REPRESENTATIONS OF METAPLECTIC GROUPS I: EPSILON DICHOTOMY AND LOCAL LANGLANDS CORRESPONDENCE

WEE TECK GAN AND GORDAN SAVIN

to Benedict H. Gross
our teacher, colleague and friend
on the occasion of his 60th birthday

Abstract. Using theta correspondence, we classify the irreducible representations of Mp_{2n} in terms of the irreducible representations of SO_{2n+1} and determine many properties of this classification. This is a local Shimura correspondence which extends the well-known results of Waldspurger for $n = 1$.

1. Introduction

Let k be a non-archimedean local field of characteristic zero and residual characteristic p. In this introduction, we assume for simplicity that p is odd. Let $(W, \langle -, - \rangle)$ be a symplectic vector space of dimension $2n$ over k, with associated symplectic group $\mathrm{Sp}(W)$. The group $\mathrm{Sp}(W)$ has a unique two-fold central extension $\mathrm{Mp}(W)$ which is called the metaplectic group:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1.$$

The purpose of this paper is to investigate the (genuine) representation theory of $\mathrm{Mp}(W)$. More precisely, we shall:

- obtain a local Langlands correspondence for $\mathrm{Mp}(W)$ and establish some of its expected properties;
- establish a result known as epsilon dichotomy, in which certain local root numbers are shown to control the non-vanishing of certain theta lifts;

The prototype of our results is the work of Waldspurger who considered the case $\dim W = 2$. If $\mathrm{Irr}(G)$ denotes the set of isomorphism clases of irreducible (genuine) representations of G, then Waldspurger showed that, with respect to any fixed additive character ψ of k, there is a natural bijection

$$\mathrm{Irr}(\mathrm{Mp}(W)) \longleftrightarrow \bigsqcup_V \mathrm{Irr}(\mathrm{SO}(V))$$

where the (disjoint) union of the RHS runs over the 3-dimensional quadratic spaces V of discriminant 1 (there are two of these) and $\mathrm{SO}(V)$ denotes the associated special orthogonal group. By combining these results with the local Langlands corespondence for $\mathrm{SO}(V)$ (with

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$\dim V = 3$), one obtains a classification of $\mathrm{Irr}(\mathrm{Mp}(W))$ in terms of L-parameters. This classification depends on the choice of ψ, but Waldspurger also determined how it changes as one varies ψ. We shall recall these results more precisely in §5. At this point, it suffices to note that Waldspurger's results were obtained by a detailed study of the local theta correspondence associated to the dual pairs $\mathrm{Mp}(W) \times \mathrm{SO}(V)$. Moreover, he was the first to realize the connection between local root numbers and theta correspondence, establishing the aforementioned result on epsilon dichotmy.

Subsequently, extensions of essentially all of Waldspurger's results mentioned above to the case of general W's were obtained in the archimedean case, by the work of Adams-Barbasch [AB1,2] and Adams [Ad]. In this paper, we shall complete this (local) story by establishing the analogous results for p-adic fields. We note that the method of proof used in the archimedean case relies crucially on the detailed analysis of harmonic K-types, and thus does not work in the p-adic setting.

More precisely, one has the following theorem, whose proof was sketched in [GGP] based on a key result of Kudla-Rallis [KR2]. We shall give a detailed proof here.

Theorem 1.1. *For each non-trivial additive character $\psi : k \to \mathbb{C}^\times$, there is a bijection*

$$\Theta_\psi : \mathit{Irr}(\mathrm{Mp}(W)) \longleftrightarrow \mathit{Irr}(\mathrm{SO}(V^+)) \sqcup \mathit{Irr}(\mathrm{SO}(V^-)),$$

where V^+ (respectively V^-) is the split (resp. non-split) quadratic space of discriminant 1 and dimension $2n+1$. This bijection is given by the theta correspondence (with respect to ψ) for the group $\mathrm{Mp}(W) \times \mathrm{SO}(V^\pm)$.

Corollary 1.2. *Assume the local Langlands correspondence for $\mathrm{SO}(V^\pm)$. Then one obtains a local Langlands correspondence for $\mathrm{Mp}(W)$, i.e. a bijection (depending on ψ)*

$$\mathcal{L}_\psi : \mathit{Irr}(\mathrm{Mp}(W)) \longleftrightarrow \Phi(\mathrm{Mp}(W))$$

where $\Phi(\mathrm{Mp}(W))$ is the set of pairs (ϕ, η) such that

- *$\phi : WD_k \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})$ is a $2n$-dimensional symplectic representation of the Weil-Deligne group WD_k of k;*
- *η is an irreducible representation of the (finite) component group*

$$A_\phi = \pi_0(Z_{\mathrm{Sp}_{2n}(\mathbb{C})}(\phi)).$$

Since the local Langlands correspondence for $\mathrm{SO}(V^\pm)$ is known for $\dim V = 5$ (by [GT] and [GTW]), the statement of the corollary is unconditional in this case. The general case should follow by combining the results of the recently released book [A] of Arthur and the results of Jiang-Soudry [JS].

One may ask if the local Langlands correspondence given in Corollary 1.2 satisfies certain typical properties. For example, for a representation σ of $\mathrm{Mp}(W)$ with L-parameter ϕ, one would expect that σ is a discrete series representation if and only if ϕ does not factor through any proper Levi subgroup. As another example, one would expect certain natural invariants, such as L-factors and ϵ-factors, to be preserved under the correspondence. To a large extent, such questions amount to whether the bijection Θ_ψ satisfies the analogous properties. We have:

Theorem 1.3. *Suppose that $\pi \in Irr(\mathrm{SO}(V))$ and $\sigma \in Irr(\mathrm{Mp}(W))$ correspond under Θ_ψ. Then we have:*

(i) π is a discrete series representation if and only if σ is a discrete series representation.

(ii) π is tempered if and only if σ is tempered. Moreover, suppose that

$$\pi \subset I_Q(\tau_1, ..., \tau_r, \pi_0),$$

where Q is a parabolic subgroup of $\mathrm{SO}(V)$ with Levi subgroup $\mathrm{GL}_{n_1} \times ... \times \mathrm{GL}_{n_r} \times \mathrm{SO}(V_0)$, the τ_i's are unitary discrete series representations of GL_{n_i}, and π_0 is a discrete series representation of $\mathrm{SO}(V_0)$. Then

$$\sigma \subset I_{\tilde{P}}(\tau_1, ..., \tau_r, \Theta_\psi(\pi_0)),$$

where $\tilde{P}$ is the parabolic subgroup of $\mathrm{Mp}(W)$ with Levi subgroup $\tilde{\mathrm{GL}}_{n_1} \times_{\mu_2} ... \times_{\mu_2} \tilde{\mathrm{GL}}_{n_r} \times \mathrm{Mp}(W_0)$. In particular, Θ_ψ gives a bijection between the (isomorphism classes of) irreducible constituents of $I_Q(\tau_1, ..., \tau_r, \pi_0)$ and $I_{\tilde{P}}(\tau_1, ..., \tau_r, \Theta_\psi(\pi_0))$.

(iii) In general, suppose that

$$\pi = J_Q(\tau_1 |\det|^{s_1}, ..., \tau_r |\det|^{s_r}, \pi_0), \quad s_1 > s_2 > .... > s_r > 0$$

is a Langlands quotient of $\mathrm{SO}(V)$, where Q is as in (ii), the τ_i's are unitary tempered representations of GL_{n_i}, and π_0 is a tempered representation of $\mathrm{SO}(V_0)$. Then

$$\sigma = J_{\tilde{P}}(\tau_1 |\det|^{s_1}, ..., \tau_r |\det|^{s_r}, \Theta_\psi(\pi_0))$$

where $\tilde{P}$ is as in (ii).

(iv) If π and σ are discrete series representations, then

$$deg(\pi) = deg(\sigma),$$

where deg denotes the formal degree with respect to the Haar measures giving

- *the Iwahori subgroup of $\mathrm{SO}(V^+)$ volume 1,*
- *the Iwahori subgroup of $\mathrm{SO}(V^-)$ volume $2 \cdot \frac{q+1}{q-1}$ (with $q =$ the size of the residue field of k),*
- *the preimage in $\mathrm{Mp}(W)$ of the Iwahori subgroup of $\mathrm{Sp}(W)$ volume 1.*

(v) If π is a generic representation of $\mathrm{SO}(V^+)$, then σ is a ψ-generic representation of $\mathrm{Mp}(W)$. If σ is ψ-generic and tempered, then π is generic.

(vi) If π is an irreducible representation of $\mathrm{SO}(V)$ and ρ is an irreducible representation of GL_r, then one has a Plancherel measure $\mu(s, \pi \times \rho, \psi)$ associated to the induced representation $I_P(s, \pi \boxtimes \rho)$. If $\sigma = \Theta_\psi(\pi)$, then one has

$$\mu(s, \pi \times \rho, \psi) = \mu(s, \sigma \times \rho, \psi).$$

(vii) If χ is a 1-dimensional character of GL_1*, then one has*

$$\begin{cases} L(s,\pi\times\chi) = L_\psi(s,\sigma\times\chi) \\ \epsilon(s,\pi\times\chi,\psi) = \epsilon(s,\sigma\times\chi,\psi) \end{cases}$$

where the local factors in question are those defined by Lapid-Rallis [LR] using the doubling method of Piatetski-Shapiro and Rallis [PSR].

(viii) Assume that π is generic, so that σ is ψ-generic. Then for any irreducible representation ρ of GL_r*, one has the equalities*

$$\begin{cases} L(s,\pi\times\rho) = L_\psi(s,\sigma\times\rho) \\ \epsilon(s,\pi\times\rho,\psi) = \epsilon(s,\sigma\times\rho,\psi). \end{cases}$$

Here the factors on the LHS are those defined by Shahidi [Sh], and those on the RHS are defined by Szpruch [Sz].

In the paper [GI] of the first author with A. Ichino, several of the results in Theorem 1.3 were established for the local theta correspondences for general dual pairs of arbitrary sizes. In fact, Theorem 1.3(iv) is one of the main results of [GI]; we do not make use of (iv) in this paper, nor will we discuss its proof.

It is not difficult to see that the bijection Θ_ψ (or L_ψ) is determined by the properties of the above theorem, at least on the level of L-packets. It has also come to our attention that Moeglin [Mo2] has given a definition of local L-packets (indeed local A-packets) of $\mathrm{Mp}(W)$ using reducibilities of generalized principal series representations, extending her approach for the linear classical groups. It follows from Theorem 1.3(i), (ii), (iii) and (vi) that our local L-packets agree with hers. Since we will not recall her intricate results, we do not elaborate on this point here.

Let us return to Theorem 1.1. The key steps in the proof of Theorem 1.1 are the following two statements:

(a) given an irreducible representation π of $\mathrm{SO}(V)$, exactly one extension of π to $\mathrm{O}(V) = \mathrm{SO}(V)\times\{\pm1\}$ has nonzero theta lift to $\mathrm{Mp}(W)$;

(b) given an irreducible representation σ of $\mathrm{Mp}(W)$, σ has nonzero theta lift to $\mathrm{O}(V)$ for exactly one V.

Now one may ask if it is possible to specify, in the context of (a), which extension $\pi^\pm$ of π participates in the theta correspondence with $\mathrm{Mp}(W)$. Analogously, given a representation σ in the context of (b), one may ask to which $\mathrm{O}(V)$ is the theta lift of σ nonzero. To describe the answers, we need to introduce some more notations.

First, let us write

$$\epsilon(V) = \begin{cases} +1, & \text{if } V = V^+; \\ -1, & \text{if } V = V^-. \end{cases}$$

Further, observe that the sign ϵ in π^ϵ simply encodes the central character of π^ϵ:

$$\epsilon = \pi^\epsilon(-1).$$

On the other hand, for an irreducible genuine representation σ of $\mathrm{Mp}(W)$, one may consider its central character ω_σ , which is a genuine character of $\tilde{Z}$ (the preimage in $\mathrm{Mp}(W)$ of the center Z of $\mathrm{Sp}(W)$). Now using the additive character ψ, one can define a genuine character χ_ψ of $\tilde{Z}$ (see 2.4). We define the central sign $z_\psi(\sigma)$ of σ by

$$z_\psi(\sigma) = \omega_\sigma(-1)/\chi_\psi(-1) \in \{\pm 1\},$$

where we note that the quotient above is independent of the choice of the preimage in $\tilde{Z}$ of $-1 \in Z$.

Now we have:

Theorem 1.4. *(i) Let π be an irreducible representation of $\mathrm{SO}(V)$. Then π^ϵ participates in theta correspondence (with respect to ψ) with $\mathrm{Mp}(W)$ if and only if*

$$\epsilon = \epsilon(V) \cdot \epsilon(1/2, \pi).$$

Here $\epsilon(s, \pi, \psi)$ is the standard epsilon factor defined by Lapid-Rallis [LR] using the doubling method; its value at $s = 1/2$ is independent of ψ.

(ii) Let σ be an irreducible representation of $\mathrm{Mp}(W)$. Then σ has nonzero theta lift (with respect to ψ) to $\mathrm{O}(V)$ if and only if the central character of σ satisfies:

$$z_\psi(\sigma) = \epsilon(V) \cdot \epsilon(1/2, \sigma, \psi) = \epsilon(V) \cdot \epsilon(1/2, \Theta_\psi(\sigma)).$$

We should mention that the analogous theorem in the context of theta correspondence for unitary groups was shown by Harris-Kudla-Sweet [HKS], at least for "most" representations.

Finally, we investigate how the local Langlands correspondence $\mathcal{L}_\psi$ depends on ψ. For this, we shall of course assume the local Langlands correspondence for $\mathrm{SO}(V^\pm)$ so that Corollary 1.2 makes sense. In addition, we assume that the local Langlands correspondence for $\mathrm{SO}(V^\pm)$ satisfies certain expected properties in relation with the theory of endoscopy; these are detailed in §12. To state the result, we recall that $\phi : WD_k \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})$ is a symplectic representation of WD_k, and if we write $\phi = \bigoplus_i n_i \cdot \phi_i$ as a direct sum of irreducible representations ϕ_i with some multiplicities n_i, then the component group A_ϕ is given by

$$A_\phi = \prod_{i:\phi_i \text{ symplectic}} \mathbb{Z}/2\mathbb{Z} a_i,$$

so that A_ϕ is a vector space over $\mathbb{Z}/2\mathbb{Z}$ with a canonical basis. Now we have:

Theorem 1.5. *For $\sigma \in Irr(\mathrm{Mp}(W))$ and $c \in k^\times$, let*

$$\mathcal{L}_\psi(\sigma) = (\phi, \eta) \text{ and } \mathcal{L}_{\psi_c}(\sigma) = (\phi_c, \eta_c).$$

Then:

(i) $\phi_c = \phi \otimes \chi_c$, where χ_c is the quadratic character associated to $c \in k^\times/k^{\times 2}$.

It follows by (i) that we have canonical identification of component groups:

$$A_\phi = A_{\phi_c} = \oplus_i \mathbb{Z}/2\mathbb{Z} a_i,$$

so that it makes sense to compare η and η_c.

(ii) the characters η and η_c are related by:

$$\eta_c(a_i)/\eta(a_i) = \epsilon(1/2, \phi_i) \cdot \epsilon(1/2, \phi_i \otimes \chi_c) \cdot \chi_c(-1)^{\frac{1}{2}\dim \phi_i} \in \{\pm\}.$$

It is interesting to note that the proof of this last theorem makes use of the Gross-Prasad conjecture for tempered representations of special orthogonal groups, which is recently demonstrated by Waldspurger in a remarkable series of articles [W5-8].

In a sequel to this paper, we shall investigate the relation of the representation theories of $\mathrm{SO}(V^{\pm})$ and $\mathrm{Mp}(W)$ from the point of view of Hecke algebra isomorphisms.

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2. **Metaplectic and Orthogonal Groups**

In this section, we establish some notations for the groups of interest in this paper. Recall that k is a non-archimedean local field of characteristic zero and residual characteristic p. Let $\mathcal{O}_k$ be the ring of integers of k with residue field $\kappa = \mathbb{F}_q$.

2.1. **Symplectic Group.** Let W be a $2n$-dimensional vector space over k equipped with a nondegenerate skew-symmetric form $\langle -, - \rangle_W$ and let $\mathrm{Sp}(W)$ be the associated symplectic group. We may fix a Witt basis of W, consisting of vectors

$$e_1, ..., e_n, e_n^*, ..., e_1^*$$

satisfying

$$\langle e_i, e_j \rangle_W = \langle e_i^*, e_j^* \rangle_W = 0 \quad \text{and} \quad \langle e_i, e_j^* \rangle_W = \delta_{ij}.$$

For any $1 \leq k \leq n$, let

$$X_k = \mathrm{Span}(e_1, ..., e_k) \quad \text{and} \quad X_k^* = \mathrm{Span}(e_1^*, ..., e_k^*),$$

so that $W = X_n \oplus X_n^*$. We also set

$$W_{n-k} = \mathrm{Span}(e_{k+1}, ..., e_n, e_n^*, ..., e_{k+1}^*)$$

so that

$$W = X_k \oplus W_{n-k} \oplus X_k^*.$$

2.2. **Parabolic Subgroups.** We now describe the parabolic subgroups of $\mathrm{Sp}(W)$ up to conjugacy. Consider the flag of isotropic subspaces

$$X_{k_1} \subset X_{k_1+k_2} \subset ... \subset X_{k_1+...+k_r} \subset W.$$

The stabilizer of such a flag is a parabolic subgroup P whose Levi factor M is given by

$$M \cong \mathrm{GL}(k_1) \times .....\mathrm{GL}(k_r) \times \mathrm{Sp}(W_{n-k_1-...-k_r}),$$

where $\mathrm{GL}(k_i)$ is the group of invertible linear maps on $\mathrm{Span}(e_{k_i+1}, ...e_{k_{i+1}})$. In particular, the maximal parabolic subgroups of $\mathrm{Sp}(W)$ are simply the stabilizers $P(X_k)$ of the isotropic spaces X_k $(1 \leq k \leq n)$. For a given k, the choice of the complementary space X_k^* gives a Levi subgroup of $P(X_k)$

$$M(X_k) = \mathrm{GL}(X_k) \times \mathrm{Sp}(W_{n-k}),$$

with $\mathrm{GL}(X_k)$ acting naturally on X_k^* by functoriality. Moreover, the unipotent radical $N(X_k)$ sits in a short exact sequence

$$1 \longrightarrow Z(X_k) \longrightarrow N(X_k) \longrightarrow \mathrm{Hom}(W_{n-k}, X_k) \longrightarrow 1$$

where $Z(X_k) \cong \mathrm{Sym}^2 X_k$ is isomorphic to the space of symmetric bilinear form on Y_k. When $k = n$, $N(X_k) = Z(X_k)$ is abelian and $P(X_n)$ is called the Siegel parabolic subgroup.

