau | - |^{-1/2}) \longrightarrow Sp(\tau) \longrightarrow 0$$

where $St(\tau)$ is a discrete series representation (a generalized Steinberg representation) and $Sp(\tau)$ is the unique Langlands quotient (a generalized Speh representation).

(ii) Suppose that $JL(\tau)$ is the twisted Steinberg representation st_χ . Then the principal series $PS(\tau| - |, \tau| - |^{-1})$ has a unique irreducible submodule which is the twisted Steinberg representation $St_\chi := St_{GL_2(D)} \otimes \chi$.

We can now state the two main theorems of this section:

Theorem 8.3. Consider the case when D is non-split.

(i) **(Principal series)** The irreducible principal series representation $PS(\tau_{D,1}, \tau_{D,2}) \boxtimes \mu$ (with $\mu^2 = \omega_{D,1} \cdot \omega_{D,2}$) participates in the local theta correspondence with GSp_4 iff one of the following holds:

- (a) $\omega_{D,1} = \omega_{D,2} = \mu$;
- (b) $\tau_{D,1}^\vee \cong \tau_{D,2} \otimes \mu^{-1}$ and $\omega_{D,1} \neq \mu$.

Moreover, if (a) holds, then $\theta(PS(\tau_{D,1}, \tau_{D,2}) \boxtimes \mu)$ is the non-generic representation of $GSp_4(F)$ which is the theta lift of the (supercuspidal) representation $\tau_{D,1} \boxtimes \tau_{D,2}$ of $GSO(D)$. If $\tau_{D,1} \neq \tau_{D,2}$, then $\theta(PS(\tau_{D,1}, \tau_{D,2}) \boxtimes \mu)$ is supercuspidal. If $\tau_{D,1} = \tau_{D,2} = \tau_D$, then $\theta(PS(\tau_{D,1}, \tau_{D,2}) \boxtimes \mu)$ is the unique non-generic summand of the tempered representation $I_{Q(Z)}(1, JL(\tau_D))$, which is denoted by $\pi_{ng}(JL(\tau_D))$ in [GT].

If (b) holds, then

$$\theta(PS(\tau_{D,1}, \tau_{D,2}) \boxtimes \mu) = I_{Q(Z)}\left(\frac{\mu}{\omega_{D,1}}, JL(\tau_{D,2}) \cdot \frac{\omega_{D,1}}{\mu}\right) = I_{Q(Z)}\left(\frac{\omega_{D,1}}{\mu}, JL(\tau_{D,2})\right)$$

which is irreducible and generic.

(ii) **(Generalized Steinberg)** If $\dim \tau_D > 1$ and $\mu = \omega_{\tau_D} \cdot \chi$ with $\chi^2 = 1$, then $\theta(St(\tau_D) \boxtimes \mu) \neq 0$ iff χ is non-trivial and $\tau_D \otimes \chi = \tau_D$, in which case

$$\Theta(St(\tau_D) \boxtimes \mu) = \theta(St(\tau_D) \boxtimes \mu) = St(\chi, JL(\tau_D)),$$

which is generic.

(iii) **(Generalized Speh)** Similarly, with $\mu = \omega_{\tau_D} \cdot \chi$, $\theta(Sp(\tau_D) \boxtimes \mu)$ is nonzero iff $\tau_D \otimes \chi = \tau_D$. Suppose that this holds. Then if $\chi \neq 1$,

$$\Theta(Sp(\tau_D) \boxtimes \mu) = \theta(Sp(\tau_D) \boxtimes \mu) = Sp(\chi, JL(\tau_D)),$$

which is non-generic. If $\chi = 1$,

$$\Theta(Sp(\tau_D) \boxtimes \omega_D) = \theta(Sp(\tau_D) \boxtimes \omega_D) = I_{Q(Z)}(| - |, JL(\tau_D) \cdot | - |^{-1/2}),$$

which is generic.

(iv) **(Twisted Steinberg)** If St_χ is the twisted Steinberg representation of $GL_2(D)$, then $\theta(St_\chi \boxtimes \mu) \neq 0$ iff $\mu = \chi^2$, in which case

$$\theta(St_\chi \boxtimes \chi^2) = St_{PGSp_4} \otimes \chi.$$

Theorem 8.4. Consider the case when D is split.

