THE METAPLECITC TENSOR PRODUCT AS AN INSTANCE OF LANGLANDS FUNCTORIALITY

WEE TECK GAN

Abstract. We interpret the metaplectic tensor product construction of Mezo for the genuine representations of the Kazhdan-Patterson covering groups in terms of the L-group formalism of Weissman.

1. Kazhdan-Patterson coverings and Metaplectic Tensor Product

Let F be a characteristic 0 local field which contains all n-th roots of unity (for a fixed $n \in \mathbb{N}$). The goal of this note is to interpret the metaplectic tensor product construction of Mezo [M] for the Kazhdan-Patterson covering groups of $\mathrm{GL}_r(F)$ in the framework of Langlands functoriality for Brylinski-Deligne extensions.

1.1. **Kazhdan-Patterson covering.** We shall be working with Brylinski-Deligne covers of the group $G_r = \mathrm{GL}_r$ over F. Let $T_r \subset B_r$ be the diagonal torus of GL_r contained in the upper triangular Borel subgroup; this defines a based root datum $(X(T_r), \Delta_r, Y(T_r), \Delta_r^\vee)$ for GL_r, and we may consider the standard pinning. Let us write

$$Y = Y(T_r) = \oplus_{i=1}^r \mathbb{Z} \cdot e_i$$

and let Y_{sc} be the sublattice spanned by the simple coroots $\Delta_r^\vee = \{e_i - e_{i+1} : i = 1, ...r-1\}$.

For $c \in \mathbb{Z}$, let Q_c be the Weyl-invariant quadratic form, whose associated symmetric bilinear form B_c is given by

$$B_c(e_i, e_j) = \begin{cases} 2c \text{ if } i = j; \\ 2c+1 \text{ if } i \neq j. \end{cases}$$

Note that for each $\alpha^\vee \in \Delta_r^\vee$,

$$Q_c(\alpha^\vee) = -1.$$

One has the (non symmetric) bilinear form D_c given by

$$D_c(e_i, e_j) = \begin{cases} c \text{ if } i = j; \\ 2c+1, \text{ if } i < j; \\ 0, \text{ if } i > j. \end{cases}$$

Hence we have $B_c(x,y) = D_c(x,y) + D_c(y,x)$, so that D_c is a bisector of B_c in the sense of [GG, §2.6]. If $\eta_0 : Y_{sc} \longrightarrow F^\times$ is the trivial map (sending every element to 1), then the pair

2010 *Mathematics Subject Classification.* Primary 11F70, Secondary 22E50.

(D_c, η_0) is an object in the category $\mathrm{Bis}_{\mathrm{GL}_r}$ in [GG, §2.6] and gives rise to a Bryinski-Deligne extension $\overline{G}_{r,c}$ of GL_r:

$$1 \longrightarrow K_2 \longrightarrow \overline{G}_{r,c} \longrightarrow \mathrm{GL}_r \longrightarrow 1.$$

Taking F-points and pushing out by the n-th Hilbert symbol $(-,-)_n$, we obtain a topological central extension

$$1 \longrightarrow \mu_n(F) \longrightarrow \overline{G}_{r,c} \stackrel{p}{\longrightarrow} \mathrm{GL}_r(F) \longrightarrow 1$$

The covering group $\overline{G}_{r,c}$ is none other than the degree n Kazhdan-Patterson cover of $\mathrm{GL}_r(F)$ associated to the twisting parameter c studied in [KP].

The bisector D_c is basically providing a cocycle for the maximal torus T_r. More precisely, one may realise

$$\overline{T}_{r,c} := p^{-1}(T_r) = T_r(F) \times \mu_n(F) \quad \text{as a set}$$

with group law given by

$$(e_i(a_i), \epsilon_1) \cdot (e_j(a_j), \epsilon_j) = \Big(e_i(a_i) e_j(a_j), \epsilon_i \epsilon_j \cdot (a_i, a_j)_n^{D(e_i,e_j)} \Big)$$

for $a_i, a_j \in F^\times$. An important observation to make here is that, with

$$T_r^n = \{t^n : t \in T_r(F)\},$$

the subset

$$T_r^n \times \mu_n \subset T_r(F) \times \mu_n(F)$$

is a subgroup. In particular, one has a natural splitting of the subgroup T_r^n into $\overline{T}_{r,c}$, giving by embedding into the first coordinate in the above presentation. Henceforth, we shall regard T_r^n as a subgroup of $\overline{T}_{r,c}$ in this way.