2.3. **Metaplectic Group.** The group $\mathrm{Sp}(W)$ has a unique two-fold cover $\mathrm{Mp}(W)$. As a set, we may write

$$\mathrm{Mp}(W) = \mathrm{Sp}(W) \times \{\pm 1\}$$

with group law given by

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \cdot c(g_1, g_2))$$

for some 2-cocycle c on $\mathrm{Sp}(W)$ valued in $\{\pm 1\}$. Without describing c explicitly, let us describe the restriction of this double cover over a maximal parabolic subgroup $P(X_k)$ of $\mathrm{Sp}(W)$.

The covering splits uniquely over the unipotent radical $N(X_k)$ of $P(X_k)$. Thus, we may regard $N(X_k)$ canonically as a subgroup of $\mathrm{Mp}(W)$ and one has a Levi decomposition

$$\widetilde{P}(X_k) = \widetilde{M}(X_k) \cdot N(X_k)$$

We need to describe the covering over $M(X_k) \cong \mathrm{GL}(X_k) \times \mathrm{Sp}(W_{n-k})$.

Not surprisingly, the restriction of the covering to $\mathrm{Sp}(W_{n-k})$ is nothing but the unique two-fold cover $\mathrm{Mp}(W_{n-k})$ of $\mathrm{Sp}(W_{n-k})$. The covering over $\mathrm{GL}(X_k)$ can be described as follows. Consider the set

$$\mathrm{GL}(X_k) \times \{\pm 1\}$$

with multiplication law

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \cdot (\det g_1, \det g_2))$$

where $(\det g_1, \det g_2)$ denotes the Hilbert symbol. Then $\widetilde{\mathrm{GL}}(X_k)$ is precisely this double cover of $\mathrm{GL}(X_k)$.

Hence, we have

$$\widetilde{M}(X_k) = \left(\widetilde{\mathrm{GL}}(X_k) \times \mathrm{Mp}(W_{n-k})\right) / \Delta\mu_2.$$

More generally, for any parabolic subgroup P, one has the Levi decomposition

$$\widetilde{P} = \widetilde{M} \cdot N$$

with

$$\widetilde{M} \cong \widetilde{\mathrm{GL}}(k_1) \times_{\mu_2} \ldots \times_{\mu_2} \widetilde{\mathrm{GL}}(k_r) \times_{\mu_2} \mathrm{Mp}(W_{n-k_1-\ldots-k_r}).$$

2.4. **Representations of $\widetilde{\mathrm{GL}}(X_k)$.** The (genuine) representation theory of $\widetilde{\mathrm{GL}}(X_k)$ can be easily related to the representation theory of $\mathrm{GL}(X_k)$. Indeed, the determinant map

$$\det : \mathrm{GL}(X_k) \longrightarrow \mathrm{GL}(1)$$

has a natural lifiting

$$\widetilde{\det} : \widetilde{\mathrm{GL}}(X_k) \longrightarrow \widetilde{\mathrm{GL}}(1)$$

given by

$$\widetilde{\det}(g, \epsilon) = (\det g, \epsilon).$$

On the other hand, if we fix an additive character ψ of k, then there is a natural genuine character of $\widetilde{\mathrm{GL}}(1)$ defined by:

$$(a, \epsilon) \mapsto \epsilon \cdot \gamma(a, \psi)^{-1}$$

with

$$\gamma(a, \psi) = \gamma(\psi_a)/\gamma(\psi)$$

and the Weil index $\gamma(\psi)$ is an 8-th root of unity associated to ψ by Weil. Composing this genuine character by $\widetilde{\det}$ gives a genuine character χ_ψ of $\widetilde{\mathrm{GL}}(X_k)$, which satisfies

$$\chi_\psi(g, \epsilon)^2 = (\det g, -1).$$

Using the genuine character χ_ψ, one obtains a bijection between $\mathrm{Irr}(\mathrm{GL}(X_k))$ and the set $\mathrm{Irr}(\widetilde{\mathrm{GL}}(X_k))$ of genuine irreducible representations of $\widetilde{\mathrm{GL}}(X_k)$, via:

$$\tau \mapsto \tilde{\tau}_\psi = \tau \otimes \chi_\psi.$$

We stress that this bijection depends on the choice of the additive character ψ.

Note that we could restrict the genuine character χ_ψ to the center $\widetilde{Z}$ of $\mathrm{Mp}(W)$. We denote this character of $\widetilde{Z}$ by χ_ψ as well. This character allows one to define a central sign for irreducible representations σ of $\mathrm{Mp}(W)$, as explained in the introduction.

2.5. **Parabolic Induction.** After the above discussion, one sees that given an irreducible representation τ of $\mathrm{GL}(X_k)$ and an irreducible representation π of $\mathrm{Mp}(W_{n-k})$, one has an irreducible representation $\tilde{\tau}_\psi \boxtimes \pi$ of $\widetilde{M}(X_k)$. Thus, one may consider the parabolically induced representation

$$I_{P(X_k),\psi}(\tau, \pi) = \mathrm{Ind}_{\widetilde{P}(X_k)}^{\mathrm{Mp}(W)} \tilde{\tau}_\psi \boxtimes \pi \quad \text{(normalized induction)}.$$

More generally, for any parabolic subgroup $P = M \cdot N$ and irreducible representation τ_i of $\mathrm{GL}(k_i)$ and π of $\mathrm{Mp}(W_{n-k_1-\ldots-k_r})$, one has the induced representation

$$I_{P,\psi}(\tau_1, ..., \tau_r, \pi).$$

A particular case of this is when $P = B$ is the Borel subgroup, so that each $k_i = 1$. In that case, given characters χ_1,...,χ_n, one has the principal series representations

$$I_{B,\psi}(\chi_1, ..., \chi_n).$$

If the χ_i's are unramified, we shall call such a representation an unramified principal series representation; note that this notion of "unramified representations" depends on the choice of ψ.

Though $\mathrm{Mp}(W)$ is not a linear group, many basic results regarding the induction and Jacquet functors remain valid. For a justification of this, the reader can consult [HM1].

2.6. **Maximal Compact Subgroup.** Let Λ be the $\mathcal{O}_k$-lattice generated by the vectors e_i's and e_j^*'s. Then Λ is a self-dual lattice and the stabilizer of Λ in $\mathrm{Sp}(W)$ is a hyperspecial maximal compact subgroup K. Note that there are two conjugacy classes of hyperspecial maximal compact subgroups in $\mathrm{Sp}(W)$; the other class of hyperspecial maximal compact subgroup is represented by the stabilizer K' of the lattice

$$\Lambda' = \langle e_i, e_j^*/\varpi \rangle.$$

The groups K and K' are conjugate by the similitude group $\mathrm{GSp}(W)$.

When $p \neq 2$, the metaplectic covering is known to split uniquely over K and K'. Thus, we may regard K and K' as subgroups of $\mathrm{Mp}(W)$. It is interesting to note that the K-spherical irreducible representations of $\mathrm{Mp}(W)$ are precisely the unique K-spherical constituents of the unramified principal series representations $I_{B,\psi}(\chi_1, ..., \chi_n)$ precisely when the conductor of ψ is of the form ϖ^{2r}. When the conductor of ψ is ϖ^{2r+1}, the analogous statement holds for the group K'. For more discussion of this, the reader can consult [GS].

2.7. **Orthogonal Groups.** Now we come to the orthogonal groups. Let V be a vector space of dimension $2n+1$ over k equipped with a nondegenerate quadratic form q_V of discriminant 1. There is a symmetric bilinear form b_q associated to q:

$$b_q(v_1, v_2) = q(v_1 + v_2) - q(v_1) - q(v_2).$$

Up to isomorphism, there are precisely two such quadratic spaces V. One of them, to be denoted by V^+, has maximal isotropic subspaces of dimension n, whereas the other has maximal isotropic subspaces of dimension $n-1$ and is denoted by V^-. As such, we call the former the split quadratic space and the latter the non-split one. We shall write

$$\epsilon(V) = \begin{cases} +1 \text{ if } V \text{ is split;} \\ -1, \text{ if } V \text{ is non-split.} \end{cases}$$

Let $\mathrm{O}(V)$ be the associated orthogonal group. Then observe that

$$\mathrm{O}(V) = \mathrm{SO}(V) \times \{\pm 1\}$$

where $\mathrm{SO}(V)$ is the special orthogonal group. The group $\mathrm{SO}(V)$ is split precisely when V is the split quadratic space.

Given any irreducible representation π of $\mathrm{SO}(V)$, there are two extensions of π to $\mathrm{O}(V)$, depending on whether the element $-1 \in \mathrm{O}(V)$ acts as $+1$ or -1. We denote these two extensions by π^+ and π^- respectively.

2.8. **Parabolic subgroups.** The parabolic subgroups of $\mathrm{O}(V)$ are stabilizers of flags of isotropic subspaces in V. More precisely, if $Y_r = \mathrm{Span}(v_1, ..., v_r)$ is a maximal isotropic subspace of V, then we may write

$$V = Y_r + V_0 + Y_r^*,$$

where V_0 is anisotropic and $Y_r^* = \mathrm{Span}(v_r^*, ..., v_1^*)$ is isotropic and satisfies

$$b_q(v_i, v_j^*) = \delta_{ij}.$$

For each $1 \leq k \leq r$, let $Y_k = \mathrm{Span}(v_1, ..., v_k)$ and $Y_k^* = \mathrm{Span}(v_1^*, ..., v_k^*)$ and let V_{n-k} be such that

$$V = Y_k + V_{n-k} + Y_k^*.$$

Now given a flag in Y_r:

$$Y_{k_1} \subset Y_{k_1+k_2} \subset .... \subset Y_{k_1+...+k_r},$$

the associated parabolic subgroup P has Levi subgroup

$$M = \mathrm{GL}(k_1) \times \mathrm{GL}(k_2) \times ... \times \mathrm{GL}(k_r) \times \mathrm{O}(V_{n-k_1-...-k_r}),$$

where $\mathrm{GL}(k_i)$ is the group of invertible linear maps on $\mathrm{Span}(v_{k_{i-1}+1}, ..., v_{k_i})$.

3. Weil Representations and Theta Correspondences

In this section, we introduce the Weil representations for $\mathrm{Mp}(W) \times \mathrm{O}(V)$ and recall the notion of theta correspondence.

3.1. **Weil Representation.** Fix an additive character ψ of k. Then the group $\mathrm{Mp}(W) \times \mathrm{O}(V)$ has a natural representation $\Omega_{V,W,\psi}$ depending on ψ. This representation can be realized on the space $S(X^* \otimes V)$ of Schwarz-Bruhat functions on $X^* \otimes V = \mathrm{Hom}(X, V)$. The action of $\mathrm{Mp}(W) \times \mathrm{O}(V)$ on $S(X^* \otimes V)$ via $\Omega_{V,W,\psi}$ is described as follows.

$$\begin{cases} (\Omega_\psi(h)\phi)(A) = \phi(h^{-1}A), \text{ if } h \in \mathrm{O}(V); \\ (\Omega_\psi(n)\phi)(A) = \psi(\frac{1}{2} \cdot \langle n(A), A\rangle) \cdot \phi(A), \text{ if } n \in N(X) = \mathrm{Sym}^2 X \subset \mathrm{Hom}(X^*, X); \\ (\Omega_\psi(m,\epsilon)\phi)(A) = \chi_\psi(m,\epsilon) \cdot |\det(m)|^{\frac{1}{2}\dim V} \cdot \phi(m^{-1} \cdot A) \text{ if } (m,\epsilon) \in \widetilde{M}(X) = \widetilde{\mathrm{GL}}(X); \\ (\Omega_\psi(w)\phi)(A) = \gamma(\psi \circ q_V)^n \cdot \int_{X^* \otimes V} \phi(B) \cdot \psi(\langle A, B\rangle)\, dB. \end{cases}$$

Here, in the last equation, w is a certain Weyl group element and $\gamma(\psi \circ q_V)$ is the Weil index associated to the pair (ψ, q_V). Moreover, in the second equation, with $A \in X^* \otimes V$, the element $n(A)$ lies in $X \otimes V$, and the pairing between $X \otimes V$ and $X^* \otimes V$ is the tensor product of the natural pairing between X and X^* and the symmetric bilinear form b_V associated to the quadratic form q_V on V:

$$b_V(v_1, v_2) = q_V(v_1 + v_2) - q_V(v_1) - q_V(v_2).$$

3.2. **Theta Correspondence.** Given an irreducible representation π of $\mathrm{O}(V)$, the maximal π-isotypic quotient of $\Omega_{V,W,\psi}$ has the form $\pi \boxtimes \Theta_{V,W,\psi}(\pi)$ for some smooth representation $\Theta_{V,W,\psi}(\pi)$ of $\mathrm{Mp}(W)$ (called the big theta lift of π). The maximal semisimple quotient of $\Theta_{V,W,\psi}(\pi)$ is denoted by $\theta_{V,W,\psi}(\pi)$ and is called the small theta lift of π.

Similarly, if σ is an irreducible genuine representation of $\mathrm{Mp}(W)$, then one has its big theta lift $\Theta_{W,V,\psi}(\sigma)$ and its small theta lift $\theta_{W,V,\psi}(\sigma)$, which are smooth representations of $\mathrm{O}(V)$.

The following theorem summarizes some basic results of Howe, Kudla [Ku], Moeglin-Vigneras-Waldspurger [MVW] and Waldspurger [W3] about the theta correspondence.

Theorem 3.1. *(i) The representation $\Theta_{V,W,\psi}(\pi)$ is either zero or has finite length.*

(ii) If π is supercuspidal, then $\Theta_{V,W,\psi}(\pi)$ is either zero or irreducible (and thus is equal to $\theta_{V,W,\psi}(\pi)$). Moreover, if π and π' are supercuspidal representations such that $\Theta_{V,W,\psi}(\pi) \cong \Theta_{V,W,\psi}(\pi')$, then $\pi \cong \pi'$.

(iii) If $p \neq 2$, then $\Theta_{V,W,\psi}(\pi)$ is either zero or has a unique irreducible quotient, so that $\theta_{V,W,\psi}(\pi)$ is irreducible. Moreover, for any irreducible representations π and π' of $\mathrm{O}(V)$,

$$\theta_{V,W,\psi}(\pi) \cong \theta_{V,W,\psi}(\pi') \Longrightarrow \pi \cong \pi'.$$

(iv) The analogous statements hold for $\Theta_{W,V,\psi}(\sigma)$ and $\theta_{W,V,\psi}(\sigma)$ if σ is an irreducible genuine representation of $\mathrm{Mp}(W)$.

3.3. **The doubling see-saw.** Given an irreducible representation π of $\mathrm{SO}(V)$, we are interested in whether $\Theta_{V,W,\psi}(\pi^\epsilon)$ is nonzero. To address this question, it is useful to introduce the "doubled space"

$$\mathbb{V} = V + (-V)$$

where $-V$ is the quadratic space $(V, -q)$. The quadratic space $\mathbb{V}$ has even dimension and is split, with a maximal isotropic subspace given by

$$V^\Delta = \{(v,v) : v \in V\} \subset \mathbb{V}.$$

Now consider the see-saw diagram:

$$\begin{tikzcd}
\mathrm{O}(\mathbb{V}) \arrow[d, dash] \arrow[drr, dash] & & \mathrm{Mp}(W) \times_{\mu_2} \mathrm{Mp}(W) \arrow[d, dash] \arrow[dll, dash] \\
\mathrm{O}(V) \times \mathrm{O}(V) & & \mathrm{Sp}(W)
\end{tikzcd}$$

Then the see-saw identity says that:

$$\mathrm{Hom}_{\mathrm{Sp}(W)}(\Theta_{V,W,\psi}(\pi) \otimes \Theta_{-V,W,\psi}(\pi^\vee), \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{O}(V)}(\Theta_{W,\mathbb{V},\psi}(1), \pi \otimes \pi^\vee).$$

Now note that if c is an element of $\mathrm{GSp}(W)$ with similitude factor -1, then

$$\Theta_{-V,W,\psi}(\pi^\vee) = \Theta_{V,W,\psi}(\pi)^c,$$

and for an irreducible representation σ of $\mathrm{Mp}(W)$, $\sigma^c \cong \sigma^\vee$ (cf. [Ku2]). From this, we deduce:

Lemma 3.2. *Let π be an irreducible representation of $\mathrm{O}(V)$, then $\Theta_{V,W,\psi}(\pi) \neq 0$ if and only if*

$$\mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{O}(V)}(\Theta_{W,\mathbb{V},\psi}(1), \pi \otimes \pi^\vee) \neq 0.$$

Similarly, starting from an irreducible representation σ of $\mathrm{Mp}(W)$ and considering the see-saw diagram

$$\begin{tikzcd}
\mathrm{Mp}(\mathbb{W}) \arrow[d, dash] \arrow[drr, dash] & & \mathrm{O}(V) \times \mathrm{O}(V) \arrow[d, dash] \arrow[dll, dash] \\
\mathrm{Mp}(W) \times \mathrm{Mp}(W) & & \mathrm{O}(V)
\end{tikzcd}$$

with

$$\mathbb{W} = W + (-W),$$

we obtain:

Lemma 3.3. *Let σ be an irreducible representation of $\mathrm{Mp}(W)$, then $\Theta_{W,V,\psi}(\sigma) \neq 0$ if and only if*

$$\mathrm{Hom}_{\mathrm{Mp}(W)\times\mathrm{Mp}(W)}(\Theta_{V,\mathbb{W},\psi}(1), \sigma \otimes \sigma^\vee) \neq 0.$$

3.4. **Degenerate Principal Series.** In order for the non-vanishing criteria given in Lemmas 3.2 and 3.3 to be useful, we need to understand the representations $\Theta_{W,\mathbb{V},\psi}(1)$ of $\mathrm{O}(\mathbb{V})$ and $\Theta_{V,\mathbb{W},\psi}(1)$ of $\mathrm{Mp}(\mathbb{W})$ more precisely. For this, we need to describe some degenerate principal series representations of $\mathrm{O}(\mathbb{V})$ and $\mathrm{Mp}(\mathbb{W})$.