(i) **(Principal series)** The irreducible principal series representation $PS(\tau_1, \tau_2) \boxtimes \mu$, with τ_i discrete series representations and $\mu^2 = \omega_1 \cdot \omega_2$, participates in the local theta correspondence with GSp_4 iff one of the following holds:

- (a) $\omega_1 = \omega_2 = \mu$;

(b) $\tau_1^\vee \cong \tau_2 \otimes \mu^{-1}$ and $\omega_1 \neq \mu$.

Moreover, if (a) holds, then $\theta(PS(\tau_1, \tau_2) \boxtimes \mu)$ is the generic representation of $GSp_4(F)$ which is the theta lift of the representation $\tau_1 \boxtimes \tau_2$ of $GSO(D)$. If $\tau_1 = \tau_2 = \tau$, then $\theta(PS(\tau_1, \tau_2) \boxtimes \mu)$ is the unique generic summand of the tempered representation $I_{Q(Z)}(1, \tau)$, which is denoted by $\pi_{gen}(\tau)$ in [GT].

If (b) holds, then

$$\theta(PS(\tau_1, \tau_2) \boxtimes \mu) = I_{Q(Z)}\left(\frac{\mu}{\omega_1}, \tau_2 \cdot \frac{\omega_1}{\mu}\right) = I_{Q(Z)}\left(\frac{\omega_1}{\mu}, \tau_2\right)$$

which is irreducible and generic.

(ii) **(Generalized Steinberg)** Suppose that τ is supercuspidal with central character ω_τ and $\mu = \omega_\tau \cdot \chi$ with $\chi^2 = 1$. Then $\theta(St(\tau) \boxtimes \mu) \neq 0$ iff χ is non-trivial and $\tau \otimes \chi = \tau$, in which case

$$\Theta(St(\tau) \boxtimes \mu) = \theta(St(\tau) \boxtimes \mu) = St(\chi, \tau),$$

which is generic.

(iii) **(Generalized Speh)** Similarly, $\theta(Sp(\tau) \boxtimes \mu)$ is nonzero iff $\tau \otimes \chi = \tau$. Suppose that this holds. Then if $\chi \neq 1$,

$$\Theta(Sp(\tau) \boxtimes \mu) = \theta(Sp(\tau) \boxtimes \mu) = Sp(\chi, \tau),$$

which is non-generic. If $\chi = 1$, then

$$\Theta(Sp(\tau) \boxtimes \omega_\tau) = \theta(Sp(\tau) \boxtimes \omega_\tau) = I_{Q(Z)}(|-|, \tau \cdot |-|^{-1/2}),$$

which is generic.

(iv) **(Twisted Steinberg)** If St_χ is the twisted Steinberg representation of GL_4 , then $\theta(St_\chi \boxtimes \mu) \neq 0$ iff $\mu = \chi^2$, in which case

$$\theta(St_\chi \boxtimes \chi^2) = St_{PGSp_4} \otimes \chi.$$

Before coming to the proofs of the Theorems, let us draw a number of consequences. Firstly, a comparison of Thm 8.3(ii, iv) and Thm. 8.4(ii, iv) gives the purely local proof of Prop. 7.10 for non-supercuspidal discrete series representations promised at the end of the previous section. Indeed, one has:

Corollary 8.5. (i) Let π be a discrete series representation of GL_4 and π_D its Jacquet-Langlands lift to $GL_2(D)$. If $\pi = St(\tau)$ with τ supercuspidal, then π (resp. π_D) has Shalika period with respect to μ iff $\mu = \omega_\tau \cdot \chi$ where χ is a non-trivial quadratic character such that $\tau \otimes \chi \cong \tau$. When this holds, the (big = small) theta lifts of π and π_D are isomorphic as representations of GSp_4 .

(ii) If $\pi = St_\chi$ is a twisted Steinberg representation, then π (resp. π_D) has Shalika period with respect to μ iff $\mu = \chi^2$, in which case the small theta lifts of π and π_D are isomorphic as representations of GSp_4 .

Secondly, Thm. 8.3(i) completes the proof of the last statement in Thm. 7.8, regarding non-genericity of the small theta lift.

Lastly, the theorems allow one to determine whether the generalized Speh representations possess Shalika periods with respect to μ . This answers a question raised by Prasad in [P2].