We shall be considering irreducible genuine representations of $\overline{G}_{r,c}$. More precisely, let us fix an embedding

$$\epsilon : \mu_n(F) \hookrightarrow \mathbb{C}^\times,$$

and let $\mathrm{Irr}_\epsilon(\overline{G}_{r,c})$ denote the set of isomorphism classes of ϵ-genuine representations of $\overline{G}_{r,c}$.

1.2. **Covers of Levi subgroups.** Now suppose that $M_r \subset \mathrm{GL}_r$ is a Levi subgroup, with

$$M_r = \mathrm{GL}_{r_1} \times .... \times \mathrm{GL}_{r_k}.$$

Note that one such M_r is the split torus T_r. Restricting the cover $\overline{G}_{r,c}$ to M_r gives a cover $\overline{M}_{r,c}$. On the other hand, for each $G_{r_i} = \mathrm{GL}_{r_i}$ in M_r, the restriction of the cover to G_{r_i} is none other than the (degree n) Kazhdan-Patterson cover $\overline{G}_{r_i,c}$, i.e.

$$p^{-1}(G_{r_i}) \cong \overline{G}_{r_i,c}.$$

While G_{r_i} and G_{r_j} commute with each other, it is no longer true in general that $p^{-1}(G_{r_i})$ and $p^{-1}(G_{r_j})$ commute. Hence , in general, there is no direct way of relating the covering groups $\overline{M}_{r,c}$ and the almost direct product

$$\overline{G}_{r_1,c} \times_{\mu_n} ..... \times_{\mu_n} \overline{G}_{r_k,c}.$$

In particular, an irreducible genuine representation of $\overline{M}_{r,c}$ is not obtained as a tensor product of irreducible genuine representations of the $\overline{G}_{r_i,c}$.

1.3. **Metaplectic tensor product.** However, in [M]. Mezo described a construction which constructs an irreducible genuine representation of $\overline{M}_{r,c}$ out of irreducible genuine representations of $\overline{G}_{r_i,c}$, for $1 \leq i \leq k$, and one extra piece of data. Let us recall his construction of this "metaplectic tensor product" briefly.

Let π_i be irreducble genuine representations of $\overline{G}_{r_i,c}$. Let

$$G_{r_i,c}^{(n)} = \{g \in G_{r_i,c} : \det(g) \in F^{\times n}\}$$

and set

$$\overline{G}_{r_i,c}^{(n)} = p^{-1}(G_{r_i,c}^{(n)}).$$

For $i \neq j$, $p^{-1}(G_{r_i,c}^{(n)})$ and $p^{-1}(G_{r_j,c}^{(n)})$ commute with each other. Hence,

$$\overline{M}_{r,c}^{(n)} := p^{-1}\left(\prod_{i=1}^{k} G_{r_i,c}^{(n)}\right) = \overline{G}_{r_1,c}^{(n)} \times_{\mu_n} \times .... \times_{\mu_n} \overline{G}_{r_i,c}^{(n)}.$$

Now consider the restriction of π_i to $\overline{G}_{r_i,c}^{(n)}$ (this restriction is semisimple of finite length since $Z(\overline{G}_{r_i,c}) \cdot \overline{G}_{r_i,c}^{(n)}$ is a finite index subgroup of $\overline{G}_{r_i,c}$) and let $\sigma_i \subset \pi_i$ be an irreducible summand in this restriction. One then has an irreducible representation

$$\sigma_1 \boxtimes .... \boxtimes \sigma_k \quad \text{of } \overline{M}_{r,c}^{(n)} = p^{-1}\left(\prod_{i=1}^{k} G_{r_i,c}^{(n)}\right).$$

Next, one picks an irreducible genuine character χ of $Z(\overline{G}_{r,c})$ such that

$$\chi = \otimes_{i=1}^{k} \omega_{\pi_i} \quad \text{on } Z(\overline{G}) \cap p^{-1}\left(\prod_{i=1}^{k} G_{r_i,c}^{(n)}\right). \tag{1.1}$$