Recall that we have the Siegel parabolic subgroup $P(V^\Delta)$ of $\mathrm{O}(\mathbb{V})$, with Levi subgroup $\mathrm{GL}(V^\Delta)$. For $s \in \mathbb{C}$, let

$$I_{P(V^\Delta)}(s) := \mathrm{Ind}_{P(V^\Delta)}^{\mathrm{O}(\mathbb{V})} |\det|^s \quad \text{(normalized induction).}$$

Similarly, we have the Siegel parabolic subgroup $\tilde{P}(W^\Delta)$ of $\mathrm{Mp}(\mathbb{W})$ with Levi subgroup $\mathrm{GL}(W^\Delta)$ and we set

$$I_{\tilde{P}(W^\Delta),\psi}(s) := \mathrm{Ind}_{\tilde{P}(W^\Delta)}^{\mathrm{O}(\mathbb{V})} \chi_\psi |\det|^s \quad \text{(normalized induction).}$$

3.5. **Theta lifts of trivial representation.** We consider the Weil representation $\Omega_{W,\mathbb{V},\psi}$ of $\mathrm{O}(\mathbb{V}) \times \mathrm{Sp}(W)$, which has a Schrodinger model realized on $S((V^\Delta)^* \otimes W)$.The action of $\mathrm{Sp}(W)$ in this model is geometric:

$$(g \cdot \phi)(a) = \phi(g^{-1} \cdot a) \quad \text{for } g \in \mathrm{Sp}(W).$$

There is a natural $\mathrm{Sp}(W)$-invariant and $\mathrm{O}(\mathbb{V})$-equivariant map

$$f : S((V^\Delta)^* \otimes W) \longrightarrow I_{P(V^\Delta)}(0)$$

which sends ϕ to the function

$$f_\phi(h) = (h \cdot \phi)(0).$$

Then we have the following proposition due to Kudla-Rallis:

Proposition 3.4. *(i) The map f induces an injection*

$$\Theta_{W,\mathbb{V},\psi}(1) \hookrightarrow I_{P(V^\Delta)}(0)$$

of $\mathrm{O}(\mathbb{V})$-modules.

(ii) The representation $\Theta_{W,\mathbb{V},\psi}(1)$ is irreducible.

(iii) One has:

$$I_{P(V^\Delta)}(0) \cong \Theta_{W,\mathbb{V},\psi}(1) \bigoplus \Theta_{W,\mathbb{V},\psi}(1) \otimes \det_{\mathrm{O}(\mathbb{V})}.$$

Similarly, with the Weil representation $\Omega_{V,\mathbb{W},\psi}$ of $\mathrm{O}(V) \times \mathrm{Mp}(\mathbb{W})$ realized on $S((W^\Delta)^* \otimes V)$, there is a natural $\mathrm{O}(V)$-invariant and $\mathrm{Mp}(\mathbb{W})$-equivariant map

$$f : S((W^\Delta)^* \otimes V) \longrightarrow I_{P(W^\Delta),\psi}(0)$$

which sends ϕ to the function:

$$f_\phi(g) = (\Omega_{V,\mathbb{W},\psi}(g)\phi)(0).$$

Then we have the following proposition, which is due to Sweet [Sw] (cf. also [Z] and [GI]):

Proposition 3.5. *(i) The map f induces an injection*

$$\Theta_{V,\mathbb{W},\psi}(1) \hookrightarrow I_{P(W^\Delta),\psi}(0).$$

(ii) The representation $\Theta_{V,\mathbb{W},\psi}(1)$ is irreducible.

(iii) One has

$$I_{P(W^\Delta),\psi}(0) \cong \Theta_{V^+,\mathbb{W},\psi}(1) \oplus \Theta_{V^-,\mathbb{W},\psi}(1).$$

4. **Doubling Zeta Integrals and Epsilon Factors**

We maintain the notations of the previous section. Propositions 3.4 and 3.5 imply that we need to understand the spaces

$$\mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{O}(V)}(I_{P(V^\Delta)}(0), \pi \otimes \pi^\vee) \quad \text{and} \quad \mathrm{Hom}_{\mathrm{Mp}(W)\times\mathrm{Mp}(W)}(I_{P(W^\Delta),\psi}(0), \sigma \otimes \sigma^\vee).$$

The doubling zeta integral allows one to write down a nonzero element in each of these two spaces.

4.1. **Doubling zeta integral.**

More precisely, for $f_s \in I_{P(V^\Delta)}(s)$, $v \in \pi$ and $v^\vee \in \pi^\vee$, we define the integral

$$Z(s, f, v, v^\vee) = \int_{\mathrm{O}(V)} f_s(g, 1) \cdot \langle g \cdot v^\vee, v \rangle \, dg.$$

The following theorem ([KR1], [LR]) summarizes the properties of this family of zeta integrals:

Theorem 4.1. *(i) There exists a constant c such that whenever $Re(s) > c$, the integral $Z(s, f, v, v^\vee)$ converges for all data f_s, v and $v^\vee$. If π is tempered, then we may take $c = -1$.*

(ii) If f_s is a standard section of $I_{P(V^\Delta)}(s)$, then the function $Z(s, f, v, v^\vee)$ is a rational function (when $Re(s) > c$) and thus admits meromorphic continuation to $\mathbb{C}$.

(iii) For each s_0, there exists data f, v and $v^\vee$ such that $Z(s_0, f, v, v^\vee)$ is finite and nonzero.

(iv) There is a non-negative integer k (depending on s_0) such that $(s - s_0)^k \cdot Z(s, f, v, v^\vee)$ is holomorphic at $s = s_0$ and is nonzero there for some choice of data.

(iv) Let $Z^*(s_0)$ denote the leading term in the Laurent expansion of $Z(s)$ as a linear form, so that

$$Z^*(s_0, f, v, v^\vee) = \Big((s-s_0)^k \cdot Z(s, f, v, v^\vee) \Big) |_{s=s_0},$$

then $Z^*(s_0)$ is a nonzero element of $\mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{O}(V)}(I_{P(V^\Delta)}(0) \otimes \pi^\vee \otimes \pi, \mathbb{C}))$. In particular, we see that

$$\mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{O}(V)}(I_{P(V^\Delta)}(0) \otimes \pi^\vee \otimes \pi, \mathbb{C})) \neq 0.$$

If π is supercuspidal, one can show that

$$\dim \mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{O}(V)}(I_{P(V^\Delta)}(0) \otimes \pi^\vee \otimes \pi, \mathbb{C})) = 1.$$

Indeed, this multiplicity one result is known to hold for most representations, and is conjectured to hold for all.

One has the analogous results for $\mathrm{Mp}(W)$, which implies:

Proposition 4.2.

$$\mathrm{Hom}_{\mathrm{Mp}(W)\times\mathrm{Mp}(W)}(I_{P(W^\Delta),\psi}(0) \otimes \sigma^\vee \otimes \sigma, \mathbb{C})) \neq 0.$$

We omit the details.

4.2. **Functional equation and standard epsilon factor.** Another important property of the doubling zeta integral is a local functional equation they satisfy. To describe this, note that there is a standard intertwining operator

$$M_\psi(s) : I_{P(V^\Delta)}(s) \longrightarrow I_{P(V^\Delta)}(-s).$$

This is defined for $Re(s) >> 0$ by the integral

$$M_\psi(s)(f)(h) = \int_{N(V^\Delta)} f(wnh)\, dn_\psi$$

and by meromorphic continuation in general, with $w = (1, -1) \in \mathrm{O}(V) \times \mathrm{O}(V)$. In [LR], Lapid-Rallis has defined a certain normalization $M_\psi^*(s)$ of $M_\psi(s)$ satisfying

$$M_\psi^*(-s) \circ M_\psi^*(s) = Id.$$

This implies that $M_\psi^*(s)$ is holomorphic at $s = 0$ and satisfies

$$M_\psi^*(0)^2 = Id.$$

In particular, $M_\psi^*(0)$ acts as $+1$ or -1 on each of the two irreducible summands of $I_{P(V^\Delta)}(0)$. We shall determine the precise action of $M_\psi^*(0)$ later on.

Refining the work of Piatetski-Shapiro and Rallis [PSR], Lapid-Rallis [LR] showed that the local zeta integeral $Z(s)$ satisfies a functional equation of the form

$$Z(-s, M_\psi^*(s)(f), v, v^\vee) = \epsilon(V) \cdot \pi(-1) \cdot \gamma(s + \frac{1}{2}, \pi, \psi) \cdot Z(s, f, v, v^\vee),$$

for some rational function $\gamma(s, \pi, \psi)$ (in q^{-s}). Following Lapid-Rallis, we have:

Definition:

(i) The function $\gamma(s,\pi,\psi)$ is called the standard γ-factor of π.

(ii) If π is tempered, we may write

$$\gamma(s,\pi,\psi)=\epsilon(s,\pi,\psi)\cdot\frac{L(1-s,\pi^\vee,\psi)}{L(s,\pi)}$$

where $\epsilon(s,\pi,\psi)$ is a monomial function of q^{-s} and $L(s,\pi)^{-1}$ is the numerator of the rational function $\gamma(s,\pi,\psi)$, normalized so that it is a polynomial in q^{-s} with constant term 1. The function $\epsilon(s,\pi,\psi)$ is called the standard epsilon factor of π and $L(s,\pi)$ is the standard L-factor of π.

(iii) If π is non-tempered, we realize π as a Langlands quotient of a standard module and define $\epsilon(s,\pi,\psi)$ and $L(s,\pi)$ by multiplicativity.

Lapid-Rallis showed that, with the above definitions, the local factors $\gamma(s,\pi,\psi)$, $\epsilon(s,\pi,\psi)$ and $L(s,\pi)$ satisfy a number of expected properties which characterize them uniquely. In particular,

$$\epsilon(1/2,\pi,\psi)=\pm 1$$

is independent of ψ. Hence, we shall simply denote it by $\epsilon(\pi)$.

4.3. **Metaplectic case.** The analogous theory of the doubling zeta integral for the metaplectic groups is obtained in [G]. We record some of the relevant facts in this subsection.

There is a standard intertwining operator

$$M_\psi(s):I_{P(W^\triangle),\psi}(s)\longrightarrow I_{P(W^\triangle),\psi}(-s).$$

One may normalize this intertwining operator to obtain the normalized operator $M_\psi^*(s)$; this normalization has been treated in [Sw] and [Z], and satisfies

$$M_\psi^*(-s)\circ M_\psi^*(s)=Id.$$

Hence $M_\psi^*(s)$ is holomorphic at $s=0$ and satisfies

$$M_\psi^*(0)^2=Id.$$

In particular, $M_\psi^*(0)$ acts as $+1$ or -1 on each of the two irreducible summands of $I_{P(W^\triangle),\psi}(0)$.

The local functional equation of the doubling zeta integral can now be written as:

$$Z(-s,M_\psi^*(s)(f),v,v^\vee)=z_\psi(\sigma)\cdot\gamma(s+\frac{1}{2},\sigma,\psi)\cdot Z(s,f,v,v^\vee),$$

for some rational function $\gamma(s,\sigma,\psi)$ (in q^{-s}), and where $z_\psi(\sigma)=\pm1$ is the central sign of σ. Following [LR], one can now make the following definition:

Definition:

(i) The function $\gamma(s,\sigma,\psi)$ is called the standard γ-factor of σ.

(ii) If σ is tempered, we may write

$$\gamma(s,\sigma,\psi)=\epsilon(s,\sigma,\psi)\cdot\frac{L(1-s,\sigma^\vee,\psi)}{L(s,\sigma,\psi)}$$

where $\epsilon(s,\sigma,\psi)$ is a monomial function of q^{-s} and $L(s,\sigma,\psi)^{-1}$ is the numerator of the rational function $\gamma(s,\sigma,\psi)$, normalized so that it is a polynomial in q^{-s} with constant term 1. The function $\epsilon(s,\sigma,\psi)$ is called the standard epsilon factor of σ and $L(s,\sigma,\psi)$ is the standard L-factor of σ relative to the choice of ψ.

(iii) If σ is non-tempered, we realize σ as a Langlands quotient of a standard module and define $\epsilon(s,\sigma,\psi)$ and $L(s,\sigma,\psi)$ by multiplicativity.

In [G], it was checked that the the above definition of $\gamma(s,\sigma,\psi)$ satisfies the analog of the "Ten Commandments" in [LR, Thm. 4] and is uniquely determined by these.

5. **Interlude: Results of Waldspurger**

Before coming to the main results of this paper, we take a short interlude to recall the results of Waldspurger [W1,2] in the case $\dim W = 2$ and $\dim V = 3$.

By studying the theta correspondence for $\mathrm{Mp}(W) \times \mathrm{SO}(V)$ in detail, Waldspurger showed:

Theorem 5.1. *Fix an additive character ψ of k.*

(i) Given any irreducible representation π of $\mathrm{SO}(V)$, the theta lift $\theta_{V,W,\psi}(\pi)$ of π to $\mathrm{Mp}(W)$ is irreducible and nonzero.

(ii) The construction in (i) gives a bijection

$$\Theta_\psi : Irr(\mathrm{SO}(V^+) \cup Irr(\mathrm{SO}(V^-)) \leftrightarrow Irr(\mathrm{Mp}(W)).$$

(iii) $\pi \in Irr(\mathrm{SO}(V))$ is a discrete series (resp. tempered) representation if and only if $\Theta_\psi(\pi)$ is a discrete series (resp. tempered) representation.

(iv) Via the local Langlands correspondence for $\mathrm{SO}(V^\pm)$, one then has a bijection

$$\mathcal{L}_\psi : \mathrm{Irr}(\mathrm{Mp}(W)) \leftrightarrow \Phi(\mathrm{Mp}(W)).$$

The above theorem says that:

(a) given $\pi \in \mathrm{Irr}(\mathrm{SO}(V))$, exactly one extension π^ϵ of π to $\mathrm{O}(V)$ participates in the theta correspondence with $\mathrm{Mp}(W)$;

(b) given $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$, σ participates in theta correspondence with exactly one of $\mathrm{O}(V^+)$ or $\mathrm{O}(V^-)$.

As a refinement of the above two statements, Waldspurger showed:

Theorem 5.2. *(i) Given $\pi \in Irr(\mathrm{SO}(V))$, π^ϵ participates in theta correspondence with $\mathrm{Mp}(W)$ if and only if*

$$\epsilon = \epsilon(V) \cdot \epsilon(1/2, \pi).$$

(ii) Given $\sigma \in Irr(\mathrm{Mp}(W))$, σ participates in theta correspondence with $\mathrm{O}(V)$ if and only if

$$z_\psi(\sigma) = \epsilon(V) \cdot \epsilon(1/2, \Theta_\psi(\sigma)) = \epsilon(V) \cdot \epsilon(1/2, \mathcal{L}_\psi(\sigma)).$$

Indeed, the two statements in the theorem are equivalent, and what Waldspurger showed is the statement (ii). The statement (ii) also has the following implication. If $\pi \in \mathrm{Irr}(\mathrm{SO}(V^+))$ has L-parameter ϕ and Jacquet-Langlands lift $\pi' \in \mathrm{Irr}(\mathrm{SO}(V^-))$ (if it exists), then the L-packet associated to ϕ is

$$\mathcal{L}_{\psi,\phi} = \{\Theta_\psi(\pi), \Theta_\psi(JL(\pi))\}.$$

The statement (ii) implies that the elements in $\mathcal{L}_{\psi,\phi}$ have different central characters.

It is instructive to examine the following example:

Example: Let St_{V^+} be the Steinberg representation of $\mathrm{SO}(V^+)$, so that its Jacquet-Langlands lift is the trivial representation 1_{V^-} of $\mathrm{SO}(V^-)$. In this case, one knows that

$$\epsilon(1/2, \mathrm{St}_{V^+}) = \epsilon(1/2, 1_{V^-}) = -1.$$

Thus, the extensions $\mathrm{St}_{V^+}^-$ and $1_{V^-}^+$ participate in the theta correspondence woith $\mathrm{Mp}(W)$. Moreover, one has

$$\Theta_\psi(\mathrm{St}_{V^+}) = \omega_\psi^o \quad \text{and} \quad \Theta_\psi(1_{V^-}) = \mathrm{St}_\psi$$

where ω_ψ^o is the odd Weil representation of $\mathrm{Mp}(W)$ associated to ψ and St_ψ is the Steinberg representation of $\mathrm{Mp}(W)$ associated to ψ. The representation St_ψ sits in a short exact sequence

$$0 \longrightarrow \mathrm{St}_\psi \longrightarrow I_{B,\psi}(|-|^{1/2}) \longrightarrow \omega_\psi^e \longrightarrow 0$$

with ω_ψ^e denoting the even Weil representation associated to ψ.

The decomposition

$$\mathrm{Irr}(\mathrm{Mp}(W)) = \bigsqcup_\phi \mathcal{L}_{\psi,\phi}$$

is a canonical decomposition, in the sense that it is independent of ψ. However, the labelling of the packets by L-parameters ϕ depends on ψ, and so does the labelling of the representations in each packet by the characters of the component group $A(\phi)$. Finally, Waldspurger determined how this dependence varies with ψ.

Theorem 5.3. *For $a \in k^\times$, let ψ_a denote the additive character given by $\psi_a(x) = \psi(ax)$ and let χ_a be the quadratic character associated to the class of $a \in k^\times/k^{\times 2}$. Suppose that*

$$\mathcal{L}_\psi(\sigma) = (\phi, \eta) \quad \textit{and} \quad \mathcal{L}_{\psi_a}(\sigma) = (\phi_a, \eta_a).$$

Then

$$\phi_a = \phi \otimes \chi_a$$

and

$$\eta_a/\eta = \epsilon(1/2, \phi \otimes \chi_a) \cdot \epsilon(1/2, \phi) \cdot \chi_a(-1).$$

The purpose of this paper is to extend Theorems 5.1, 5.2 and 5.3 to the case of higher rank (when $\dim V = 2n + 1$).