Theorem 8.6. (i) *The generalized Speh representation $Sp(\tau)$ has Shalika period with respect to μ if and only if $\mu = \omega_\tau$.*

(ii) *The generalized Speh representation $Sp(\tau_D)$ has Shalika period with respect to μ if and only if $\mu = \omega_{\tau_D}$.*

Proof. The proofs of (i) and (ii) are similar, so we shall only address (i). By Thm. 8.4(iii), we see that with $\mu = \omega_\tau \cdot \chi$, $\Theta(Sp(\tau) \boxtimes \mu)$ is generic if and only if $\chi = 1$. Thus, $Sp(\tau)$ has Shalika period with respect to μ if and only if $\chi = 1$. \square

Remark: The local analog of the proof of Proposition 6.3 gives an alternative proof that $Sp(\tau)$ has Shalika period with respect to ω_τ . The fact that $Sp(\tau)$ does not have Shalika period with respect to any other μ seems a more delicate matter.

We must now prove Thms. 8.3 and 8.4. Given the essential similarity in the statements of the two theorems, it is not surprising that one can execute their proofs concurrently. Thus, in the remainder of the section, D is a possibly split quaternion algebra.

The key step is the computation of the normalized Jacquet module of Ω_D with respect to $Q(Z)$ and $P(J)$. This is a by-now-standard computation, following the lines of [K], and we shall simply state the results below. For the computations, it is in fact better not to identify $GSO(V_D)$ with a quotient of $GL_2(D) \times GL_1$. Thus, we shall work directly with the parabolic $P(J) = M(J) \cdot N(J)$ with $M(J) = GL(J) \times GSO(D)$, and we represent an element of $M(J)$ by (a, α, β) with

$$(\alpha, \beta) \in GSO(D) \cong (D^\times \times D^\times) / \{(z, z^{-1}) : z \in F^\times\}.$$

For a character χ and a representation $\tau_1 \boxtimes \tau_2$ of $GSO(D)$, one may consider the normalized induced representation $I_{P(J)}(\chi, \tau_1 \boxtimes \tau_2)$.

The relation of the two descriptions of principal series representations of $GSO(V_D)$ is as follows. Suppose that under the natural map $P_D \times GL_1 \rightarrow P(J)$,

$$\left(\left(\begin{array}{c} \alpha \\ \beta \end{array} \right), z \right) \mapsto (a, \alpha', \beta'),$$

then we have:

$$\begin{cases} \alpha = a \cdot \overline{\alpha'}^{-1} \\ \beta = \beta' \\ z = a^{-1} \cdot \mathbb{N}(\alpha'). \end{cases}$$

From this, one deduces that

$$PS(\tau_1, \tau_2) \boxtimes \mu \cong I_{P(J)}(\omega_1 \mu^{-1}, \tau_1^\vee \mu \boxtimes \tau_2).$$

Now we have:

Proposition 8.7. *Let $R_{P(J)}(\Omega_D)$ denote the normalized Jacquet module of Ω_D along $P(J)$ (where D is possibly split). Then we have a short exact sequence of $M(J) \times GSp(W)$ -modules:*

$$0 \longrightarrow A \longrightarrow R_{P(A)}(\Omega_D) \longrightarrow B \longrightarrow 0.$$

Here,

$$B \cong \Omega_{W,D},$$

where $\Omega_{W,D}$ is the induced Weil representation for $GS\!p(W) \times GSO(D)$, and

$$A \cong I_{Q(Z)}(S(F^\times) \otimes \Omega_{W',D} \otimes |\lambda_{W'}|^{-1} \otimes |\lambda_D|^{-1}).$$

The action of $(GL(J) \times GSO(D)) \times (GL(Z) \times GS\!p(W'))$ on $S(F^\times)$ is given by:

$$((a, h), (b, g)) \cdot f(x) = f(b^{-1} \cdot x \cdot a \cdot \lambda_{W'}(g)),$$

and $\Omega_{W',D}$ denotes the induced Weil representation of $GS\!p(W') \times GSO(D)$.