One then obtains an irreducible representation

$$\chi \boxtimes \sigma_i \boxtimes .... \boxtimes \sigma_k \text{ of } Z(\overline{G}_{r,c}) \cdot p^{-1}\left(\prod_{i=1}^{k} G_{r_i,c}^{(n)}\right) \subset \overline{M}_{r,c}.$$

One extends this irreducible representation as much as possible to a subgroup $\overline{M}'_{r,c}$ of $\overline{M}_{r,c}$ and sets

$$\Pi = \mathrm{ind}_{\overline{M}'_{r,c}}^{\overline{M}_{r,c}} \chi \boxtimes (\boxtimes_{i=1}^{k} \sigma_i).$$

It was shown in [M] that the above construction gives a well-defined surjective map

$$\tilde{\otimes} : \left(\mathrm{Irr}_\epsilon(\overline{G}_{r_1,c}) \times .... \times \mathrm{Irr}_\epsilon(\overline{G}_{r_k,c}) \times \mathrm{Irr}_\epsilon(Z(\overline{G}_{r,c}))\right)^{\heartsuit} \longrightarrow \mathrm{Irr}_\epsilon(\overline{M}_{r,c}). \tag{1.2}$$

Here the superscript in $(....)^{\heartsuit}$ indicates that one is considering tuples $(\pi_1, ..., \pi_k, \chi)$ satisfying (1.1). This map is the so-called metaplectic tensor product. It is not injective: replacing each π_i by $\pi_i \otimes (\chi_i \circ \det)$ where χ_i is a character of $F^\times$ such that $\chi_i^n = 1$ would give the same output, but this is the only reason for the non-injectivity.

There is a global analog of the metaplectic tensor product for automorphic representations which has been developed by Takeda; see [T1, T2].

2. L-group Formalism

The goal of this appendix is to give an interpretation of this construction of Mezo in the framework of Langlands functoriality, as developed in [We] and [GG]. To do this, we shall need to recall briefly the theory of dual groups and L-groups for Brylinski-Deligne extensions.

2.1. **Dual group.** Set

$$Y^{\#} = \{y \in Y : B_c(y,z) \in n\mathbb{Z} \text{ for all } z \in Y\} \subset Y \otimes_{\mathbb{Z}} \mathbb{Q},$$

and let $X^{\#}$ be its dual lattice in $X(T_r) \otimes_{\mathbb{Z}} \mathbb{Q}$. For $\alpha \in \Delta_r$, put

$$n_\alpha = n/(n, Q_c(\alpha^\vee)) = n \quad (\text{since } Q_c^(\alpha^\vee) = -1),$$

and set

$$\alpha^\vee_{\#} = n \cdot \alpha^\vee, \quad \text{and} \quad \alpha_{\#} = n^{-1} \cdot \alpha.$$

Denote by $\Delta^\vee_{\#}$ and $\Delta_{\#}$ the sets of these modified coroots and roots. Then $(Y^{\#}, \Delta^\vee_{\#}, X^{\#}, \Delta_{\#})$ is a based root datum and the associated connected reductive group $\overline{G}^\vee_{r,c}$ over $\mathbb{C}$ is the Langlands dual group of $\overline{G}_{r,c}$. It is explicitly given by [GG, §16.2]

$$\overline{G}^\vee_{r,c} \cong \{(g,\lambda) : \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : \det(g) = \lambda^d\} \subset \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$$

with

$$d = \mathrm{GCD}(n, (2c+1)r - 1).$$

2.2. **Structural facts.** Let $H_{r,c}$ be the split linear algebraic group (pinned) whose dual group is $\overline{G}^\vee_{r,c}$. Then

$$H_{r,c} = (\mathrm{GL}_r \times \mathrm{GL}_1)/\{(\lambda, \lambda^{-d}) : \lambda \in \mathrm{GL}_1\}$$

contains the (diagonal) maximal split torus

$$A_{r,c} = Y^{\#} \otimes \mathbb{G}_m \cong (T_r \times \mathrm{GL}_1)/\{(\lambda, \lambda^{-d}) : \lambda \in \mathrm{GL}_1\}.$$

Since $Y^{\#} \subset Y$ and $Y^{\#}_{sc} = \mathbb{Z} \cdot \Delta^\vee_{\#} \subset Y_{sc}$, one may restrict the bisector D_c to $Y^{\#}$ and η_0 to $Y^{\#}_{sc}$. The data $(D_c|_{Y^{\#}}, \eta_0)$ then gives rise to a Brylinski-Deligne cover $\overline{H}_{r,c}$ of $H_{r,c}$ whose dual group is $\overline{H}^\vee_{r,c} = \overline{G}^\vee_{r,c}$.