6. The Local Langlands Correspondence for Mp_{2n}

The goal of this section is to prove Theorem 1.1. The proof of this theorem was sketched in [GGP], with the key step being the following special case of more general results of Kudla-Rallis [KR2]:

Theorem 6.1. *Let σ be an irreducible (genuine) representation of $\mathrm{Mp}(W)$. Then at most one of $\Theta_{W,V^+,\psi}(\sigma)$ or $\Theta_{W,V^-,\psi}(\sigma)$ is nonzero.*

Corollary 6.2. *For $\sigma \in Irr(\mathrm{Mp}(W))$, exactly one of $\Theta_{W,V^+,\psi}(\sigma)$ or $\Theta_{W,V^-,\psi}(\sigma)$ is nonzero.*

Proof. Given $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$, Lemma 3.3, Prop. 3.5(iii) and Prop. 4.2 imply that at least one of $\Theta_{W,V^+,\psi}(\sigma)$ or $\Theta_{W,V^-,\psi}(\sigma)$ is nonzero. Thus the corollary follows by Thm. 6.1. $\square$

When p is odd, the small theta lift $\theta_{W,V^\epsilon,\psi}(\sigma)$ is irreducible or zero. Thus, the corollary implies that one has an injective map

$$\mathrm{Irr}(\mathrm{Mp}(W)) \longrightarrow \mathrm{Irr}(\mathrm{O}(V^+)) \sqcup \mathrm{Irr}(\mathrm{O}(V^-)).$$

By restriction of representations of $\mathrm{O}(V)$ to $\mathrm{SO}(V)$, one obtains a map (not necessarily injective at this point)

$$\Theta_\psi : \mathrm{Irr}(\mathrm{Mp}(W)) \longrightarrow \mathrm{Irr}(\mathrm{SO}(V^+)) \sqcup \mathrm{Irr}(\mathrm{SO}(V^-)).$$

We need to show that the map Θ_ψ is bijective. For this, we note:

Proposition 6.3. *Given $\pi \in Irr(\mathrm{O}(V))$, with extensions $\pi^\pm$ to $\mathrm{O}(V)$, at most one of $\Theta_{V,W,\psi}(\pi^\pm)$ is nonzero.*

Proof. Suppose on the contrary that $\pi^\pm$ both participate in theta correspondence with $\mathrm{Mp}(W)$, say

$$\sigma^+ = \theta_{V,W,\psi}(\pi^+) \quad \text{and} \quad \sigma^- = \theta_{V,W,\psi}(\pi^-).$$

Observe also that

$$\pi^- = \pi^+ \otimes \det.$$

Now consider the doubling seesaw diagram:

$$\begin{tikzcd}
\mathrm{Mp}(W+(-W)) \arrow[d, dash] \arrow[dr, dash] & \mathrm{O}(V)\times\mathrm{O}(V) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{Mp}(W)\times\mathrm{Mp}(W) & \mathrm{O}(V).
\end{tikzcd}$$

The seesaw identity implies that

$$\mathrm{Hom}_{\mathrm{Mp}(W)\times\mathrm{Mp}(W)}(\Theta_{V,W+(-W),\psi}(\det), \sigma^+ \boxtimes (\sigma^-)^\vee) \supset \mathrm{Hom}_{\mathrm{O}(V)}(\pi^+ \otimes (\pi^-)^\vee, \det) \neq 0.$$

This implies that

$$\Theta_{V,W+(-W),\psi}(\det) \neq 0.$$

However, a classical result of Rallis [R, Appendix] says that the determinant character of $\mathrm{O}(V)$ does not participate in the theta correspondence with $\mathrm{Mp}(4r)$ for $r \leq n$. This gives the desired contradiction and the proposition is proved. $\square$

Corollary 6.4. *Given $\pi \in Irr(\mathrm{SO}(V))$, with extensions $\pi^{\pm}$ to $\mathrm{O}(V)$, exactly one of $\Theta_{V,W,\psi}(\pi^{\pm})$ is nonzero.*

Proof. By Lemma 3.2, Prop. 3.4(iii) and Thm. 4.1(iv), we see that at least one of $\Theta_{V,W,\psi}(\pi^{\pm})$ is nonzero. Thus the corollary follows by Prop. 6.3. $\square$

This corollary implies that the map Θ_{ψ} is bijective (when p is odd). Thm. 1.1 is proved.

Remarks: The only reason for the assumption of odd residue characteristic in Theorem 1.1 and its proof is that Howe's conjecture for local theta correspondence is only known under this assumption.

7. **Theta Dichotomy and Epsilon Factor**

In this section, we shall prove Thm. 1.4, which we restate here for ease of reference:

Theorem 7.1. *(i) Let π be an irreducible representation of $\mathrm{SO}(V)$. Then π^{ϵ} participates in theta correspondence (with respect to ψ) with $\mathrm{Mp}(W)$ if and only if*

$$\epsilon = \epsilon(V) \cdot \epsilon(1/2, \pi).$$

(ii) Let σ be an irreducible representation of $\mathrm{Mp}(W)$. Then σ has nonzero theta lift (with respect to ψ) to $\mathrm{O}(V)$ if and only if the central sign of σ satisfies:

$$z_{\psi}(\sigma) := \omega_{\sigma}(-1)/\chi_{\psi}(-1) = \epsilon(V) \cdot \epsilon(1/2, \sigma, \psi) = \epsilon(V) \cdot \epsilon(1/2, \Theta_{\psi}(\sigma)).$$

This theorem refines the results of Cors. 6.2 and 6.4. Moreover, we do not need to assume that p is odd here.

We first prove statement (i) in the theorem. Assume first that $\pi \in \mathrm{Irr}(\mathrm{SO}(V))$ is tempered. Then the doubling zeta integral $Z(s, f, v, v^{\vee})$ is holomorphic at $s = 0$ for any $v \in \pi$ and $v^{\vee} \in \pi^{\vee}$, and $f \in I_{P(V^{\triangle})}(0)$. Moreover,

$$0 \neq Z(0) \in \mathrm{Hom}_{\mathrm{O}(V)\times\mathrm{O}(V)}(I_{P(V^{\triangle})}(0) \otimes \pi^{\epsilon} \otimes (\pi^{\epsilon})^{\vee}, \mathbb{C}).$$

We need to determine whether $Z(0)$ is nonzero when restricted to the irreducible submodule $\Theta_{W,\mathbb{V},\psi}(1)$. For this, we take note of the local functional equation

$$Z(-s, M^*_{\psi}(s)(f), v, v^{\vee}) = \epsilon(V) \cdot \pi^{\epsilon}(-1) \cdot \gamma(s + \frac{1}{2}, \pi, \psi) \cdot Z(s, f, v, v^{\vee}).$$

Specializing to $s = 0$, and noting that $\pi^{\epsilon}(-1) = \epsilon$, one obtains:

$$Z(0) \circ M^*_{\psi}(0) = \epsilon(V) \cdot \epsilon \cdot \gamma(1/2, \pi, \psi) \cdot Z(0).$$

Since the central L-value $L(1/2, \pi)$ is finite when π is tempered, we see that $\gamma(1/2, \pi, \psi) = \epsilon(1/2, \pi, \psi)$, so that the local functional equation reads:

$$Z(0) \circ M^*_{\psi}(0) = \epsilon(V) \cdot \epsilon \cdot \epsilon(1/2, \pi, \psi) \cdot Z(0).$$

Now we have the following crucial Lemma 7.2, which implies that $Z(0)$ is nonzero when restricted to $\Theta_{W,\mathbb{V},\psi}(1)$ if and only if

$$\epsilon(V) \cdot \epsilon \cdot \epsilon(1/2, \pi, \psi) = 1.$$

Hence, assuming Lemma 7.2 for the moment, we see that π^ϵ participates in theta correspondence with $\mathrm{Mp}(W)$ if and only if

$$\epsilon = \epsilon(V) \cdot \epsilon(1/2, \pi).$$

This proves Thm. 7.1(i) for tempered π's.

We shall now prove:

Lemma 7.2. *The normalized intertwining operator $M_\psi^*(0)$ acts as $+1$ on $\Theta_{W,\mathbb{V},\psi}(1)$ and as -1 on $\Theta_{W,\mathbb{V},\psi}(1) \otimes \det_{\mathrm{O}(\mathbb{V})}$.*

Proof. We first claim that $M_\psi^*(0)$ acts by opposite signs on the two irreducible summands $\Theta_{W,\mathbb{V},\psi}(1)$ and $\Theta_{W,\mathbb{V},\psi}(1) \otimes \det_{\mathrm{O}(\mathbb{V})}$. Indeed, if f lies in $\Theta_{W,\mathbb{V},\psi}(1)$, then the function $f' = f \cdot \det_{\mathrm{O}(\mathbb{V})}$ lies in $\Theta_{W,\mathbb{V},\psi}(1) \otimes \det_{\mathrm{O}(\mathbb{V})}$. Now, for $Re(s) >> 0$, we have

$$M_\psi(s)(f')(g) = \int_{N(V^\Delta)} f'(wng)\, dn_\psi = \int_{N(V^\Delta)} f(wng) \cdot \det_{\mathrm{O}(\mathbb{V})}(wng)\, dn_\psi,$$

where

$$w = (-1, 1) \in \mathrm{O}(V) \times \mathrm{O}(V) \subset \mathrm{O}(\mathbb{V}).$$

Since $\det(w) = -1$, we see that

$$M_\psi(s)(f') = -M_\psi(s)(f) \cdot \det_{\mathrm{O}(\mathbb{V})},$$

which proves our claim.

Now we can complete the proof of the lemma in two different ways. For the first proof, one computes the effect of $M_\psi(s)$ on the spherical vector f_0 by the Gindikin-Karpelevich formula. Taking into account the normalizing factor in $M_\psi^*(s)$, one then sees that

$$M_\psi^*(0)(f_0) = f_0,$$

so that $M_\psi^*(0)$ acts as $+1$ on $\Theta_{W,\mathbb{V},\psi}(1)$ and as -1 on $\Theta_{W,\mathbb{V},\psi}(1) \otimes \det_{\mathrm{O}(\mathbb{V})}$.

For the second proof, we exploit the theta correspondence to come to the same conclusion. More precisely, we shall see that by Kudla's cuspidal support theorem ([Ku, Thm 2.5] and [Ku2, Thms. 7.1 and 7.2]), almost all unramified tempered representations of $\mathrm{O}(V^+)$ participate in theta correspondence with $\mathrm{Mp}(W)$. In other words, for almost all unramified tempered representations π of $\mathrm{SO}(V^+)$, the extension of π which has nonzero theta lift to $\mathrm{Mp}(W)$ is π^+.

To see this, consider an unramified irreducible representation $\pi = I_B(\chi_1, ..., \chi_n)$ of $\mathrm{SO}(V^+)$ induced from an unramified character $\chi_1 \times .... \times \chi_n$ of the Borel subgroup B. For its two extensions π^ϵ to $\mathrm{O}(V^+)$, the cuspidal support of the resulting representation π^ϵ is $(\epsilon, \chi_1, ..., \chi_n)$ with $\epsilon = \pm$ regarded as a representation of $\mathrm{O}(1)$. By Kudla's cuspidal support theorem ([Ku, Thm 2.5] and [Ku2, Thms. 7.1 and 7.2]), the theta lift of π^ϵ to $\mathrm{Mp}(W)$ (if nonzero) has cuspidal support depending on the first occurrence of the representation ϵ of $\mathrm{O}(1)$. If $\epsilon = +$, then its first occurrence is $\mathrm{Mp}(0) = \{1\}$ (by convention), and so if $\Theta_{V^+,W,\psi}(\pi^+) \neq 0$, it has cuspidal support $(\chi_1, ..., \chi_n)$. On the other hand, if $\epsilon = -$, its first occurrence is Mp(2), where its theta lift there is an odd Weil representation ω_ψ^o, which is supercuspidal. If $\Theta_{V^+,W,\psi}(\pi^-) \neq 0$, then we must have (without loss of generality) $\chi_1 = \chi|-|^{\pm 1/2}$, with χ a quadratic character, and the cuspidal support of $\Theta_{V^+,W,\psi}(\pi^-)$ would be $(\omega_\psi^o, \chi_2, ..., \chi_n)$. In particular, in this case, χ_1 is not unitary.

Now if the χ_i's are unitary, the second option cannot happen. Moreover, for almost all unramified unitary χ_i's, $\pi = I_B(\chi_1, ..., \chi_n)$ is irreducible. Thus, for such unramified tempered π's, the first option must happen, i.e. $\Theta_{V^+,W,\psi}(\pi^+) \neq 0$, since we already know that exactly one of π^+ or π^- has nonzero theta lift to $\mathrm{Mp}(W)$.

Hence, when we consider the doubling zeta integral for associated to such π^+'s, the linear functional $Z(0)$ is nonzero when restricted to $\Theta_{W,\mathbb{V},\psi}(1) \otimes \pi^+ \otimes (\pi^+)^\vee$. Now examine the local functional equation:

$$Z(0) \circ M_\psi^*(0) = \epsilon(V^+) \cdot \epsilon(1/2, \pi, \psi) \cdot Z(0).$$

Since $\epsilon(V^+) = 1 = \epsilon(1/2, \pi)$ for unramified π, we conclude from the local functional equation that $M_\psi^*(0)$ acts as $+1$ on $\Theta_{W,\mathbb{V},\psi}(1)$. □

When π is non-tempered, we may express π as the unique irreducible submodule of an induced representation

$$I_P(\tau_1|\det|^{-s_1}, \tau_2|\det|^{-s_2}, ..., \tau_r|\det|^{-s_r}, \pi_0)$$

where the τ_i's are unitary discrete series representations of $\mathrm{GL}(k_i)$, π_0 is a tempered representation of $\mathrm{O}(V_0)$ (with $n = k_1 + ... + k_r + m$) and the numbers s_i's satisfy

$$s_1 \geq s_2 \geq ... \geq s_r > 0.$$

Then for $\epsilon = \pm$, we have

$$\pi^\epsilon \hookrightarrow I_P(\tau_1|\det|^{-s_1}, \tau_2|\det|^{-s_2}, ..., \tau_r|\det|^{-s_r}, \pi_0^{\epsilon_0})$$

with

$$\epsilon = \epsilon_0 \cdot \prod_{i=1}^r \tau_i(-1).$$

Moreover, by multiplicativity of standard epsilon factors [LR],

$$\epsilon(1/2, \pi, \psi) = \epsilon(1/2, \pi_0, \psi) \cdot \prod_{i=1}^r \tau_i(-1).$$

In view of the facts that $\epsilon(V) = \epsilon(V_0)$ and that the theorem has been proven for π_0, it remains to show that

$$\Theta_{V,W,\psi}(\pi^\epsilon) \neq 0 \Longrightarrow \Theta_{V_0,W_0,\psi}(\pi_0^{\epsilon_0}) \neq 0,$$

where $\dim V_0 = \dim W_0 + 1$.

First note that by induction in stages,

$$\pi^\epsilon \hookrightarrow I_{P(Y_k)}(\tau|\det|^{-s}, \pi_1^{\epsilon_1})$$

where $s>0$, τ is a unitary discrete series representation of $\mathrm{GL}(Y_k)$ and $\pi_1^{\epsilon_1}$ is an irreducible representation of $\mathrm{O}(V_{n-k})$ with $\epsilon=\epsilon_1\cdot\tau(-1)$. Now we have

$$0\neq\Theta_{V,W,\psi}(\pi^\epsilon)^*=\mathrm{Hom}_{\mathrm{O}(V)}(\Omega_{V,W,\psi},\pi^\epsilon)\hookrightarrow\mathrm{Hom}_{\mathrm{O}(V)}(\Omega_{V,W,\psi},I_{P(Y_k)}(\tau|\det|^{-s},\pi_1^{\epsilon_1}))$$

$$=\mathrm{Hom}_{M(Y_k)}(R_{P(Y_k)}(\Omega_{V,W,\psi}),\tau|\det|^{-s}\boxtimes\pi_1^{\epsilon_1})$$

where $R_{P(Y_k)}$ denotes the normalized Jacquet functor with respect to the parabolic $P(Y_k)$ with Levi subgroup $M(Y_k)=\mathrm{GL}(Y_k)\times\mathrm{O}(V_{n-k})$.

Now the normalized Jacquet module $R_{P(Y_k)}(\Omega_{V,W,\psi})$ has been computed by Kudla [K, Thm. 2.8]:

Proposition 7.3. *The normalized Jacquet module $R_{P(Y_k)}(\Omega_{V,W,\psi})$ has a $M(Y_k)\times\mathrm{Mp}(W)$-invariant filtration*

$$0\subset R^k\subset .....\subset R^1\subset R^0=R_{P(Y_k)}(\Omega_{V,W,\psi}),$$

with successive quotient (for $0\leq r\leq k$) given by:

$$J^r:=R^r/R^{r+1}\cong Ind_{Q(Y_{k-r},Y_k)\times\mathrm{O}(V_{n-k})\times P(X_r)}^{\mathrm{GL}(Y_k)\times\mathrm{O}(V_{n-k})\times\mathrm{Mp}(W)}S(Isom(Y_r',X_r))\otimes\Omega_{V_{n-k},W_{n-r},\psi}.$$

Here, we have:

(a) *$Q(Y_{k-r},Y_k)$ is the maximal parabolic subgroup of $\mathrm{GL}(Y_k)$ which stabilizes the flag*

$$0\subset Y_{k-r}\subset Y_k,$$

so that its Levi subgroup is $\mathrm{GL}(Y_{k-r})\times\mathrm{GL}(Y_r')$ with $Y_r'=\langle v_{k-r+1},...,v_k\rangle$;

(b) *$Isom(Y_r',X_r)$ is the set of invertible linear maps from Y_r' to X_r and $S(Isom(Y_r',X_r))$ is the space of locally constant compactly supported functions on $Isom(Y_r',X_r)$;*

(c) *the action of $\mathrm{GL}(Y_{k-r})\times\mathrm{GL}(Y_r')\times\mathrm{O}(V_{n-k})\times\mathrm{GL}(X_r)\times\mathrm{Mp}(W_{n-r})$ on $S(Isom(Y_r',X_r))\otimes\Omega_{V_{n-k},W_{n-r},\psi}$ is given by:*

  (i) *$\mathrm{GL}(Y_{k-r})$ acts by the character $|\det_{Y_{k-r}}|^{\frac{k-r}{2}}$;*

  (ii) *$(b,c)\in\mathrm{GL}(Y_r')\times\mathrm{GL}(X_r)$ acts on $S(Isom(Y_r',X_r))$ by*

$$(b,c)\varphi(t)=\varphi(c^{-1}tb),$$

  so that if we identify $\mathrm{GL}(Y_r')$ and $\mathrm{GL}(X_r)$ with $\mathrm{GL}(r)$ by the given bases on Y_r' and X_r, then this is simply the regular representation of $\mathrm{GL}(r)\times\mathrm{GL}(r)$ on $S(\mathrm{GL}(r))$;

  (iii) *$\mathrm{O}(V_{n-k})\times\mathrm{Mp}(W_{n-r})$ acts on $\Omega_{V_{n-k},W_{n-r},\psi}$ by the Weil representation associated to these data.*

In particular, the bottom piece of the filtration is:

$$R^k=Ind_{\mathrm{GL}(Y_k)\times\mathrm{O}(V_{n-k})\times P(X_k)}^{\mathrm{GL}(Y_k)\times\mathrm{O}(V_{n-k})\times\mathrm{Mp}(W)}S(Isom(Y_k',X_k))\otimes\Omega_{V_{n-k},W_{n-k},\psi}.$$

Using the notations in the above proposition, we have:

Lemma 7.4. *Suppose that*

$$Hom_{M(Y_k)}(R_{P(Y_k)}(\Omega_{V,W,\psi}), \tau|\det|^{-s} \boxtimes \pi_1^{\epsilon_1}) \neq 0$$

with $s \geq 0$ and τ a unitary discrete series representation of $\mathrm{GL}(Y_k)$*. Then for $r < k$,*

$$\mathrm{Hom}_{M(Y_k)}(J^r, \tau|\det|^{-s} \boxtimes \pi_1^{\epsilon_1}) = 0,$$

so that

$$\mathrm{Hom}_{M(Y_k)}(R^k, \tau|\det|^{-s} \boxtimes \pi_1^{\epsilon_1}) \neq 0.$$

In particular,

$$\mathrm{Hom}_{\mathrm{O}(V_{n-k})}(\Omega_{V_{n-k},W_{n-k},\psi}, \pi_1^{\epsilon_1}) \neq 0.$$

Proof. Assume that $r < k$ and write $Q = Q(Y_{k-r}, Y_k)$ for ease of notations. Then

$$\mathrm{Hom}_{M(Y_k)}(J^r, \tau|\det|^{-s} \boxtimes \pi_1^{\epsilon_1}) =$$

$$= \mathrm{Hom}_{\mathrm{GL}(Y_{k-r})\times\mathrm{GL}(Y_r')\times\mathrm{O}(V_{n-k},W_{n-r})}(S(\mathrm{Isom}(Y_r', X_r)) \otimes \Omega_{V_{n-k},W_{n-r}}, |\det_{Y_k}|^{-s} \cdot R_{\overline{Q}}(\tau) \boxtimes \pi_1^{\epsilon_1}).$$

Now since τ is a unitary discrete series representation, $R_{\overline{Q}}(\tau)$ is an irreducible discrete series representation $|\det|^{t_1}\tau_1 \boxtimes |\det|^{t_2}\tau_2$, with τ_i unitary and $t_i \in \mathbb{R}$ satisfying:

$$t_1 < t_2 \quad \text{and} \quad t_1 \cdot (k-r) + t_2 \cdot r = 0.$$

In particular, we must have

$$t_1 \leq 0.$$

Thus, on $|\det_{Y_k}|^{-s} \cdot R_{\overline{Q}}(\tau)$, the center of $\mathrm{GL}(Y_{k-r})$ acts by the character $|\det|^{-s+t_1}$ (up to a unitary character), whereas on $S(\mathrm{Isom}(Y_r', X_r)) \otimes \Omega_{V_{n-k},W_{n-r}}$, $\mathrm{GL}(Y_{k-r})$ acts by $|\det|^{(k-r)/2}$ by Prop. 7.3. Since

$$-s + t_1 \leq 0 \quad \text{and} \quad k - r > 0,$$

we deduce that the above Hom space must be 0. $\square$

Using this lemma, we deduce inductively that since

$$0 \neq \Theta_{V,W,\psi}(\pi^\epsilon) \subset \mathrm{Hom}_{\mathrm{O}(V)}(\Omega_{V,W,\psi}, I_P(\tau_1|\det|^{-s_1}, \tau_2|\det|^{-s_2}, ..., \tau_r|\det|^{-s_r}, \pi_0)),$$

we have

$$\mathrm{Hom}_{\mathrm{O}(V_0)}(\Omega_{V_0,W_0,\psi}, \pi_0^{\epsilon_0}) \neq 0,$$

so that $\Theta_{V_0,W_0,\psi}(\pi_0^{\epsilon_0}) \neq 0$, as desired.