Proposition 8.8. *Let $R_{Q(Z)}(\Omega_D)$ denote the normalized Jacquet module of Ω_D along $Q(Z)$ (where D is possibly split). Then we have a short exact sequence of $GSO(V_D) \times L(Z)$ -modules:*

$$0 \longrightarrow A' \longrightarrow R_{Q(Z)}(\Omega_D) \longrightarrow B' \longrightarrow 0.$$

Here,

$$B' \cong |\det_Z| \boxtimes \Omega_{W',V_D}$$

where Ω_{W',V_D} is the induced Weil representation of $GS\!p(W') \times GSO(V_D)$ and

$$A' \cong I_{P(J)}(S(F^\times) \otimes \Omega_{W',D} \otimes |\det_Z| \cdot |\det_J|^{-1} \cdot |\lambda_{W'}|^{-2} \cdot |\lambda_D|^{-1}).$$

The action of $(GL(J) \times GSO(D)) \times (GL(Z) \times GS\!p(W'))$ on $S(F^\times)$ is given by

$$((a, h), (b, g)) \cdot f(x) = f(a^{-1} \cdot \lambda_{W'}(g)^{-1} \cdot x \cdot b),$$

and $\Omega_{W',D}$ is the induced Weil representation of $GS\!p(W') \times GSO(D)$.

Applying Frobenius reciprocity and Props. 8.7 and 8.8, we obtain:

Proposition 8.9. (i) *Consider the space*

$$\mathrm{Hom}_{GSO(V_D)}(\Omega_D, I_{P(J)}(\chi, \tau_1 \boxtimes \tau_2))$$

as a representation of $GS\!p(W)$. Then we have:

(a) *If $\chi \neq 1$, then*

$$\mathrm{Hom}_{GSO(V_D)}(\Omega_D, I_{P(J)}(\chi, \tau_1 \boxtimes \tau_2)) = 0$$

unless

$$\tau_1 = \tau_2 = \tau,$$

in which case

$$\mathrm{Hom}_{GSO(V_D)}(\Omega_D, I_{P(J)}(\chi, \tau \boxtimes \tau)) = I_{Q(Z)}(\chi^{-1}, JL(\tau) \cdot \chi)^* \quad (\text{full linear dual}).$$

(b) *If $\chi = 1$ but $\tau_1 \neq \tau_2$, then*

$$\mathrm{Hom}_{GSO(V_D)}(\Omega_D, I_{P(J)}(\chi, \tau_1 \boxtimes \tau_2)) = \Theta_{W,D}(\tau_1 \boxtimes \tau_2)^*,$$

where $\Theta_{W,D}(\tau_1 \boxtimes \tau_2)$ denotes the big theta lift of $\tau_1 \boxtimes \tau_2$ from $GSO(D)$ to $GS\!p(W)$.

(c) *If $\chi = 1$ and $\tau_1 = \tau_2 = \tau$, then we have an exact sequence:*

$$0 \longrightarrow \Theta_{W,D}(\tau \boxtimes \tau)^* \longrightarrow \mathrm{Hom}_{GSO(V_D)}(\Omega_D, I_{P(J)}(\chi, \tau \boxtimes \tau)) \longrightarrow (I_{Q(Z)}(1, JL(\tau)))^*.$$

(ii) Assume that $\chi \neq | - |$. Then as a representation of $GSO(V_D)$,

$$\mathrm{Hom}_{GSp(W)}(\Omega_D, I_{Q(Z)}(\chi, \tau)) = I_{P(J)}(\chi^{-1}, (JL(\tau) \cdot \chi) \boxtimes (JL(\tau) \cdot \chi))^*.$$

Proof of Theorems 8.3 and 8.4. Now we can prove Thms. 8.3 and 8.4. In the following, we shall use the fact that if π is an irreducible representation of $GSO(V_D)$, then

$$\Theta(\pi)^* \cong \mathrm{Hom}_{GSO(V_D)}(\Omega_D, \pi).$$

We consider the different cases separately.

Principal Series

Suppose that

$$PS(\tau_1, \tau_2) \boxtimes \mu \cong I_{P(J)}(\omega_1 \mu^{-1}, \tau_1^\vee \mu \boxtimes \tau_2)$$

is an irreducible principal series representation with $JL(\tau_i)$ discrete series representations. If $\omega_1 \neq \mu$, then by Prop. 8.9(i)(a), we deduce that

$$\Theta(PS(\tau_1, \tau_2) \boxtimes \mu) = 0$$

unless $\tau_1^\vee \cdot \mu = \tau_2$, in which case

$$\Theta(PS(\tau_1, \tau_2) \boxtimes \mu) = I_{Q(Z)}\left(\frac{\mu}{\omega_1}, JL(\tau_2) \cdot \frac{\omega_1}{\mu}\right).$$

Since the latter is irreducible also, it is isomorphic to $I_{Q(Z)}(\omega_1 \mu^{-1}, JL(\tau_2))$.