The inclusion $Y^{\#} \hookrightarrow Y$ induces an isogeny

$$i : A_{r,c} \longrightarrow T_r$$

which is explicitly given by

$$i(t,\lambda) = \lambda^{n/d} \cdot t^n.$$

This isogeny plays a crucial role in the structure theory and representation theory of $\overline{G}_{r,c}$.

For example, one has

$$i(A_{r,c}(F)) = p(Z(\overline{T}_{r,c})),$$

where $Z(\overline{T}_{r,c})$ denotes the center of $\overline{T}_{r,c}$. Alternatively, one may pullback the cover $\overline{T}_{r,c}$ to $A_{r,c}$ via i, yielding a cover $\overline{A}_{r,c} \subset \overline{H}_{r,c}$. Then one has

$$i(\overline{A}_{r,c}) = Z(\overline{T}_{r,c}).$$

Observe that

$$Z(\overline{T}_{r,c}) \supset T_r^n.$$

On the other hand, let $Z(\overline{G}_{r,c})$ be the center of $\overline{G}_{r,c}$. Then one has

$$i(Z(H_{r,c})) = p(Z(\overline{G}_{r,c})) = Z(G_r) \cap p(Z(\overline{T}_{r,c})).$$

While the second equality is true in general, the first is a special feature of Kazhdan-Patterson covers. In any case, we have

$$Z(\overline{T}_{r,c}) = Z(\overline{G}_{n,r}) \cdot T_r^n.$$

Because of the above, the central character of an irreducible genuine representation of $\overline{G}_{r,c}$ is a genuine character of $Z(\overline{H}_{r,c})$ which is trivial on $\mathrm{Ker}(i)$. Note that

$$p(Z(\overline{H}_{r,c})) = \{1\} \times \mathrm{GL}_1(F) \subset A_{r,c}(F).$$

and

$$Z(\overline{H}_{r,c}) = \overline{Z(H_{r,c})}.$$

In particular, $Z(\overline{H}_{r,c})$ is an example of a Brylinski-Deligne cover of GL_1, and its associated dual group is

$$\overline{Z(H_{r,c})}^{\vee} = \overline{H}_{r,c}^{\vee}/[\overline{H}_{r,c}^{\vee}, \overline{H}_{r,c}^{\vee}] = \overline{G}_{r,c}^{\vee}/[\overline{G}_{r,c}^{\vee}, \overline{G}_{r,c}^{\vee}].$$

Thus, its L-group is the pushout of ${}^L\overline{G}_{r,c}$ by the natural map $\overline{G}_{r,c}^{\vee} \longrightarrow \overline{G}_{r,c}^{\vee}/[\overline{G}_{r,c}^{\vee}, \overline{G}_{r,c}^{\vee}]$.

2.3. **L-group and LLC.** In a foundational paper [We] of Weissman, the dual group $\overline{G}_{r,c}^{\vee}$ is enhanced to give an L-group extension ${}^L\overline{G}_{r,c}$:

$$1 \longrightarrow \overline{G}_{r,c}^{\vee} \longrightarrow {}^L\overline{G}_{r,c} \longrightarrow W_F \longrightarrow 1$$

where W_F is the Weil group of F. A more down-to-earth construction of ${}^L\overline{G}_{r,c}$, also due to Weissman, is described in [GG, §5], where it is shown that this L-group extension is split. The set $Spl({}^L\overline{G}_{r,c})$ of splittings over the Weil-Deligne group $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$, modulo the conjugation action of $\overline{G}_{r,c}^{\vee}$, is the set of L-parameters for $\overline{G}_{r,c}$. These L-parameters are expected to classify the irreducible genuine representations of $\overline{G}_{r,c}$.

More precisely, the local Langlands correspondence (LLC) predicts that there is a natural map

$$\mathcal{L} : \mathrm{Irr}_{\epsilon}(\overline{G}_{r,c}) \longrightarrow Spl({}^L\overline{G}_{r,c}).$$

Unlike the case of linear reductive groups, this map is not expected to be surjective, as a consequence of the fact that the isogeny $i : A_{r,c} \longrightarrow T_r$ is not an isomorphism. It is however expected to be injective for the groups $\overline{G}_{r,c}$.