We have thus completed the proof of Thm. 7.1(i). This also allows us to deduce one of the equalities in Thm. 7.1(ii). Indeed, from the definition of the Weil representation given in (3.1), one sees that the action of $-1 \in \mathrm{O}(V)$ on $\Omega_{V,W,\psi}$ differs from that of the central element $(-1, 1) \in \mathrm{Mp}(W)$ by $\chi_\psi(-1)$. Thus, if $\Theta_\psi(\sigma) = \pi \in \mathrm{Irr}(\mathrm{SO}(V))$, then the result of (i) implies that the central element $(-1, 1) \in \mathrm{Mp}(W)$ must act on σ by

$$\chi_\psi(-1) \cdot \epsilon(V) \cdot \epsilon(1/2, \pi).$$

Thus, the central sign of σ is

$$z_\psi(\sigma) = \epsilon(V) \cdot \epsilon(1/2, \pi) = \epsilon(V) \cdot \epsilon(1/2, \Theta_\psi(\sigma)).$$

Thus we have established the analog of Waldspurger's Thm. 5.2.

To complete the proof of Thm. 7.1(ii), we need to show that

$$z_\psi(\sigma) = \epsilon(V) \cdot \epsilon(1/2, \sigma, \psi).$$

This is equivalent to $\epsilon(1/2, \sigma, \psi) = \epsilon(1/2, \Theta_\psi(\sigma))$, which is itself a consequence of the identity $\epsilon(s, \sigma, \psi) = \epsilon(s, \Theta_\psi(\sigma), \psi)$ which we will show in §11.

However, we could also give a proof of the desired displayed identity by an argument analogous to that for (i). Such a proof of this result has been given by Zorn [Z], though the notion of epsilon factors $\epsilon(s, \sigma, \psi)$ used in his paper differs from ours. More precisely, his definition of the standard L-factors and epsilon factors for $\mathrm{Mp}(W)$ is based on the approach of "good sections". The problem with such an approach is that one does not know how to show that these local factors are multiplicative when they should be.

In any case, let us give a sketch of the proof of the remaining part of (ii) here: it is merely a mirror reflection of the argument of (i). Assume first that σ is tempered. In this case, the doubling zeta integral $Z(s)$ for $\mathrm{Mp}(W) \times_{\mu_2} \mathrm{Mp}(W) \subset \mathrm{Mp}(W + (-W))$ is holomorphic at $s = 0$ and so is the local L-factor $L(s, \sigma, \psi)$. Moreover, we know that $\Theta_{W,V,\psi}(\sigma) \neq 0$ if and only if the linear form

$$Z(0) : I_{\tilde{P}(W^\Delta)}(0) \otimes \sigma^\vee \otimes \sigma \longrightarrow \mathbb{C}$$

is nonzero when restricted to the submodule $\Theta_{V,\mathbb{W},\psi}(1)$. On the other hand, the local functional equation of the doubling zeta integral reads:

$$Z(0) \circ M_\psi^*(0) = z_\psi(\sigma) \cdot \epsilon(1/2, \sigma, \psi) \cdot Z(0).$$

Now suppose that $M_\psi^*(0)$ acts by the sign $\alpha_\epsilon = \pm 1$ on the submodule $\Theta_{V^\epsilon,\mathbb{W},\psi}(1)$ of $I_{\tilde{P}(W^\Delta)}(0)$. Then the local functional equation shows that

$$\Theta_{W,V^\epsilon,\psi}(\sigma) \neq 0 \Longleftrightarrow \alpha_\epsilon = z_\psi(\sigma) \cdot \epsilon(1/2, \sigma, \psi).$$

Hence, it remains to show the following analog of Lemma 7.2:

Lemma 7.5. *The normalized intertwining operator $M_\psi(0)^*$ acts by $+1$ on $\Theta_{V^+,\mathbb{W},\psi}(1)$ and by -1 on $\Theta_{V^-,\mathbb{W},\psi}(1)$.*

Proof. When p is odd, this was shown by Zorn [Z], who proved a Gindikin-Karpelevich type formula by direct computation. However, one can give a proof which works for all p by making use of information from the theta correspondence, based on our discussion before the lemma.

More precisely, to show that $\alpha_\epsilon = \epsilon$, it suffices to find a representation σ of $\mathrm{Mp}(W)$ which participates in theta correspondence with $\mathrm{O}(V^\epsilon)$ and verify for this σ that

$$\epsilon = z_\psi(\sigma) \cdot \epsilon(1/2, \sigma, \psi).$$

When $\epsilon = +1$, one simply takes

$$\sigma = I_{B,\psi}(\chi_1, ... \chi_n)$$

where B is a Borel subgroup of $\mathrm{Mp}(W)$ and each χ_i is an unramified unitary character of $k^\times$. For generic choices of χ_i, we know by Kudla' cuspidal support theorem that such a σ participates in theta correspondence with $\mathrm{O}(V^+)$, and it follows by multiplicativity that

$$z_\psi(\sigma) = \epsilon(1/2, \sigma, \psi) = 1.$$

When $\epsilon = -1$, one takes

$$\sigma = I_{P,\psi}(\chi_1, ...\chi_{n-1}, \mathrm{St}_\psi)$$

where P is a parabolic subgroup with Levi factor $(\mathrm{GL}_1)^{n-1} \times \mathrm{Mp}_2$, each χ_i is an unramified unitary character of $k^\times$ and St_ψ is the Steinberg representation of $\mathrm{Mp}_2(k)$ with respect to ψ (see the example in §5). For generic choices of χ_i, one knows by Kudla's cuspidal support theorem ([Ku, Thm. 2.5] and [Ku2, Thms. 7.1 and 7.2]) and the example in §5 that σ participates in theta correspondence with $\mathrm{O}(V^-)$ and

$$\Theta_{\psi,V^-,W}(\sigma) = I_Q(\chi_1, ..., \chi_{n-1}, 1_{\mathrm{O}(V_1^-)}),$$

where Q is the parabolic subgroup of $\mathrm{SO}(V^-)$ with Levi factor $(\mathrm{GL}_1)^{n-1} \times \mathrm{SO}(V_1^-)$. Further, it is easy to check that

$$z_\psi(\sigma) = 1 \quad \text{and} \quad \epsilon(1/2, \sigma, \psi) = -1.$$

This proves the lemma. $\square$

Together with the local functional equation, the lemma implies immediately that $Z(0)$ is non-zero on $\Theta_{V,\mathbb{W},\psi}(1)$ if and only if

$$\epsilon(V) = z_\psi(\sigma) \cdot \epsilon(1/2, \sigma, \psi),$$

thus proving (ii) when σ is tempered.

Suppose now that σ is non-tempered. Then by [BJ, Thm. 4.1],

$$\sigma \hookrightarrow I_{\tilde{P},\psi}(\tau_1|-|^{-s_1}, ..., \tau_r|-|^{-s_r}; \sigma_0)$$

where τ_i is a unitary discrete series representation of $\mathrm{GL}(k_i)$, σ_0 is a tempered representation of $\mathrm{Mp}(W_0)$ and

$$s_1 \geq s_2 \geq ... \geq s_r > 0.$$

Moreover, the central signs of σ and σ_0 are related by

$$z_\psi(\sigma) = z_\psi(\sigma_0) \cdot \prod_{i=1}^r \tau_i(-1).$$

Similarly, by multiplicativity, the epsilon factor satisfies

$$\epsilon(1/2, \sigma, \psi) = \epsilon(1/2, \sigma_0, \psi) \cdot \prod_{i=1}^r \tau_i(-1).$$

Since we already know the desired result for the tempered representation σ_0, it remains to show that

$$\Theta_{W,V,\psi}(\sigma) \neq 0 \Longrightarrow \Theta_{W_0,V_0,\psi}(\sigma_0) \neq 0,$$

where $\dim V_0 = \dim W_0 + 1$. For this, the argument proceeds by a Jacquet module computation, analogous to the proof of (i); we omit the details.

8. Discrete Series and Langlands Data

The purpose of this section is to prove Theorem 1.3(i)-(iii). We first show the following crucial result about the theta correspondence, which holds for all primes p.

Theorem 8.1. *Let $\pi \in Irr(\mathrm{O}(V))$ and suppose that its big theta lift $\Theta_{\psi,V,W}(\pi)$ on $\mathrm{Mp}(W)$ is nonzero.*

(i) If π is a discrete series representation, then $\Theta_{\psi,V,W}(\pi)$ is a direct sum of irreducible discrete series representations of $\mathrm{Mp}(W)$. In particular, when $p \neq 2$, $\Theta_{\psi,V,W}(\pi) = \theta_{\psi,V,W}(\pi)$ is an irreducible discrete series representation.

(ii) Let $\pi \in Irr(\mathrm{O}(V))$ be tempered and suppose that

$$\pi \subset I_Q(\tau_1, ..., \tau_r, \pi_0),$$

where Q is a parabolic subgroup of $\mathrm{O}(V)$ with Levi subgroup $\mathrm{GL}_{n_1} \times ... \times \mathrm{GL}_{n_r} \times \mathrm{O}(V_0)$, the τ_i's are unitary discrete series representations of GL_{n_i}, and π_0 is a discrete series representation of $\mathrm{O}(V_0)$. Then

$$\Theta_{\psi,V,W}(\pi) \subset I_{\tilde{P}}(\tau_1, ..., \tau_r, \Theta_{\psi,V_0,W_0}(\pi_0)),$$

where $\tilde{P}$ is the parabolic subgroup of $\mathrm{Mp}(W)$ with Levi subgroup $\tilde{\mathrm{GL}}_{n_1} \times_{\mu_2} ... \times_{\mu_2} \tilde{\mathrm{GL}}_{n_r} \times \mathrm{Mp}(W_0)$. In particular, $\Theta_{\psi,V,W}(\pi)$ is a direct sum of irreducible tempered representations, and when $p \neq 2$, $\Theta_{\psi,V,W}(\pi) = \theta_{\psi,V,W}(\pi)$ is irreducible.

(iii) More generally, suppose that

$$\pi = J_Q(\tau_1|\det|^{s_1}, ..., \tau_r|\det|^{s_r}, \pi_0), \quad s_1 > s_2 > .... > s_r > 0$$

is a Langlands quotient of $\mathrm{O}(V)$, where Q is as in (ii), the τ_i's are unitary tempered representations of GL_{n_i}, and π_0 is a tempered representation of $\mathrm{O}(V_0)$. Then

$$I_{\tilde{P}}(\tau_1|\det|^{s_1}, ..., \tau_r|\det|^{s_r}, \Theta_{\psi,V_0,W_0}(\pi_0)) \twoheadrightarrow \Theta_{\psi,V,W}(\pi).$$

where $\tilde{P}$ is the parabolic subgroup of $\mathrm{Mp}(W)$ with Levi subgroup $\tilde{\mathrm{GL}}_{n_1} \times_{\mu_2} ... \times_{\mu_2} \tilde{\mathrm{GL}}_{n_r} \times \mathrm{Mp}(W_0)$. In particular, when $p \neq 2$, $\theta_{\psi,V,W}(\pi)$ is the unique Langlands quotient of the standard module $I_{\tilde{P}}(\tau_1|\det|^{s_1}, ..., \tau_r|\det|^{s_r}, \theta_{\psi,V_0,W_0}(\pi_0))$

The analogous assertions in (i), (ii) and (iii) hold if one starts with $\sigma \in Irr(\mathrm{Mp}(W))$ and considers its big theta lift $\Theta_{\psi,V,W}(\sigma)$.

Proof. The proof of this follows that of an analogous theorem of Muić [M, Thm. 4.1], but with significant simplifications. Before going to the proof, let us note a lemma which will be frequently used in the proof and is a direct consequence of the Casselman square-integrability criterion (see [BJ, Thm 3.4] for the case of covering groups).

Lemma 8.2. *Let π be a discrete series representation of $\mathrm{O}(V)$ and let $Q = Q(Y_t)$ be a maximal parabolic subgroup of $\mathrm{O}(V)$ with Levi factor $\mathrm{GL}(Y_t) \times \mathrm{O}(V_0)$. On any irreducible constituent of the normalized Jacquet module $R_Q(\pi)$, the center of $\mathrm{GL}(Y_t)$ acts by a character of the form $\chi \cdot |-|^{\alpha}$ where χ is unitary and $\alpha > 0$.*

The analogous result holds for discrete series representations of $\mathrm{Mp}(W)$.

(i) Pick any $\mathrm{Mp}(W)$-equivariant filtration

$$0 \subset \Sigma_r \subset \Sigma_{r-1} \subset \ldots \subset \Sigma_1 = \Theta_{\psi,V,W}(\pi)$$

whose successive quotients

$$\Pi_i = \Sigma_i / \Sigma_{i+1}$$

are irreducible. We shall argue by contradiction that each of these successive quotients is square-integrable. By Casselman's square-integrability criterion [BJ, Thm. 3.4], this will show that the representation $\Theta_{\psi,V,W}(\pi)$ is itself square-integrable, and thus is semisimple. Indeed, it is a basic result of Harish-Chandra that the irreducible discrete series representations are projective objects in the category of admissible tempered representations.

Suppose then that k is the smallest index such that $\Pi_k = \Sigma_k/\Sigma_{k+1}$ is non-square-integrable. Then one has

$$\Pi_k \hookrightarrow I_{\tilde{P}(X_t),\psi}(\tau|\det_{X_t}|^{-s}, \sigma_0)$$

where τ is a unitary discrete series representation of $\mathrm{GL}(X_t)$, $s \geq 0$ and π_0 is an irreducible representation of $\mathrm{Mp}(W_0)$. Here, $\dim W_0 + 2t = \dim W$. To ease notation, let us write P in place of $\tilde{P}(X_t)$. Then by Frobenius reciprocity, one has

$$R_P(\Pi_k) \twoheadrightarrow \tau|\det|^{-s} \boxtimes \sigma_0.$$

Now by the exactness of Jacquet modules, one has

$$0 \subset R_P(\Sigma_r) \subset \ldots. \subset R_P(\Sigma_1)$$

with

$$R_P(\Sigma_i)/R_P(\Sigma_{i+1}) = R_P(\Pi_i).$$

Thus one obtains a short exact sequence of representations of $\mathrm{GL}(X_t) \times \mathrm{Mp}(W_0)$:

$$0 \longrightarrow \tau|\det|^{-s} \boxtimes \sigma_0 \longrightarrow A \longrightarrow B \longrightarrow 0$$

where A is a quotient of $R_P(\Theta_{\psi,V,W}(\pi))$ and B is a finite length representation equipped with a filtration with successive quotients $R_P(\Pi_i)$ for $i < k$. Now the key observation is that this short exact sequence splits.

To see this, note that for $i < k$, each Π_i is square-integrable by assumption. Lemma 8.2 implies that on each irreducible constituent of $R_P(\Pi_i)$ $(i < k)$, the center of $\mathrm{GL}(X_t)$ acts by a character of the form $\chi \cdot |-|^{\alpha}$ with $\alpha > 0$ and χ unitary. Since the center of $\mathrm{GL}(X_t)$ acts on $\tau|\det|^{-s} \boxtimes \sigma_0$ by $|-|^{-st}$ (up to a unitary character), we conclude that the above short exact sequence splits, so that one has a nonzero map

$$R_P(\Theta_{\psi,V,W}(\pi)) \longrightarrow \tau|\det|^{-s} \boxtimes \sigma_0.$$

By Frobenius reciprocity, one obtains a nonzero map

$$\Omega_{\psi,V,W} \twoheadrightarrow \pi \boxtimes \Theta_{\psi,V,W}(\pi) \longrightarrow \pi \boxtimes I_P(\tau|\det|^{-s}, \sigma_0),$$

so that

$$\pi^* \hookrightarrow \mathrm{Hom}_{\mathrm{Mp}(W)}(\Omega_{\psi,V,W}, I_P(\tau|\det|^{-s}, \sigma_0)) = \mathrm{Hom}_{\mathrm{GL}(X_t)\times\mathrm{Mp}(W_0)}(R_P(\Omega_{\psi,V,W}), \tau|\det|^{-s} \boxtimes \sigma_0),$$

where π^* denotes the full linear dual of π. By the analogs of Prop. 7.3 and Lemma 7.4 (with the roles of $\mathrm{O}(V)$ and $\mathrm{Mp}(W)$ exchanged), one concludes that

$$\mathrm{Hom}_{\mathrm{GL}(X_t)\times\mathrm{Mp}(W_0)}(R_P(\Omega_{\psi,V,W}), \tau|\det|^{-s}\boxtimes\sigma_0) = I_{Q(Y_t)}(\tau^\vee|\det_{Y_t}|^s, \Theta_{\psi,V_0,W_0}(\sigma_0))^*.$$

Thus, one has

$$\pi^\vee \hookrightarrow I_{Q(Y_t)}(\tau|\det_{Y_t}|^{-s}, \Theta_{\psi,V_0,W_0}(\sigma_0)^\vee),$$

so that there is a nonzero map

$$R_{Q(Y_t)}(\pi^\vee) \twoheadrightarrow \tau|\det_{Y_t}|^{-s}\boxtimes\pi_0$$

for some irreducible representation π_0 of $\mathrm{O}(V_0)$ and with $s \geq 0$. Since π and hence $\pi^\vee$ are square-integrable by assumption, this contradicts Lemma 8.2. With this contradiction, (i) is proved.