On the other hand, suppose that $\omega_1 = \mu$ but $\tau_1 = \tau_1^\vee \cdot \mu \neq \tau_2$. Then Prop. 8.9(i)(b) shows that

$$\Theta(PS(\tau_1, \tau_2) \boxtimes \mu) = \Theta_{W,D}(\tau_1 \boxtimes \tau_2).$$

Finally, suppose that $\omega_1 = \mu$ and $\tau_1 = \tau_2 = \tau$. By Prop. 8.9(i)(c), we obtain

$$I_{Q(Z)}(1, JL(\tau)) \longrightarrow \Theta(PS(\tau, \tau) \boxtimes \mu) \longrightarrow \Theta_{W,D}(\tau \boxtimes \tau) \longrightarrow 0.$$

This shows that

$$\theta(PS(\tau, \tau) \boxtimes \mu) \supset \theta_{W,D}(\tau \boxtimes \tau).$$

We now have to examine if the two constituents of $I_{Q(Z)}(1, JL(\tau))$ contribute to $\Theta(PS(\tau, \tau) \boxtimes \mu)$ or even $\theta(PS(\tau, \tau) \boxtimes \mu)$.

We shall suppose that D is non-split; the split case is similar and so we omit the details. Then $\theta_{W,D}(\tau \boxtimes \tau)$ is the unique non-generic summand of $I_{Q(Z)}(1, JL(\tau))$. Now by Prop. 7.1(ii), one knows that $PS(\tau \boxtimes \tau)$ has Shalika period with respect to $\mu = \omega_1$. Thus, $\Theta(PS(\tau, \tau) \boxtimes \mu)$ is generic, so that the generic summand of $I_{Q(Z)}(1, JL(\tau))$ does occur as a submodule of $\Theta(PS(\tau, \tau) \boxtimes \mu)$. However, it does not occur as a quotient of $\Theta(PS(\tau, \tau) \boxtimes \mu)$. This follows by [KR, Thm. 3.8]: since the generic summand of $I_{Q(Z)}(1, JL(\tau))$ has nonzero theta lift to $GSO(2, 2)$, it cannot participate in the theta correspondence with $GSO(V_D)$. Thus, we now know:

$$\theta(PS(\tau, \tau) \boxtimes \mu) = \theta_{W,D}(\tau \boxtimes \tau) \quad \text{or} \quad 2 \cdot \theta_{W,D}(\tau \boxtimes \tau).$$

To show that the latter is not possible, we use Prop. 8.9(ii) to see that

$$\mathrm{Hom}_{GSp(W)}(\Omega_D, I_{Q(Z)}(1, JL(\tau)) = I_{P(J)}(1, \tau \boxtimes \tau)^* = (PS(\tau, \tau) \boxtimes \omega_\tau)^*$$

so that

$$PS(\tau, \tau) \boxtimes \omega_\tau \twoheadrightarrow \Theta(\theta_{W,D}(\tau \boxtimes \tau)).$$

This shows that

$$\begin{cases} \theta(PS(\tau, \tau) \boxtimes \mu) = \theta_{W,D}(\tau \boxtimes \tau); \\ \Theta(\theta_{W,D}(\tau \boxtimes \tau)) = PS(\tau, \tau) \boxtimes \mu. \end{cases}$$

This completes the proof of Thms. 8.3(i) and 8.4(i).

Generalized Steinberg and Speh

Now we consider the theta lift of the generalized Steinberg representation $St(\tau) \otimes \mu$ and the generalized Speh representation $Sp(\tau) \boxtimes \mu$, where

$$\mu = \omega_\tau \cdot \chi \quad \text{with } \chi^2 = 1.$$

Since

$$St(\tau) \boxtimes \mu \hookrightarrow I_{P(J)}(\chi| - |, (\tau \cdot \chi| - |^{-1/2}) \boxtimes (\tau| - |^{-1/2})),$$

we deduce by Prop. 8.9(i)(a) that

$$\Theta(St(\tau) \boxtimes \mu)^* \hookrightarrow \text{Hom}_{GSO(V_D)}(\Omega_D, I_{P(J)}(\chi| - |, (\tau \cdot \chi| - |^{-1/2}) \boxtimes (\tau| - |^{-1/2})),$$

which vanishes unless $\tau \otimes \chi \cong \tau$, in which case one has:

$$I_{Q(Z)}(\chi| - |^{-1}, JL(\tau) \cdot | - |^{1/2}) \twoheadrightarrow \Theta(St(\tau) \boxtimes \mu).$$