Likewise, if we consider the cover $\overline{M}_{r,c}$ of the Levi subgroup M_r, then

$$\overline{M}_{r,c}^{\vee} \hookrightarrow \overline{G}_{r,c}^{\vee}$$

is the Levi subgroup of type $(r_1, ..., r_k)$ and one has [GG, Lemma 5.3]

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \overline{M}_{r,c}^{\vee} & \longrightarrow & {}^L\overline{M}_{r,c} & \longrightarrow & W_F & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \overline{G}_{r,c}^{\vee} & \longrightarrow & {}^L\overline{G}_{r,c} & \longrightarrow & W_F & \longrightarrow & 1. \end{array} \tag{2.1}$$

The LLC predicts a natural map

$$\mathcal{L}_M : \mathrm{Irr}_\epsilon(\overline{M}_{r,c}) \longrightarrow Spl({}^L\overline{M}_{r,c}).$$

2.4. **Desiderata.** We highlight some expected properties of the LLC which will be used later on.

- (Central characters) If $\pi \in \mathrm{Irr}_\epsilon(\overline{G}_{r,c})$ has central character ω_π, regarded as a genuine character of $Z(\overline{H}_{r,c}) = \overline{Z(H_{r,c})}$, then the L-parameter of ω_π is deduced from that of π by the pushout via $\overline{G}_{r,c}^{\vee} \longrightarrow \overline{G}_{r,c}^{\vee}/[\overline{G}_{r,c}^{\vee}, \overline{G}_{r,c}^{\vee}]$. One way of expressing this is that one has commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}_\epsilon(\overline{G}_{r,c}) & \xrightarrow{\mathcal{L}} & Spl({}^L\overline{G}_{r,c}) \\ \downarrow & & \downarrow \\ \mathrm{Irr}_\epsilon(Z(\overline{G}_{r,c})) & \xrightarrow{\mathcal{L}} & Spl({}^L\overline{Z(H_{r,c})}) \end{array}$$

where the first vertical arrow is the central character map and the second vertical arrow is induced by the natural map from $\overline{G}_{r,c}^{\vee}$ to its cocenter $\overline{G}_{r,c}^{\vee}/[\overline{G}_{r,c}^{\vee}, \overline{G}_{r,c}^{\vee}]$.

- (Twisting) If $\pi \in \mathrm{Irr}_\epsilon(\overline{G}_{r,c})$ and $\chi : G_r \longrightarrow \mathbb{C}^\times$ is a 1-dimensional character, then $\pi \otimes (\chi \circ \det) \in \mathrm{Irr}_\epsilon(\overline{G}_{r,c})$ also. If the L-parameter of π is $\phi : WD_F \longrightarrow {}^L\overline{G}_{r,c}$ and that of χ is

$$\phi_\chi : W_F \longrightarrow Z(G_r^\vee) \cong \mathbb{C}^\times \subset G_r^\vee \cong \mathrm{GL}_r(\mathbb{C}),$$

then the L-parameter of $\pi \otimes (\chi \circ \det)$ should be given by [GG, §12.2]. Specializing to the case of interest here, we have a natural map

$$\delta : Z(G_r^\vee) = \mathbb{C}^\times \longrightarrow Z(\overline{G}_{r,c}^{\vee}) = \{(a \cdot I_r, b) \in \mathbb{C}^\times \times \mathbb{C}^\times : a^r = b^d\}$$

given by

$$\delta(z) = (z^n, z^{r \cdot \frac{n}{d}}).$$

Then the L-parameter of $\pi \otimes (\chi \circ \det)$ is given by $\phi \otimes (\delta \circ \phi_\chi)$, and for $w \in W_F$, one has

$$\delta \circ \phi_\chi(w) = (\chi(w)^n I_r, \chi(w)^{rn/d}) \in \overline{G}_{r,c}^{\vee}.$$