(ii) Suppose that

$$\pi \subset I_Q(\tau_1, ..., \tau_r, \pi_0)$$

is tempered, as in the statement of (ii). Using Proposition 7.3 and Lemma 7.4 and arguing as in (i), one sees that

$$I_{\tilde{P}}(\tau_1, ..., \tau_r, \Theta_\psi(\pi_0^{\epsilon_0})) \twoheadrightarrow \Theta_{\psi,V,W}(\pi^\epsilon).$$

By (i), $\Theta_\psi(\pi_0)$ is a direct sum of irreducible discrete series representations. This proves (ii).

(iii) This is similar to (ii). Suppose that

$$\pi = J_Q(\tau_1|\det|^{s_1}, ..., \tau_r|\det|^{s_r}, \pi_0), \quad s_1 > s_2 > .... > s_r > 0.$$

Then

$$\pi \hookrightarrow I_P(\tau_1^\vee|\det|^{-s_1}, \tau_2^\vee|\det|^{-s_2}, ..., \tau_r^\vee|\det|^{-s_r}, \pi_0).$$

Using Proposition 7.3 and Lemma 7.4, one sees that

$$I_{\tilde{P}}(\tau_1|\det|^{s_1}, ..., \tau_r|\det|^{s_r}, \Theta_\psi(\pi_0)) \twoheadrightarrow \Theta_{\psi,V,W}(\pi^\epsilon).$$

This proves (iii).

□

The above theorem implies Theorem 1.3(i), (ii) and (iii). We recall that the equality of formal degrees described in Theorem 1.3(iv) is one of the main results of [GI] (with an input from [GS2]); we do not prove (iv) in this paper.

Remarks: When $p \neq 2$, a different proof of the fact that $\pi \in \mathrm{Irr}(\mathrm{SO}(V))$ is discrete series if and only if $\sigma = \theta_{\psi,V,W}(\pi) \in \mathrm{Irr}(\mathrm{Mp}(W))$ is discrete series can be found in [GI]. However, the proof in [GI] does not show the equality

$$\Theta_{\psi,V,W}(\pi) = \theta_{\psi,V,W}(\pi),$$

when π is discrete series. This equality is necessary to establish the results in Theorem 8.1(ii) and (iii).

We also note the following corollary:

Corollary 8.3. *Suppose that $\tau_1,\ldots,\tau_r$ are discrete series representations of GL_{n_i}, and π_0 and $\sigma_0 = \Theta_\psi(\pi_0)$ are discrete series representations of $\mathrm{SO}(V_0)$ and $\mathrm{Mp}(W_0)$ respectively. Then, when $p \neq 2$, the induced representations $I_Q(\tau_1,\ldots,\tau_r,\pi_0)$ and $I_{P,\psi}(\tau_1,\ldots,\tau_r,\sigma_0)$ have the same number of irreducible summands (up to equivalence and ignoring multiplicities).*

Proof. This is an immediate consequence of Thm 8.1(ii). □

9. **Generic Representations**

In this section, we study how the bijection Θ_ψ treats the subset of generic representations. In particular, we prove Theorem 1.3(v).

Let U be the unipotent radical of a Borel subgroup $B = T \cdot U$ of $\mathrm{SO}(V^+)$ and let λ be any generic character of U. Any two such generic characters are in the same orbit under the adjoint action of the maximal torus T, so the choice of λ is not important. By definition, a representation π of $\mathrm{SO}(V^+)$ is generic if $\mathrm{Hom}_U(\pi,\lambda) \neq 0$.

Similarly, let U' be the unipotent radical of a Borel subgroup $\tilde{B}' = \tilde{T}' \cdot U'$ of $\mathrm{Mp}(W)$. The T'-orbits of generic characters of U' are naturally indexed by non-trivial characters of k modulo the action of $k^{\times 2}$ (see [GGP, §12]). Thus the additive character ψ of k gives rise to a T'-orbit of generic characters λ'_ψ of U'. A representation σ of $\mathrm{Mp}(W)$ is said to be ψ-generic if $\mathrm{Hom}_{U'}(\sigma,\lambda'_\psi) \neq 0$.

The following is Theorem 1.3(iv):

Theorem 9.1. *If π is a generic representation of $\mathrm{SO}(V^+)$, then $\sigma = \Theta_\psi(\pi)$ is a ψ-generic representation of $\mathrm{Mp}(W)$. If σ is ψ-generic and tempered, then π is generic.*

The theorem is a consequence of the computation of the Whittaker modules of the Weil representation $\Omega_{\psi,V,W}$:

Proposition 9.2. *Using the above notations,*

$$(\Omega_{V^+,W,\psi})_{U,\lambda} \cong \mathrm{ind}_{U'}^{\mathrm{Mp}(W)} \lambda'_\psi.$$

Proof. See [MS1] and [Fu] for an analogous computation. We omit the details. □

Note that the above proposition uniquely specifies the T'-orbit of λ'_ψ without recourse to [GGP, §12]. The following corollary of Proposition 9.2 establishes Theorem 1.3(iv).

Corollary 9.3. *Suppose that p is odd.*

(i) Let $\pi \in Irr(\mathrm{SO}(V^+)$ be generic. Then $\sigma = \Theta_\psi(\pi)$ is ψ-generic.

(ii) Let $\sigma \in Irr(\mathrm{Mp}(W))$ be ψ-generic. Then the big theta lift $\Theta_{\psi,V,W}(\sigma)$ of σ to $\mathrm{O}(V^+)$ has a unique generic constituent. In particular, if σ is tempered, then $\pi = \Theta_\psi(\sigma)$ is generic.

Remark: In (ii) above, it is not true that if $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$ is ψ-generic, then $\pi = \Theta_\psi(\sigma)$ is generic. Indeed, a counterexample can already be found when $n = 1$. In that case, if

$\sigma = \omega_\psi^e$ is the even Weil representation of $\mathrm{Mp}(W)$ associated to ψ, then σ is ψ-generic, but $\pi = \Theta_\psi(\sigma)$ is the trivial representation of $\mathrm{SO}(V^+) \cong \mathrm{PGL}_2(k)$.

Consider now the dual pair

$$\mathrm{SO}(V_n^+) \times \mathrm{Mp}(W_{n-1}),$$

and the associated Weil representation $\Omega_{V_n^+, W_{n-1},\psi}$. By a computation similar to the proof of Proposition 9.2, one has:

Proposition 9.4. *As a representation of* $\mathrm{Mp}(W_{n-1})$*,*

$$(\Omega_{V_n^+, W_{n-1},\psi})_{U,\lambda} = 0.$$

Corollary 9.5. *If π is a generic representation of* $\mathrm{SO}(V^+)$*, then π does not participate in the theta correspondence with* $\mathrm{Mp}(W_{n-1}) = \mathrm{Mp}_{2n-2}$*. In particular, if π is generic and supercuspidal, then $\Theta_{V^+,W,\psi}(\pi)$ is an irreducible supercuspidal ψ-generic representation of* $\mathrm{Mp}(W)$*.*

10. **Plancherel Measures**

In this section, we shall see that the bijection Θ_ψ respects a family of invariants known as the Plancherel measures attached to representations of $\mathrm{SO}(V) \times \mathrm{GL}_r$ and $\mathrm{Mp}(W) \times \mathrm{GL}_r$, thus establishing Theorem 1.3(vi). We begin by recalling the definition of the Plancherel measure. For the basic properties of Plancherel measures that will be used this section, we refer the reader to [GI, Appendix B].

Suppose that π is an irreducible representation of $\mathrm{SO}(V)$ and ρ is an irreducible representation of GL_r. Since $L_r = \mathrm{SO}(V) \times \mathrm{GL}_r$ is a Levi subgroup of a parabolic $Q_r = L_r \cdot U_r$ of $\mathrm{SO}(V_{n+r})$, one has the induced representation

$$I_{Q_r}(s, \pi \boxtimes \rho) = I_{Q_r}(|\det_{\mathrm{GL}_r}|^s, \pi \boxtimes \rho).$$

If $\bar{Q}_r = L_r \cdot \bar{U}_r$ is the opposite parabolic, then we similarly have the induced representation $I_{\bar{Q}_r}(s, \pi \boxtimes \rho)$. Then there is a standard intertwining operator

$$A_\psi(s, \pi \boxtimes \rho, U_r, \bar{U}_r) : I_{Q_r}(s, \pi \boxtimes \rho) \longrightarrow I_{\bar{Q}_r}(s, \pi \boxtimes \rho).$$

The composite $A_\psi(s, \pi \boxtimes \rho, \bar{U}_r, U_r) \circ A_\psi(s, \pi \boxtimes \rho, U_r, \bar{U}_r)$ is a scalar operator on $I_{Q_r}(s, \pi \boxtimes \rho)$ and the Plancherel measure is the scalar-valued meromorphic function defined by

$$\mu(s, \pi \boxtimes \rho, \psi)^{-1} = A_\psi(s, \pi \boxtimes \rho, \bar{U}_r, U_r) \circ A_\psi(s, \pi \boxtimes \rho, U_r, \bar{U}_r).$$

Similarly, if σ is an irreducible representation of $\mathrm{Mp}(W)$ and ρ is an irreducible representation of GL_r, then one may define the associated Plancherel measure $\mu(s, \sigma \times \rho, \psi)$, as the composition of two standard intertwining operators as above.

The factorization of the intertwining operators into the product of "rank 1" operators corresponding to simple reflections implies an inductive property of the Plancherel measures known as multiplicativity. This is described in [GI, Appendix B], along with other properties of the Plancherel measure.

The main result of this section is:

Proposition 10.1. *If π is an irreducible representation of* $\mathrm{SO}(V)$ *with* $\sigma = \Theta_\psi(\pi)$ *and* ρ *is an irreducible representation of* GL_r*, then one has*

$$\mu(s, \pi \times \rho, \psi) = \mu(s, \sigma \times \rho, \psi).$$

Corollary 10.2. *Suppose that both π and $\sigma = \Theta_\psi(\pi)$ are supercuspidal. Then for any supercuspidal representation ρ of* GL_r*, $I_{Q_r}(s, \pi \boxtimes \rho)$ reduces if and only if $I_{P_r,\psi}(s, \sigma \boxtimes \rho)$ reduces. In particular, when $\rho^\vee = \rho$, $I_{P_r,\psi}(s, \sigma \boxtimes \rho)$ reduces for a unique $s \geq 0$.*

To infer this corollary, one only needs to take into account of [GI, Prop. B.7]. In particular, the corollary gives an alternative proof of a special case of the main results of [HM2]. In [GI, Thm. 12.1and Cor. 12.2], an identity of Plancherel measures as in Prop. 10.1 was established for the theta correspondence associated to a general dual pair $\mathrm{Mp}(2m) \times \mathrm{SO}(2n+1)$, which gives an alternative proof of the general results of [HM2].

The rest of this section is devoted to the proof of the proposition. We first draw two consequences of the property of multipicativity [GI, Props. B.4 and B.5] and Kudla's cuspidal support theorem ([Ku, Thm. 2.5] and [Ku2, Thms. 7.1 and 7.2]):

(i) The desired identity of Plancherel measure in Proposition 10.1 holds when $V = V^+$ and π and ρ have nonzero Iwahori-fixed vectors, where the Iwahori subgroup in question is the setwise stabilizer of a fundamental chamber in the building of $\mathrm{SO}(V)$. Indeed, such a π is contained in a principal series representation induced from a Borel subgroup of $\mathrm{SO}(V)$:

$$\pi \subset I_B(\mu_1, ..., \mu_n).$$

Kudla's cuspidal support theorem ([Ku, Thm. 2.5] and [Ku2, Thms. 7.1 and 7.2]) then implies that σ is a subquotient of the principal series representation $I_{B',\psi}(\mu_1, ..., \mu_n)$ of $\mathrm{Mp}(W)$. Using [GI, Prop. B.6], which expresses the relevant Plancherel measures in terms of Tate's γ-factors, it is then easy to establish the desired identity in this case.

(ii) Proposition 10.1 is reduced to the case when π and ρ are both supercuspidal.

In the basic case where π and ρ are both supercuspidal, the proof is via a global-to-local argument.

More precisely, we can find the following data:

- F is a totally complex number field, with two places v_0 and v_1 such that $F_{v_i} \cong k$;
- Ψ is an additive character of $F \backslash \mathbb{A}$ (where $\mathbb{A}$ is the ring of adeles of F) such that $\Psi_{v_0} = \Psi_{v_1} = \psi$;
- $\mathbb{V}$ is a quadratic space over F of dimension $2n+1$ and discriminant 1 such that $\mathbb{V} \otimes_F F_{v_i} \cong V$ for $i = 0$ or 1; moreover, we may assume that $\epsilon(\mathbb{V} \otimes F_v) = +$ for all finite places v outside v_0 and v_1.

Given the above data, one can find cuspidal representations Π of $\mathrm{O}(\mathbb{V})$ and Ξ of $\mathrm{GL}_r(\mathbb{A})$ such that

- $\Pi_{v_0} = \Pi_{v_1} = \pi^\epsilon$ and $\Xi_{v_0} = \Xi_{v_1} = \rho$, where π^ϵ is the unique extension of π to $\mathrm{O}(V)$ which participates in the theta correspondence with $\mathrm{Mp}(W)$;
- for all finite $v \neq v_0$ or v_1, Π_v and Ξ_v are unramified;

The simplest way to find such a Π is to use a construction of Henniart [He, Appendice 1] via Poincare series; in the proof of [He, Appendice 1, Thm.], it suffices to pick the test function f_v to be the characteristic function of a hyperspecial maximal compact subgroup at all finite places $v \neq v_0$ or v_1 (and the test functions f_{v_0} and f_{v_1} to be matrix coefficients of π). In the following, we shall write Π_v to denote the restriction of the representation Π_v of $\mathrm{O}(\mathbb{V}_v)$ to $\mathrm{SO}(\mathbb{V}_v)$.

Now consider the global theta lift $\Theta_{\mathbb{V},\mathbb{W}_k,\Psi}(\Pi)$ of Π to the tower of metaplectic groups $\mathrm{Mp}(\mathbb{W}_k)$. Let k be the first index such that the global theta lift $\Theta_{\mathbb{V},\mathbb{W}_k,\Psi}(\Pi)$ is nonzero. Suppose that Σ is an irreducible summand of $\Theta_{\mathbb{V},\mathbb{W}_k,\Psi}(\Pi)$. Then for all places v, one can relate Σ_v with $\Theta_{\psi_v}(\Pi_v)$. Indeed, for all finite $v \neq v_0$ or v_1, since Π_v is unramified, one knows how to compute the theta lift of Π_v to any $\mathrm{Mp}(W_k)$. Similarly, one understands the theta correspondence for complex groups completely. At the places v_0 and v_1, where $\Pi_{v_i} = \pi^\epsilon$ is supercuspidal, one can appeal to the cuspidal support theorem of Kudla [Ku]. The following lemma summarizes these results:

Lemma 10.3. *Write $k = n + t$ with $t \in \mathbb{Z}$.*

(i) If $k > n$ so that $t > 0$, then

$$\Sigma_v \subset I(|-|^{1/2} \boxtimes |-|^{3/2} \boxtimes .... \boxtimes |-|^{(2t-1)/2} \boxtimes \Theta_{\psi_v}(\Pi_v)).$$

(ii) If $k = n$, then $\Sigma_v = \Theta_{\psi_v}(\Pi_v)$.

(iii) If $k < n$, so that $t < 0$, then

$$\Theta_{\psi_v}(\Pi_v) \subset I(|-|^{1/2} \boxtimes |-|^{3/2} \boxtimes .... \boxtimes |-|^{(2|t|-1)/2} \boxtimes \Sigma_v).$$

As a consequence of the multiplicative property of the Plancherel measure [GI, Prop. B.4 and B.5], we deduce that for all places v,

(10.4)
$$\frac{\mu(s, \Sigma_v \boxtimes \Xi_v, \Psi_v)}{\mu(s, \Theta_{\psi_v}(\Pi_v) \boxtimes \Xi_v, \Psi_v)} = \begin{cases} \prod_{i=1}^{t} \mu(s, \Xi_v \boxtimes |-|^{(2i-1)/2}) \cdot \mu(s, \Xi_v \boxtimes |-|^{-(2i-1)/2}); \\ 1 \\ \left(\prod_{i=1}^{|t|} \mu(s, \Xi_v \boxtimes |-|^{(2i-1)/2}) \cdot \mu(s, \Xi_v \boxtimes |-|^{-(2i-1)/2})\right)^{-1} \end{cases}$$

in the three respective cases of the lemma. Now for $v \neq v_0$ or v_1, we already know that

$$\mu(s, \Theta_{\psi_v}(\Pi_v) \boxtimes \Xi_v, \Psi_v) = \mu(s, \Pi_v \boxtimes \Xi_v, \Psi_v).$$

To prove this identity for the places v_0 and v_1, and thus completing the proof of Proposition 10.1, it suffices to show that for $v = v_0$ or v_1, $\mu(s, \Sigma_v \boxtimes \Xi_v, \Psi_v)/\mu(s, \Pi_v \boxtimes \rho_v, \Psi_v)$ is given by the same formulas as in (10.4).