Recall that the above induced representation is irreducible if $\chi = 1$ and has $St(\chi, JL(\tau))$ as unique irreducible quotient if $\chi \neq 1$. From this, we conclude that if $\chi \neq 1$, then one has:

- $\theta(St(\tau) \boxtimes \mu) \subset St(\chi, JL(\tau))$;
- $\theta(St(\chi, JL(\tau))) \neq 0$,

whereas if $\chi = 1$, one has

- $\theta(St(\tau) \boxtimes \mu) \subset I_{Q(Z)}(| - |, JL(\tau) \cdot | - |^{-1/2})$
- $\theta(I_{Q(Z)}(| - |, JL(\tau) \cdot | - |^{-1/2})) \neq 0$.

On the other hand, if $\chi \neq 1$, one may apply Prop. 8.9(ii) to $I_{Q(Z)}(\chi| - |, JL(\tau) \cdot | - |^{-1/2})$ and arguing as above, one obtains:

- $\theta(St(\chi, JL(\tau))) \subset St(\tau) \boxtimes \mu$;
- $\theta(St(\tau) \boxtimes \mu) \neq 0$.

Hence, we have shown that when $\chi \neq 1$,

$$\begin{cases} \theta(St(\tau) \boxtimes \mu) = St(\chi, JL(\tau)); \\ \theta(St(\chi, JL(\tau))) = St(\tau) \boxtimes \mu. \end{cases}$$

Similarly, by applying Prop. 8.9(i)(a) to $I_{P(J)}(\chi| - |^{-1}, (\tau \cdot \chi| - |^{-1/2}) \boxtimes (\tau| - |^{1/2}))$ and Prop. 8.9(ii) to $I_{Q(Z)}(\chi| - |^{-1}, JL(\tau) \cdot | - |^{1/2})$, one deduces that

$$\Theta(Sp(\tau) \boxtimes \mu) = 0$$

unless $\tau \otimes \chi = \tau$, in which case one has, if $\chi \neq 1$,

$$\begin{cases} \theta(Sp(\tau) \boxtimes \mu) = Sp(\chi, JL(\tau)); \\ \theta(Sp(\chi, JL(\tau))) = Sp(\tau) \boxtimes \mu, \end{cases}$$

whereas if $\chi = 1$,

$$\begin{cases} \theta(Sp(\tau) \boxtimes \mu) = I_{Q(Z)}(|-|, JL(\tau) \cdot |-|^{-1/2}); \\ \theta(I_{Q(Z)}(|-|, JL(\tau) \cdot |-|^{-1/2})) = Sp(\tau) \boxtimes \mu. \end{cases}$$

This then implies that (for $\chi = 1$)

$$\Theta(St(\tau) \boxtimes \omega_\tau) = 0.$$

We have more or less completed the proof of Thm. 8.3(ii) and (iii), as well as Thm. 8.4(ii) and (iii), except that we still need to check that the small theta lifts and the big theta lifts are equal.

For that, we argue by contradiction. Suppose for example that (when $\chi \neq 1$)

$$\Theta(St(\tau) \boxtimes \mu) \neq \theta(St(\tau) \boxtimes \mu).$$

Then we must have

$$\Theta(St(\tau) \boxtimes \mu) = I_{Q(Z)}(\chi|-|^{-1}, JL(\tau) \cdot |-|^{1/2}).$$

This means that

$$(St(\tau) \boxtimes \mu)^* \hookrightarrow \text{Hom}_{GSp(W)}(\Omega_D, I_{Q(Z)}(\chi|-|^{-1}, JL(\tau) \cdot |-|^{1/2})).$$

But by Prop. 8.9(ii),

$$\text{Hom}_{GSp(W)}(\Omega_D, I_{Q(Z)}(\chi|-|^{-1}, JL(\tau) \cdot |-|^{1/2})) = I_{P(J)}(\chi|-|, \tau \cdot |-|^{-1/2})^*.$$

Thus, we would conclude that

$$I_{P(J)}(\chi|-|, \tau \cdot |-|^{-1/2}) \twoheadrightarrow St(\tau) \boxtimes \mu,$$

which is a contradiction. The other cases are treated similarly; we omit the details. This completes the proof of the parts of Thms. 8.3 and 8.4 pertaining to the generalized Steinberg and generalized Sp $_{2n}$ representations.