2.5. **LLC for covering tori.** We now specialize to the case when $M_r = T_r$. In this case, the LLC has been shown, i.e. the map $\mathcal{L}_T$ has been constructed. More precisely, since $\overline{T}_{r,c}$ is a Heisenberg group, an irreducible genuine representation is determined by its central character. Hence we have natural maps:

$$\mathrm{Irr}_\epsilon(\overline{T}_{r,c}) \longleftrightarrow \mathrm{Irr}_\epsilon(Z(\overline{T}_{r,c})) \hookrightarrow \mathrm{Irr}_\epsilon(\overline{A}_{r,c})$$

where the inclusion is induced by $i : \overline{A}_{r,c} \longrightarrow \overline{T}_{r,c}$. It was shown in [GG, §8] that one has a map

$$\mathrm{Irr}_\epsilon(\overline{A}_{r,c}) \longleftrightarrow Spl({}^L\overline{A}_{r,c}) = Spl({}^L\overline{T}_{r,c}).$$

The composite of these maps give the desired

$$\mathcal{L}_T : \mathrm{Irr}_\epsilon(\overline{T}_{r,c}) \longrightarrow Spl({}^L\overline{T}_{r,c}).$$

The above construction of the LLC for $\overline{T}_{r,c}$ does not care that T_r is a maximal split torus of G_r. Let us take that into account now. In this case, the Weyl group $W = N_{G_r}(T_r)/T_r$ acts naturally on $\mathrm{Irr}_\epsilon(\overline{T}_{r,c})$ and $Spl({}^L\overline{T}_{r,c})$. It was shown in [GG, §9.3 and Prop. 9.5] that the LLC map $\mathcal{L}_T$ is W-equivariant.

2.6. **LLC for principal series.** The above properties of the LLC for $\overline{T}_{r,c}$ allows us to define the LLC map $\mathcal{L}$ for principal series representations of $\overline{G}_{r,c}$. Namely, one expects a commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}_\epsilon(\overline{T}_{r,c}) & \xrightarrow{\mathcal{L}_T} & Spl({}^L\overline{T}_{r,c}) \\ \downarrow & & \downarrow \\ \mathrm{Irr}_\epsilon(\overline{G}_{r,c}) & \xrightarrow{\mathcal{L}} & Spl({}^L\overline{G}_{r,c}) \end{array} \tag{2.2}$$

Here the first vertical arrow is via parabolic induction and taking Langlands quotient whereas the second is by the natural inclusion of L-groups. Because the LLC map $\mathcal{L}_T$ is W-equivariant and the two vertical arrows are W-invariant, this commutativity diagram serves to define the map $\mathcal{L}$ on the set $\mathrm{Irr}_{\epsilon,ps}(\overline{G}_{r,c})$ of those genuine representations of $\overline{G}_{r,c}$ which are Langlands quotient of standard modules induced from the Borel subgroup B_r.

Explicitly, a principal series representation of $\overline{G}_{r,c}$ is of the form

$$I(\chi) = \mathrm{Ind}_{\overline{B}_r}^{\overline{G}_{r,c}} \tau(\chi)$$

where $\tau(\chi)$ is the irreducible representation of $\overline{T}_{r,c}$ with central character χ on $Z(\overline{T}_{r,c})$, or equivalently χ is a character of $\overline{A}_{r,c}$ which is trivial on $\mathrm{Ker}(i)$. By replacing χ by a W-translate, we may assume $I(\chi)$ is a standard module and denote its unique irreducible quotient by $J(\chi)$. If the L-parameter of χ is

$$\phi_\chi : W_F \longrightarrow {}^L A_{r,c} = {}^L\overline{T}_{r,c},$$

then the L-parameter of $J(\chi)$ is

$$\mathcal{L}(J(\chi)) : W_F \xrightarrow{\phi_\chi} {}^L T_{r,c} \longrightarrow {}^L\overline{G}_{r,c}.$$

Likewise, one has a classification of the set $\mathrm{Irr}_{\epsilon,ps}(\overline{M}_{r,c})$ of (Langlands quotients of) principal series representations of $\overline{M}_{r,c}$, since T_r is a maximal split torus of M_r. In other words, one has a commutative diagram

$$\begin{array}{ccc} \mathrm{Irr}_\epsilon(\overline{T}_{r,c}) & \longrightarrow & Spl({}^L\overline{T}_{r,c}) \\ \downarrow & & \downarrow \\ \mathrm{Irr}_{\epsilon,ps}(\overline{M}_{r,c}) & \xrightarrow{\mathcal{L}_M} & Spl({}^L\overline{M}_{r,c}) \end{array} \tag{2.3}$$

where the first vertical row is parabolic induction (and taking Langlands quotient) and is W_M-invariant.