For this, we appeal to the global functional equation for Plancherel measures [GI, Prop. B.8] to the representation $\Sigma \boxtimes \Xi$ and $\Pi \boxtimes \Xi$. We deduce that for any finite set S of places of

F containing v_0 and v_1, $\mu_S(s, \Sigma_v \boxtimes \Xi_v, \Psi_v)/\mu_S(s, \Pi_v \boxtimes \Xi_v, \Psi_v)$ is a given in (10.4). Thus, we conclude that

$$\mu(s, \pi \boxtimes \rho, \psi)^2 = \mu(s, \Theta_{\psi_v}(\pi) \boxtimes \rho, \psi)^2.$$

Since the Plancherel measures are ≥ 0 on the imaginary axis [GI, Prop. B.7], we have:

$$\mu(s, \pi \boxtimes \rho, \psi) = \mu(s, \Theta_{\psi_v}(\pi) \boxtimes \rho, \psi).$$

This completes the proof of Proposition 10.1.

11. **Local Factors**

In this section, we show that the bijection Θ_ψ respects γ-factors, L-factors and ϵ-factors associated to representations of $\mathrm{Mp}(W)$ and $\mathrm{SO}(V)$. We assume the following working hypotheses:

Working Hypotheses:

(i) there is a theory of γ-factors $\gamma(s, \pi \times \rho, \psi)$ for irreducible representations $\pi \boxtimes \rho$ of $\mathrm{SO}(V) \times \mathrm{GL}_r$;

(ii) there is a theory of γ-factors $\gamma(s, \sigma \times \rho, \psi)$ for irreducible representations $\sigma \boxtimes \rho$ of $\mathrm{Mp}(W) \times \mathrm{GL}_r$,

Moreover, the theories of γ-factors satisfy the following conditions:

(a) (Multiplicativity) If $\pi = \mathrm{Ind}_Q^{\mathrm{SO}(V)} \tau \boxtimes \pi_0$, with τ a representation of GL_k and π_0 a representation of $\mathrm{SO}(V_0)$, then

$$\gamma(s, \pi \times \rho, \psi) = \gamma(s, \tau \times \rho, \psi) \cdot \gamma(s, \tau^\vee \times \rho, \psi) \cdot \gamma(s, \pi_0 \times \rho, \psi),$$

where the first two γ-factors on the RHS are the Rankin-Selberg γ-factors of $\mathrm{GL}_k \times \mathrm{GL}_r$. If $\rho = \mathrm{Ind}_R \rho_1 \times \rho_2$, with ρ_i an irreducible representation of GL_{r_i}, then

$$\gamma(s, \pi \times \rho, \psi) = \gamma(s, \rho_1 \times \rho, \psi) \cdot \gamma(s, \rho_2 \times \rho, \psi),$$

where the twoγ-factors on the RHS are Rankin-Selberg γ-factors. The similar identities hold for $\gamma(s, \sigma \times \rho, \psi)$.

(b) (Minimal Case) Suppose that $V = V_1^-$, the rank 3 non-split quadratic space of discriminant 1. If $\mathbf{1}$ denotes the trivial representation of the compact group $\mathrm{SO}(V)$ and χ denotes any character of GL_1, then

$$\gamma(s, \mathbf{1} \times \chi, \psi) = \gamma(s + 1/2, \chi, \psi) \cdot \gamma(s - 1/2, \chi, \psi),$$

where the γ-factors on the RHS are those of GL_1.

(c) (Global Functional Equation) Suppose that F is a number field with ring of adeles $\mathbb{A}$, and V is a quadratic space over F of dimension $2n+1$ and discriminant 1. Let $\Psi = \otimes_v \psi_v$ be a non-trivial additive character of $F\backslash\mathbb{A}$, $\Pi = \otimes_v \pi_v$ a cuspidal representation of $\mathrm{SO}(V)(\mathbb{A})$ and $\Xi = \otimes_v \xi_v$ a cuspidal representation of $\mathrm{GL}_r(\mathbb{A})$. If S is a finite set of places of F containing all archimedean places and all finite places where Ψ, Π or Ξ is ramified, then one has a functional equation

$$L^S(s, \Pi \times \Xi) = \prod_{v \in S} \gamma(s, \pi_v \times \xi_v, \psi_v) \cdot L^S(1-s, \Pi^\vee \times \Xi^\vee).$$

Likewise, if W is a symplectic space over F and Σ is a cuspidal representation of $\mathrm{Mp}(W)(\mathbb{A})$, then one has

$$L^S(s, \Sigma \times \Xi, \Psi) = \prod_{v \in S} \gamma(s, \sigma_v \times \xi_v, \psi_v) \cdot L^S(1-s, \Sigma^\vee \times \Xi^\vee).$$

Now we have:

Proposition 11.1. *Suppose that one has:*

(i) *a theory of γ-factors $\gamma(s, \pi \times \rho, \psi)$ for irreducible representations $\pi \boxtimes \rho$ of $\mathrm{SO}(V) \times \mathrm{GL}_r$;*

(ii) *a theory of γ-factors $\gamma(s, \sigma \times \rho, \psi)$ for irreducible representations $\sigma \boxtimes \rho$ of $\mathrm{Mp}(W) \times \mathrm{GL}_r$,*

satisfying the above working hypotheses. Then if $\sigma = \Theta_\psi(\pi)$, we have

$$\gamma(s, \pi \times \rho, \psi) = \gamma(s, \sigma \times \rho, \psi).$$

Proof. The proof is similar but simpler than that of Proposition 10.1. By properties (a) and (b), and Kudla's cuspidal support theorem ([Ku, Thm. 2.5] and [Ku2, Thms. 7.1 and 7.2]), one knows that the desired identity holds when π has nonzero Iwahori-fixed vectors. Further, one is reduced to the case when the representations π and ρ are supercuspidal. For the supercuspidal case, the proof is via a global argument similar to, but simpler than, that of Proposition 10.1 (see also [MS1, Prop. 5.4]). More precisely, let $\mathbb{V}$ be as in the proof of Proposition 10.1, so that $\mathbb{V}_{v_0} = V$ for a finite place v_0. By a construction of Henniart [He, Appendice 1] via Poincare series, one can find cuspidal representations Π and Ξ of $\mathrm{O}(\mathbb{V})$ and $\mathrm{GL}_r(\mathbb{A})$ such that $\Pi_{v_0} = \pi^\epsilon$ (where π^ϵ is the unique extension of π to $\mathrm{O}(V)$ which participates in the theta correspondence with $\mathrm{Mp}(W)$) and $\Xi_{v_0} = \rho$, and such that for all other finite places v, Π_v and Ξ_v have nonzero Iwahori-fixed vectors. Indeed, in the proof of [He, Appendice 1, Thm.], one simply picks the test function f_v to be a matrix coefficient of π at $v = v_0$, to be the characteristic function of a hyperspecial maximal compact subgroup for almost all $v \neq v_0$ and to be the characteristic function of an Iwahori subgroup at all other $v \neq v_0$. Then the argument in Prop. 10.1, using property (c) and our knowledge of the desired identity at all places outside v_0, implies that

$$\gamma(s, \pi \times \rho, \psi) = \gamma(s, \sigma \times \rho, \psi).$$

□

Corollary 11.2. *Assume the hypotheses of Proposition 11.1. If one defines the local L-factors $L(s, \pi \times \rho)$ and $L(s, \sigma \times \rho, \psi)$, as well as local epsilon factors $\epsilon(s, \pi \times \rho, \psi)$ and $\epsilon(s, \sigma \times \rho, \psi)$ following the approach of Shahidi (i.e. analogous to that in [LR, §10] or §4.2), then one has:*

$$\begin{cases} L(s, \pi \times \rho) = L_\psi(s, \sigma \times \rho) \\ \epsilon(s, \pi \times \rho, \psi) = \epsilon(s, \sigma \times \rho, \psi). \end{cases}$$

In particular, a theory of γ-factors satisfying the working hypotheses has been developed in the following cases:

(a) for generic representations $\pi \boxtimes \rho$ of $\mathrm{SO}(V^+) \times \mathrm{GL}_r$ by Shahidi [Sh] and Soudry [So];
(b) for ψ-generic representations $\sigma \otimes \rho$ of $\mathrm{Mp}(W) \times \mathrm{GL}_r$ by D. Szpruch [Sz];
(c) for all irreducible representations $\pi \times \chi$ of $\mathrm{SO}(V^\pm) \times \mathrm{GL}_1$ and $\sigma \boxtimes \chi$ of $\mathrm{Mp}(W) \times \mathrm{GL}_1$ via the doubling method (cf. [PSR], [LR] and [G]).

Thus we have:

Corollary 11.3. *(i) Suppose that $\sigma = \Theta_\psi(\pi)$, with $\pi \in Irr(\mathrm{SO}(V^+))$ generic. Then the equalities of L- and ϵ-factors in Corollary 11.2 hold.*

(ii) Th equalities L- and ϵ-factors in Corollary 11.2 hold for representations of $\mathrm{SO}(V) \times \mathrm{GL}_1$ and $\mathrm{Mp}(W) \times \mathrm{GL}_1$.

12. **Variation of ψ**

One remaining issue is the dependence of the bijection $\mathcal{L}_\psi$ when ψ varies. More precisely, for $c \in k^\times$, let ψ_c be the character $\psi_c(x) = \psi(cx)$. Then we would like to know the relation between $\mathcal{L}_\psi(\sigma)$ and $\mathcal{L}_{\psi_c}(\sigma)$. Here, recall that

$$\mathcal{L}_\psi(\sigma) = (\phi, \eta),$$

where

$$\phi : WD_k \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C})$$

and η is an irreducible character of the component group $A_\phi = \pi_0(Z_{\mathrm{Sp}_{2n}}(\phi))$. The component group A_ϕ can be explicitly described as follows. If we decompose ϕ as a $2n$-dimensional representation:

$$\phi = \bigoplus_i n_i \phi_i,$$

then (cf. [GGP])

$$A_\phi = \bigoplus_{i:\phi_i \text{ is symplectic}} \mathbb{Z}/2\mathbb{Z} a_i.$$

In [GGP], a conjecture was stated for the relation between $\mathcal{L}_\psi(\sigma)$ and $\mathcal{L}_{\psi_c}(\sigma)$. The purpose of this section is to verify this conjecture. Of course, to address this issue, one would need to assume that the local Langlands correspondence for $\mathrm{SO}(V^\pm)$ is known. Hence, let us begin by setting down the precise hypotheses we shall require:

<u>Hypothesis LLC</u>

(a) We assume the local Langlands correspondence for the classical groups $G = \mathrm{SO}_{2n+1}$, SO_{2n} and Sp_{2n} as supplied by the recently released book of Arthur [A] and supplemented by the results of Jiang-Soudry [JS]. In particular, each irreducible representation of this group is indexed by a pair (ϕ, η) consisting of an L-parameter ϕ for the group G and a character η of the component group A_ϕ.

(b) Moreover, we suppose that the local Langlands correspondence satisfies the desiderata in [B] and preserves local L-factors and ϵ-factors as in Theorem 1.3(vii) and (viii).

(c) In addition, for representations π_1 and π_2 which have the same L-parameter, one has an equality of Plancherel measures:

$$\mu(s, \pi_1 \times \rho, \psi) = \mu(s, \pi_2 \times \rho, \psi).$$

In particular, by Shahidi [Sh], such Plancherel measures can be expressed in terms of γ-factors of Artin type associated to the L-parameters.

Under the above hypothesis, one has the following highly non-trivial results:

- (GP) The Gross-Prasad conjecture [GP] for tempered representations of special orthogonal groups holds by the recent work of Waldspurger [W5-9]. More precisely, suppose that π is an irreducible tempered representation of $\mathrm{SO}(V)$ (with $\dim V = 2n+1$) with $\mathcal{L}_\psi(\pi) = (\phi, \eta)$ as above, and τ is an irreducible tempered representation of $\mathrm{SO}(U)$ with $U \subset V$ of codimension 1. Suppose further that

$$\mathrm{Hom}_{\mathrm{SO}(U)}(\pi \otimes \tau, \mathbb{C}) \neq 0.$$

Then

$$\eta(a_i) = \epsilon(1/2, \phi_i \otimes \phi_\tau) \cdot \chi_U(-1)^{\frac{1}{2} \cdot \dim \phi_i},$$

where χ_U is the quadratic character of $k^\times$ associated to the disc(U). Similarly, if $V \subset U$ with codimension 1, then $\mathrm{Hom}_{\mathrm{SO}(V)}(\pi \otimes \tau, \mathbb{C}) \neq 0$ implies that $\eta(a_i)$ is given by the above formula as well.

- (Θ) Consider the theta correspondence for $\mathrm{Sp}(W) \times \mathrm{O}(U)$ with $\dim W = 2n$ and $\dim U = 2n+2$ with discriminant χ_U. For an irreducible tempered representation of τ of $\mathrm{Sp}(W)$ which participates in this theta correspondence, it was shown by Muić [M1,2] that

$$\Theta_{W,U,\psi}(\tau) = \theta_{W,U,\psi}(\tau) =: \theta(\tau).$$

Moreover, $\theta(\tau)$ is irreducible when restricted to $\mathrm{SO}(U)$. Finally, it was shown by Muić [M1,2] and Moeglin [Mo1] that the L-parameters ϕ_τ and $\phi_{\theta(\tau)}$ are related by

$$\phi_{\theta(\tau)} = 1 \oplus \chi_U \cdot \phi_\tau.$$

Similarly, consider the theta correspondence for $\mathrm{Sp}(W) \times \mathrm{O}(U)$ with $\dim U = 2n$ and let τ be an irreducible tempered representation of $\mathrm{O}(U)$. Then

$$\Theta_{U,W,\psi}(\tau) = \theta_{U,W,\psi}(\tau) =: \theta(\tau).$$

Moreover, the L-parameters of τ and $\theta(\tau)$ are related by

$$\phi_{\theta(\tau)} = \chi_U \cdot (1 \oplus \phi_\tau).$$

Under the hypothesis LLC and the above theorems (GP) and (Θ), one has:

Theorem 12.1. *For $\sigma \in Irr(\mathrm{Mp}(W))$ and $c \in k^\times$, let $\mathcal{L}_\psi(\sigma) = (\phi, \eta)$ and $\mathcal{L}_{\psi_c}(\sigma) = (\phi_c, \eta_c)$. Then:*

(i) $\phi_c = \phi \otimes \chi_c$, where χ_c is the quadratic character associated to $c \in k^\times / k^{\times 2}$.

It follows by (i) that we have canonical identification of component groups:

$$A_\phi = A_{\phi_c} = \oplus_i \mathbb{Z}/2\mathbb{Z} a_i,$$

so that it makes sense to compare η and η_c.

(ii) the characters η and η_c are related by:

$$\eta_c(a_i)/\eta(a_i) = \epsilon(1/2, \phi_i) \cdot \epsilon(1/2, \phi_i \otimes \chi_c) \cdot \chi_c(-1)^{\frac{1}{2}\dim \phi_i} \in \{\pm 1\}.$$

When $\dim W = 2$, this reduces to Theorem 5.3 of Waldspurger. The remainder of this paper is devoted to the proof of the theorem. We first note the following reduction.

Proposition 12.2. *If Theorem 12.1 holds for tempered representations of* $\mathrm{Mp}(W)$*, then it holds for all representations.*

Proof. Suppose that $\sigma = J_{P,\psi}(\tau_1, ..., \tau_r, \sigma_0)$. Write $\mathcal{L}_\psi(\sigma) = (\phi, \eta)$ and $\mathcal{L}_{\psi_c}(\sigma) = (\phi_c, \eta_c)$ as in the conjecture, and similarly, write $\mathcal{L}_\psi(\sigma_0) = (\phi_0, \eta_0)$ and $\mathcal{L}_{\psi_c}(\sigma_0) = (\phi_{0,c}, \eta_{0,c})$. We are assuming that the pairs (ϕ_0, η_0) and $(\phi_{0,c}, \eta_{0,c})$ are related as in Theorem 12.1.

Now Theorem 1.3(iii) implies that

$$\phi = \phi_1 \oplus ... \oplus \phi_r \oplus \phi_0 \oplus \phi_1^\vee \oplus ... \oplus \phi_r^\vee,$$

where ϕ_i is the L-parameter of τ_i for $i \geq 1$. Moreover, there is a natural identification $A_\phi = A_{\phi_0}$ under which one has $\eta = \eta_0$.

On the other hand, as genuine characters of $\tilde{\mathrm{GL}}(X)$, one has

$$\chi_{\psi_c} = \chi_\psi \cdot (\chi_c \circ \det_X).$$

Thus, one also has

$$\sigma = J_{P,\psi_c}(\tau_1 \otimes \chi_c, ..., \tau_r \otimes \chi_c, \sigma_0).$$

By Theorem 1.3(iii) again, one has

$$\phi_c \otimes \chi_c = \phi_1 \oplus ... \oplus \phi_r \oplus (\phi_{0,c} \otimes \chi_c) \oplus \phi_1^\vee \oplus ... \oplus \phi_r^\vee,$$

and $\eta_c = \eta_{0,c}$. Since $(\phi_{0,c}, \eta_{0,c})$ and (ϕ_0, η_0) are related as in Theorem 12.1, so are (ϕ_c, η_c) and (ϕ, η). $\square$

Now suppose that $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$ is tempered, and let

$$\pi = \Theta_{W,V,\psi}(\sigma) \quad \text{and} \quad \pi_c = \Theta_{W,V,\psi_c}(\sigma).$$

Note that π and π_c are both irreducible by Theorem 8.1. To prove Theorem 12.1, we need to show that their L-parameters ϕ and ϕ_c are related by

$$\phi \otimes \chi_c = \phi_c.$$

By Theorem 1.3(ii), this identity for tempered representations follows from the case of discrete series representations. Hence, we may assume that σ is discrete series. Now by Proposition 10.1, one has the following identities of Plancherel measures:

$$\mu(s, (\pi \otimes \chi_c) \times \rho, \psi) = \mu(s, \pi \times (\rho \otimes \chi_c), \psi) = \mu(s, \sigma \times (\rho \otimes \chi_c), \psi)$$

and

$$\mu(s, \pi_c \times \rho, \psi_c) = \mu(s, \sigma \times \rho, \psi_c),$$

where ρ is any supercuspidal representation of GL_r (for any r).