Twisted Steinberg Representations

Now consider the twisted Steinberg representation St_χ with $\chi^4 = \mu^2$. We assume that D is non-split, since the case for split D is similar. Since

$$St_\chi \boxtimes \mu \hookrightarrow I_{P(J)}\left(\frac{\chi^2}{\mu}, \frac{\mu}{\chi}|-|^{-1} \boxtimes \chi|-|^{-1}\right),$$

and

$$St_{PGSp_4} \otimes \chi \hookrightarrow I_{Q(Z)}(|-|^2, JL(\chi)|-|^{-1}),$$

we may apply Prop. 8.9(i) and (ii) to conclude that a necessary condition for the non-vanishing of theta lifts is $\mu = \chi^2$, in which case a similar argument as the above cases shows that

$$\theta(St_\chi \boxtimes \chi^2) = St_{PGSp_4} \otimes \chi$$

and

$$\theta(St_{PGSp_4} \otimes \chi) = St_\chi \boxtimes \chi^2.$$

This completes the proof of Thms. 8.3(iv) and 8.4(iv). \square

Explicit Theta Correspondence for $GL_2 \times GSO(V_D)$

We conclude this section by describing the local theta correspondence for $GL_2 \times GSO(V_D)$, where D is possibly split. This was needed at certain places in Section 6. Hence, let $\Omega_{W', D}$ be the induced Weil representation for this dual pair which can be realized on $S(V_D)$. As in Section 6, we have $W' = F \cdot e \oplus F \cdot f$ and we let $B = T \cdot U$ be the Borel subgroup stabilizing $F \cdot e$. Then we have

Proposition 8.10. *Let $R_B(\Omega_{W',D})$ be the normalized Jacquet module of $\Omega_{W',D}$ with respect to the unipotent radical U of B . There is a short exact sequence of representations of $T \times GSO(V_D)$:*

$$0 \longrightarrow I_{P(J)}(S(F^\times \times F^\times)) \longrightarrow R_B(\Omega_{W',D}) \longrightarrow S(F^\times) \longrightarrow 0.$$

Here, the actions of $t(a,b) \in T$ and $h \in GSO(V_D)$ on $S(F^\times)$ are given by

$$\begin{cases} (t(a,b) \cdot \phi)(t) = |a|^{-1/2} \cdot |b|^{-5/2} \cdot \phi(tab) \\ (h \cdot \phi)(t) = |\lambda(h)|^{-3/2} \cdot \phi(t\lambda(h)). \end{cases}$$

On the other hand, the action of $T \times M(J) = T \times GL(J) \times GSO(D)$ on $S(F^\times \times F^\times)$ is given as follows. For $t(a,b) \in T$,

$$(t(a,b) \cdot \phi)(t,x) = |a|^{-1/2} \cdot |b|^{-5/2} \cdot \phi(tab, b^{-1}x).$$

For $\alpha \in GL(J)$,

$$(\alpha \cdot \phi)(t,x) = |\alpha|^{-2} \cdot \phi(t, \alpha^{-1}x),$$

and for $h \in GSO(D)$,

$$(h \cdot \phi)(t,x) = |\lambda_D(h)|^{-1/2} \cdot \phi(t\lambda_D(h), x).$$

Using this proposition, we deduce (at least for parts (i) and (ii)):

Theorem 8.11. *Let τ be an irreducible infinite dimensional unitary (up to twisting) representation of GL_2 .*

(i) *If $\tau = \pi(\chi_1, \chi_2)$, then*

$$\Theta_D(\tau) = PS(\chi_1 \circ \det, \chi_2 \circ \det) \boxtimes (\chi_1 \chi_2)$$

which is irreducible.

(ii) *If $\tau = st_\chi$ is a twisted Steinberg representation, then when D is non-split,*

$$\Theta_D(\tau) = PS(JL(\tau)| - |^{1/2}, JL(\tau)| - |^{-1/2}) \boxtimes \omega_\tau$$

which is irreducible. On the other hand, when D is split, then $\theta_D(\tau)$ is the unique irreducible quotient of $PS(\tau| - |^{1/2}, \tau| - |^{-1/2})$.

(iii) *If τ is supercuspidal, then $\Theta_D(\tau) = Sp(JL(\tau)) \boxtimes \omega_\tau$.*

In the interest of space and time, we leave the details of the proof to the reader.