On the other hand, it follows from [GI, §B.2] that

$$\mu(s, \sigma \times \rho, \psi_c) = |c|^{2nr + r(r+1)/2} \cdot \mu(s, \sigma \times (\rho \otimes \chi_c), \psi)$$

and

$$\mu(s, \pi_c \times \rho, \psi_c) = |c|^{(2n+1)r + r(r-1)/2} \cdot \mu(s, \pi_c \times \rho, \psi).$$

Hence, we deduce that

$$\mu(s, \pi_c \times \rho, \psi) = \mu(s, (\pi \otimes \chi_c) \times \rho, \psi).$$

By Hypothesis (LLC), we may express these Plancherel measures in terms of Shahidi's γ-factors, which can in turn be expressed as γ-factors of L-parameters. This gives the identity

$$\gamma(s, \phi_c \otimes \phi_\rho, \psi) \cdot \gamma(-s, \phi_c \otimes \phi_\rho^\vee, \overline{\psi}) = \gamma(s, (\phi \otimes \chi_c) \otimes \phi_\rho, \psi) \cdot \gamma(-s, (\phi \otimes \chi_c) \otimes \phi_\rho^\vee, \overline{\psi}).$$

The following lemma then allows one to conclude that

$$\phi_c = \phi \otimes \chi_c.$$

Lemma 12.3. *Suppose that ϕ_1 and ϕ_2 are $2n$-dimensional semisimple representations of WD_k, each of which is a multiplicity-free sum of irreducible symplectic summands. If,*

$$\gamma(s, \phi_1 \otimes \phi_\rho, \psi) \cdot \gamma(-s, \phi_1 \otimes \phi_\rho^\vee, \overline{\psi}) = \gamma(s, \phi_2 \otimes \phi_\rho, \psi) \cdot \gamma(-s, \phi_2 \otimes \phi_\rho^\vee, \overline{\psi}) \tag{12.4}$$

for every irreducible representation ϕ_ρ of W_k, then

$$\phi_1 \cong \phi_2$$

as representations of WD_k.

Proof. We shall proceed by induction on $\dim \phi_i = 2n$. Suppose that ϕ_0 is an irreducible representation of W_k such that $\phi_0 \boxtimes S_r$ is contained in ϕ_1 for some $r \geq 1$. Here S_r is the r-dimensional irreducible repersentation of $\mathrm{SL}_2(\mathbb{C})$. Let r_0 be the smallest such r. Taking $\phi_\rho = \phi_0$ and evaluating at $s = (r_0 - 1)/2 \geq 0$, one sees that the LHS of (12.4) has a zero at $s = (r_0 - 1)/2$, and hence so must the RHS. This implies that $L(-s, \phi_2 \otimes \phi_0^\vee) \cdot L(s, \phi_2 \otimes \phi_0)$ must have a pole at $s = (r_0 - 1)/2$. It is not difficult to see that this can only happen if ϕ_2 contains $\phi_0 \boxtimes S_{r_0}$ as well. Thus, we may cancel $\phi_0 \boxtimes S_r$ from both ϕ_1 and ϕ_2, and still have the analog of (12.4). The lemma then follows by induction. $\square$

Now let

$$U_c \subset V$$

be a quadratic subspace of discriminant c and codimension 1. Then we have:

$$V = U_c + L_c$$

where L_c is a nondegenerate line of discriminant c, and $\mathrm{SO}(U_c) \subset \mathrm{SO}(V)$. We have:

Lemma 12.5. *Given an irreducible tempered representation π of $\mathrm{SO}(V)$, there exists an irreducible tempered representation ξ_c of $\mathrm{SO}(U_c)$ such that*

$$\mathrm{Hom}_{\mathrm{SO}(U_c)}(\pi, \xi_c) \neq 0.$$

Proof. Let f_π be a matrix coefficient of π, which is a smooth function on $\mathrm{SO}(V)$. By replacing f_π by a $\mathrm{SO}(V)$-translate if necessary, we may assume that f_π has nonzero restriction to the subgroup $\mathrm{SO}(U_c)$. Then there is an element $\phi \in C_c^\infty(\mathrm{SO}(U_c))$ such that

$$\int_{\mathrm{SO}(U_c)} f_\pi(h) \cdot \overline{\phi(h)}\, dh \neq 0.$$

By the Plancherel theorem for Schwarz-Harish-Chandra functions [W4, Thm. VIII.1.1],

$$\phi = \bigoplus_M \phi_M$$

as M ranges over conjugacy classes of Levi subgroups of $\mathrm{SO}(U_c)$ and each ϕ_M is a finite sum of "wave packets" associated to discrete series representations of M. More precisely, a "wave packet" is a function on $\mathrm{SO}(U_c)$ of the form:

$$h \mapsto \int_{X_0(M)} \beta(s) \cdot \langle I(s,\tau)(h)(\mathcal{F}, \mathcal{F}'\rangle\, \mu(s) \cdot ds$$

where

- $X_0(M)$ denotes the compact torus of unramified unitary characters of M;
- β is a smooth function on $X_0(M)$;
- τ is a discrete series representation (with unitary central character) of M;
- $I(s,\tau)$ is the family of tempered representations of $\mathrm{SO}(U_c)$ parabolically induced from the unramified twists $\tau \otimes \chi_s$ for $s \in X_0(M)$, which are all realized on the same space $I(\tau)$ of functions on a maximal compact subgroup K of $\mathrm{SO}(U_c)$;
- $\mathcal{F}$ and $\mathcal{F}'$ are elements of $I(\tau)$;
- $\langle -,-\rangle$ is the standard inner product on $I(\tau)$ induced by an inner product on τ and integration over K;
- $\mu(s)\, ds$ is the Plancherel measure associated to (M,τ), with ds a Haar measure of $X_0(M)$ and $\mu(s)$ a smooth function.

Hence, we conclude that

$$\int_{\mathrm{SO}(U_c)} f_\pi(h) \cdot \int_{X_0(M)} \beta(s) \cdot \langle I(s,\tau)(h)(\mathcal{F}, \mathcal{F}'\rangle\, \mu(s) \cdot ds\, dh \neq 0 \tag{12.6}$$

for some choice of $(M, \tau, \mathcal{F}, \mathcal{F}')$. We note also that the function

$$s \mapsto \langle I(s,\tau)(h)\mathcal{F}, \mathcal{F}'\rangle$$

is continuous in s, as is easy to see from the definitions.

Now in [II], it was shown that

$$\int_{\mathrm{SO}(U_c)} f_\pi(h) \cdot \overline{f_{\xi_c}(h)}\, dh$$

is absolutely convergent for any tempered representation ξ_c of $\mathrm{SO}(U_c)$. Hence the double integral in (12.6) is absolutely convergent. On exchanging the order of integration in (12.6), we deduce that

$$\int_{\mathrm{SO}(U_c)} f_\pi(h) \cdot \langle I(s,\tau)(h)(\mathcal{F}, \mathcal{F}'\rangle\, dh \neq 0$$

for some $(\tau, s, \mathcal{F}, \mathcal{F}')$. This implies that there is a tempered representation ξ_c of $\mathrm{SO}(U_c)$ and a matrix coefficient f_{ξ_c} such that

$$\int_{\mathrm{SO}(U_c)} f_\pi(h) \cdot \overline{f_{\xi_c}(h)}\, dh \neq 0.$$

This proves the lemma. $\square$

By (GP), one has

$$\eta(a_i) = \epsilon(1/2, \phi_i \otimes \phi_{\xi_c}) \cdot \chi_c(-1)^{1/2 \dim \phi_i}.$$

Let $\tau = \Theta_{U_c,W,\psi}(\xi_c)$. By (Θ), τ is either zero or irreducible tempered. Now by the see-saw identity associated to the see-saw diagram:

$$\begin{tikzcd} \mathrm{Sp}(W) \times \mathrm{Mp}(W) \arrow[d, dash] \arrow[dr, dash] & \mathrm{SO}(V) \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{Mp}(W) & \mathrm{SO}(U_c) \times \mathrm{SO}(L_c) \end{tikzcd}$$

we deduce that

$$\mathrm{Hom}_{\mathrm{Sp}(W)}(\tau \otimes \omega_{W,\psi_c}, \sigma) \neq 0.$$

In particular, τ must be nonzero and by (Θ), its L-parameter is

$$\phi_\tau = \chi_c \cdot (1 + \phi_{\xi_c}).$$

Moreover, since the representations τ, σ and ω_{W,ψ_c} are unitary, one sees by taking complex conjugate that

$$\mathrm{Hom}_{\mathrm{Sp}(W)}(\tau^\vee \otimes \omega_{W,\psi_{-c}}, \sigma^\vee) \neq 0.$$

Since

$$\mathrm{Hom}_{\mathrm{Sp}(W)}(\tau^\vee \otimes \omega_{W,\psi_{-c}}, \sigma^\vee) = \mathrm{Hom}_{\mathrm{Sp}(W)}(\tau^\vee \otimes \omega_{W,\psi_{-c}} \otimes \sigma, \mathbb{C}) = \mathrm{Hom}_{\mathrm{Sp}(W)}(\omega_{W,\psi_{-c}} \otimes \sigma, \tau),$$

we conclude that

$$\mathrm{Hom}_{\mathrm{Sp}(W)}(\sigma \otimes \omega_{W,\psi_{-c}}, \tau) \neq 0.$$

Now consider the see-saw diagram

$$\begin{tikzcd} \mathrm{Mp}(W) \times \mathrm{Mp}(W) \arrow[d, dash] \arrow[dr, dash] & \mathrm{SO}(V + L_{-1}) \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{Sp}(W) & \mathrm{SO}(V) \times \mathrm{SO}(L_{-1}) \end{tikzcd}$$

where L_{-1} is a quadratic line of discriminant -1, and examine the theta correspondence with respect to the additive character ψ_c. Set

$$\xi = \Theta_{W,V+L_{-1},\psi_c}(\tau).$$

By (Θ) again, ξ is either zero or irreducible tempered, in which case its L-parameter is

$$\phi_\xi = 1 + \phi_\tau = 1 + \chi_c + \chi_c \cdot \phi_{\xi_c}.$$

Now the see-saw identity says that

$$\mathrm{Hom}_{\mathrm{SO}(V)}(\xi, \Theta_{\psi_c}(\sigma)) = \mathrm{Hom}_{\mathrm{Sp}(W)}(\sigma \otimes \omega_{W,\psi_{-c}}, \tau) \neq 0.$$

Since $\mathcal{L}_{\psi_c}(\sigma) = (\phi \otimes \chi_c, \eta_c)$, it follows by (GP) that

$$\eta_c(a_i) = \epsilon(1/2, \phi_i \otimes \chi_c \otimes \phi_\xi).$$

Thus,

$$\eta_c(a_i)/\eta(a_i) = \epsilon(1/2, \phi_i) \cdot \epsilon(1/2, \phi_i \otimes \chi_c) \cdot \chi_c(-1)^{1/2 \dim \phi_i}.$$

This completes the proof of Theorem 12.1.

References

[A] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, preprint, available at http://www.claymath.org/cw/arthur/pdf/Book.pdf.

[Ad] J. Adams, *Genuine representations of the metaplectic group and epsilon factors*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 721731, Birkhäuser, Basel, 1995.

[AB1] J. Adams and D. Barbasch, *Reductive dual pair correspondence for complex groups*, J. Funct. Anal. 132 (1995), no. 1, 1–42.

[AB2] J. Admas and D. Barbasch, *Genuine representations of the metaplectic group*, Compositio Math 113 (1998), no. 1, 23-66.

[B] A. Borel, *Automorphic L-functions*, in *Automorphic forms, representations and L-functions*, Proc. Sympos. Pure Math., XXXIII, Part II, Amer. Math. Soc., Providence, R.I. (1977), 27-61.

[BJ] D. Ban and C. Jantzen, *The Langlands quotient theorem for finite central extensions of p-adic groups*, preprint.

[Fu] M. Furusawa, *On the theta lift from* SO_{2n+1} *to* $\widetilde{\mathrm{Sp}}_n$, J. Reine Angew. Math. 466 (1995), 87–110.

[G] W. T. Gan, *Doubling zeta integrals and local factors for metaplectic groups*, to appear in Nagoya Math. Journal (Hiroshi Saito memorial volume), available at http://www.math.ucsd.edu/ wgan/multiplicativity.pdf

[GGP] W. T. Gan, B. H. Gross and D. Prasad, *Symplectic local root numbers, central critical L-values and restriction problems in the representation theory of classical groups*, to appear in Asterisque 180, Vol. 1.

[GI] W. T. Gan and A. Ichino, *Formal degrees and local theta correspondences*, submitted, available at http: www.math.ucsd.edu/ wgan/formal-deg-7.pdf.

[GP] B. Gross and D. Prasad, *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}. Canad. J. Math. 44 (1992), no. 5, 974-1002.

[GS] W. T. Gan and G. Savin, *Representations of metaplectic groups II: Hecke algebra correspondences*, submitted, available at http://www.math.ucsd.edu/ wgan/hecke$_3$.1.pdf.

[GT] W. T. Gan and S. Takeda, *The local Langlands conjecture for* GSp(4), Annals of Math. Vol 173, issue 3 (2011), 1841-1882.

[GTW] W. T. Gan and W. Tantono, *The local Langlands conjecture for* GSp(4) *II: the case of inner forms*, preprint, available at http://www.math.ucsd.edu/ wgan/LLC-GSp4-inner.pdf.

[He] G. Henniart, *La conjecture de Langlands locale pour GL(3)*, Mém. Soc. Math. France (N.S.) No. 11-12 (1984), 186 pp.

[HKS] M. Harris, S. Kudla and J. Sweet, *Theta dichotomy for unitary groups*, J. Amer. Math. Soc. **9** (1996), 941–1004.

[HM1] M. Hanzer and G. Muić, *Parabolic induction and Jacquet functors for metaplectic groups*, Journal of Algebra 323 No.1, 241-260, (2010).

[HM2] M. Hanzer and G. Muić, *Rank one reducibility for metapletic groups via theta correspondence*, Canadian J. Math. 63 (2011), no. 3, 591615.

[II] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, Geom. Funct. Anal. **19** (2010), 1378–1425.

[JS] D. H. Jiang and D. Soudry, *Generic representations and local Langlands reciprocity law for p-adic* SO_{2n+1}, in *Contributions to automorphic forms, geometry, and number theory*, 457–519, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[Ku] S. Kudla, *On the local theta correspondence.* Invent. Math. 83 (1986) 229-255.

[Ku2] S. Kudla, *Notes on the local theta correspondence*, available at www.math.toronto.edu/ ssk.

[KR1] S. Kudla and S. Rallis, *Poles of Eisenstein series and L-functions*, in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, 81–110, Israel Math. Conf. Proc., 3, Weizmann, Jerusalem, 1990

[KR2] S. Kudla and S. Rallis, *On first occurrence in the local theta correspondence*, in *Automorphic representations, L-functions and applications: progress and prospects*, 273–308, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[LR] E. Lapid and S. Rallis, *On the local factors of representations of classical groups*, in *Automorphic representations, L-functions and applications: progress and prospects*, 309–359, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005.

[M1] G. Muić, *On the structure of theta lifts of discrete series for dual pairs* $(\mathrm{Sp}(n), \mathrm{O}(V))$, Israel J. Math. 164 (2008), 87–124

[M2] G. Muić, *Theta lifts of tempered representations for dual pairs* $(\mathrm{Sp}_{2n}, \mathrm{O}(V))$, Canadian J. Math. 60 No. 6 (2008), 1306-1335.

[MS1] G. Muić and G. Savin, *Symplectic-orthogonal theta lifts of generic discrete series*, Duke Math. J. 101 (2000), no. 2, 317–333.

[MS2] G. Muić and G. Savin, *Complementary series for Hermitian quaternionic groups*, Canad. Math. Bull. 43 (2000), no. 1, 90–99.

[Mo1] C. Moeglin, *Conjecture d'Adams pour la correspondance de Howe et filtration de Kudla*, in *Arithmetic Geometry and Automorphic Forms*, 445-503, Advanced Lectures in Math. Vol. 19, Higher Education Press (Beijing) and International Press 2011.

[Mo2] C. Moeglin, *Multiplicité 1 dans les paquets d'Arthur aux places P-adiques*, in *On certain L-functions*, 333374, Clay Math. Proc., 13, Amer. Math. Soc., Providence, RI, 2011.

[MVW] C. Moeglin, M.-F. Vigneras and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, Lecture Notes in Mathematics, 1291. Springer-Verlag, Berlin, 1987.

[MW] C. Moeglin and J. L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Tracts in Mathemtics 113, Cambridge University Press (1995).

[PSR] I. Piatetski-Shapiro and S. Rallis, ϵ *factor of representations of classical groups*, Proc. Nat. Acad. Sci. U.S.A. 83 (1986), no. 13, 4589–4593.

[R] S. Rallis, *On the Howe duality conjecture*, Compositio Math. 51 (1984), no. 3, 333–399.

[Sh] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2) 132 (1990), no. 2, 273–330.

[Si] A. Silberger, *Introduction to harmonic analysis on reductive p-adic groups*, Mathematical Notes, 23. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979.

[So] D. Soudry, *Rankin-Selberg convolutions for* $\mathrm{SO}(2n+1) \times \mathrm{GL}(n)$*: local theory*, Mem. Amer. Math. Soc., Vol. 105, no. 500, Amer. Math. Soc., Providence, R. I., 1993.

[Sw] J. Sweet, *Functional equations of p-adic zeta integrals and representations of the metaplectic group*, preprint (1995).

[Sz] D. Szpruch, *The Langlands-Shahidi method for the metaplectic group and applications*, thesis (Tel Aviv University), available at arXiv:1004.3516v1.

[W1] J.-L. Waldspurger, *Correspondance de Shimura*, J. Math. Pures Appl. (9) 59 (1980), no. 1, 1–132.

[W2] J.-L. Waldspurger, *Correspondances de Shimura et quaternions*, Forum Math. 3 (1991), no. 3, 219–307.

[W3] J.-L. Waldspurger, *Demonstration d'une conjecture de dualite de Howe dans le cas p-adique,* $p \neq 2$, in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989)*, 267–324, Israel Math. Conf. Proc., 2, Weizmann, Jerusalem, 1990.

[W4] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques dapres Harish-Chandra*, Journal of Institut Math Jussieu 2(2) (2003), 235333.

[W5] J.-L.Waldspurger, *Une Formule intégrale reliée à la conjecture locale de Gross-Prasad*, Compositio Math. 146 (2010), no. 5, 11801290.

[W6] J.-L. Waldspurger, *Une Formule intégrale reliée à la conjecture locale de Gross-Prasad, 2ème partie: extension aux représentations tempérées*, to appear in Asterisque 180, Vol. 1.

[W7] J.-L.Waldspurger, *Calcul d'une valeur d'un facteur epsilon par une formule intégrale*, to appear in Asterisque 180, Vol. 2.

[W8] J.-L.Waldspurger, *La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes spciaux orthogonaux*, to appear in Asterisque 180, Vol. 2.

[W9] J.-L. Waldspurger, *Une variante d'un résultat de Aizenbud, Gourevitch, Rallis et Schiffmann*, Asterisque 180, Vol. 1.

[Z] C. Zorn, *Theta dichotomy and doubling epsilon factors for* $\mathrm{Mp}(n, F)$, American Journal of Math. 133 (2011), no. 5, 13131364.

W.T.G.: Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, 92093;

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road Singapore 119076

E-mail address: wgan@math.ucsd.edu

G. S.: Department of Mathematics, University of Utah, Salt Lake City, UT

E-mail address: savin@math.utah.edu