TABLE 1. Explicit theta lifts from $GSO(V_D)$ to $GS p_4$

$\pi \boxtimes \mu \in \text{Irr}(GSO(V_D))$			$\theta(\pi \boxtimes \mu) \in \text{Irr}(GS p_4)$		
$PS(\tau_{D,1}, \tau_{D,2}) \boxtimes \mu$	a	$\omega_{D,1} = \omega_{D,2} = \mu$	$\tau_{D,1} \neq \tau_{D,2}$	$\theta(\tau_{D,1} \boxtimes \tau_{D,2})$	non-generic S.C.
	b		$\tau_{D,1} = \tau_{D,2} = \tau_D$		$\pi_{ng}(JL(\tau_D))$
	c	$\tau_{D,1}^\vee \cong \tau_{D,2} \otimes \mu^{-1}, \omega_{D,1} \neq \mu$		$I_{Q(Z)}(\omega_{D,1} \cdot \mu^{-1}, JL(\tau_{D,2}))$	
	d	otherwise		0	
$St(\tau_D) \boxtimes \mu, \dim \tau_D > 1$ $\mu = \omega_D \cdot \chi, \chi^2 = 1$	a	$\chi \neq 1, \tau_D \otimes \chi = \tau_D$		$St(\chi, JL(\tau_D))$	
	b	otherwise		0	
$Sp(\tau_D) \boxtimes \mu, \dim \tau_D > 1$ $\mu = \omega_D \cdot \chi, \chi^2 = 1$	a	$\tau_D \otimes \chi = \tau_D$	$\chi \neq 1$	$Sp(\chi, JL(\tau_D))$	
	b		$\chi = 1$	$I_{Q(Z)}(- , JL(\tau_D) \cdot - ^{-1/2})$	
	c	otherwise		0	
$St_\chi \boxtimes \mu$	a	$\mu = \chi^2$		$St_{PGSp_4 \otimes \chi}$	
	b	otherwise		0	

TABLE 2. Explicit theta lifts from $GSO(V)$ to $GS p_4$

$\pi \boxtimes \mu \in \text{Irr}(GSO(V))$			$\theta(\pi \boxtimes \mu) \in \text{Irr}(GS p_4)$		
$PS(\tau_1, \tau_2) \boxtimes \mu$ τ_1, τ_2 discrete series	a	$\omega_1 = \omega_2 = \mu$	$\tau_1 \neq \tau_2$	$\theta(\tau_1 \boxtimes \tau_2)$	generic
	b		$\tau_1 = \tau_2 = \tau$		$\pi_{gen}(\tau)$
	c	$\tau_1^\vee \cong \tau_2 \otimes \mu^{-1}, \omega_1 \neq \mu$		$I_{Q(Z)}(\omega_1 \cdot \mu^{-1}, \tau_2)$	
	d	otherwise		0	
$St(\tau) \boxtimes \mu, \tau$ supercuspidal $\mu = \omega \cdot \chi, \chi^2 = 1$	a	$\chi \neq 1, \tau \otimes \chi = \tau$		$St(\chi, \tau)$	
	b	otherwise		0	
$Sp(\tau) \boxtimes \mu, \tau$ supercuspidal $\mu = \omega_D \cdot \chi, \chi^2 = 1$	a	$\tau \otimes \chi = \tau$	$\chi \neq 1$	$Sp(\chi, \tau)$	
	b		$\chi = 1$	$I_{Q(Z)}(- , \tau \cdot - ^{-1/2})$	
	c	otherwise		0	
$St_\chi \boxtimes \mu$	a	$\mu = \chi^2$		$St_{PGSp_4 \otimes \chi}$	
	b	otherwise		0	

TABLE 3. Explicit theta lifts from GL_2 to $GSO(V_D)$

$\tau \in \text{Irr}(GL_2)$		$\theta(\tau) \in \text{Irr}(GSO(V_D))$
$\pi(\chi_1, \chi_2)$		$PS(\chi_1 \circ \det, \chi_2 \circ \det) \boxtimes (\chi_1 \chi_2)$
st_χ	D non-split	$PS(JL(\tau) - ^{1/2}, JL(\tau) - ^{-1/2}) \boxtimes \omega_\tau$
	D split	unique quotient of $PS(\tau - ^{1/2}, \tau - ^{-1/2}) \boxtimes \omega_\tau$
Supercuspidal		$Sp(JL(\tau)) \boxtimes \omega_\tau$

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