2.7. **Distinguished splittings.** In [GG, §6 and §7], we have defined, constructed and classified a set of so-called distinguished splittings of the L-group extension ${}^L\overline{G}_{r,c}$. It was shown that a distinguished splitting of ${}^L\overline{G}_{r,c}$ takes value in ${}^L\overline{T}_{r,c}$ and gives rise to the following:

- it gives an isomorphism
$$^L\overline{G}_{r,c} \cong \overline{G}_{r,c}^\vee \times W_F = {}^L H_{r,c},$$
and hence a bijective map (depending on the distinguished splitting).
$$\{L\text{-parameters of } \overline{G}_{r,c}\} \longleftrightarrow \{L\text{-parameters of } H_{r,c}\}.$$
- it gives a distinguished W-invariant genuine character χ of $Z(\overline{T}_{r,c})$, or equivalently a genuine character of $\overline{A}_{r,c}$, which is trivial on the kernel of i. One can restrict such a distinguished W-invariant character of $Z(\overline{T}_{r,c})$ to the center $Z(\overline{G}_{r,c})$.

We highlight a key property of these distinguished characters in the context of the Kazhdan-Patterson covers, which follows from their definition and construction; see [GG, §7 and §16.2]:

Lemma 2.4. *The distinguished characters of $Z(\overline{T}_{r,c})$ are trivial on the subgroup $T_r^n \subset Z(\overline{T}_{r,c})$. Any two distinguished characters differ from each other by twisting by a character of $p(Z(\overline{T}_{r,c}))/T_r^n \cong (F^\times)^{n/d}/F^{\times n}$. Pulled back to $A_{r,c}$ via i, this gives a character of $Z(H_{r,c})/Z(H_{r,c})^d$ (which is a quotient of $A_{r,c}$ by the second projection).*

Moreover, it was shown in [GG, §7 and §16.2] that given an additive character ψ of F, one can construct an associated distinguished splitting and hence a W-invariant genuine character χ_ψ of $Z(\overline{T}_{r,c})$. Using this, one has an associated bijection (depending on ψ)

$$Spl({}^L\overline{G}_{r,c}) \longleftrightarrow \mathrm{Hom}(W_F, \overline{G}_{r,c}^\vee)/\overline{G}_{r,c}^\vee\text{-conjugacy}.$$

The analogous statement holds for any of the Levi covers $\overline{M}_{r,c}$. Thus, the use of a distinguished splitting (or equivalently a distinguished W-invariant genuine character of $\overline{T}_{r,c}$) is to serve as a base-point and thus allow one to work with the dual group instead of the L-group extensions.

3. L-group interpretation of metaplectic tensor product

With the above preparation, we are now ready to formulate an interpretation of the metaplectic tensor product using the L-group.

3.1. **Setup.** Recall that $M \subset \mathrm{GL}_r$ is a Levi subgroup, with

$$M = \mathrm{GL}_{r_1} \times .... \times \mathrm{GL}_{r_k}.$$

Restricting the cover $\overline{G}_{r,c}$ to M gives a cover $\overline{M}_{r,c}$, whose dual group is

$$\overline{M}_{r,c}^{\vee} = \{(g_1, ..., g_k, \lambda) \in \prod_{i=1}^{k} \mathrm{GL}_{r_i}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : \prod_{i=1}^{k} \det(g_i) = \lambda^d\}.$$

On the other hand, for each $G_{r_i} = \mathrm{GL}_{r_i}$ in M, we have the (degree n) Kazhdan-Patterson cover $p^{-1}(G_{r_i,c}) \cong \overline{G}_{r_i,c}$, with its own dual group $\overline{G}_{r_i,c}^{\vee}$. Setting

$$d_i = \mathrm{GCD}(n, (2c+1)r_i - 1),$$

one has

$$\overline{G}_{r_i,c}^{\vee} = \{(g, \lambda) : \mathrm{GL}_{r_i}(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : \det(g) = \lambda^{d_i}\}.$$

We shall write det for the character of $\overline{G}_{r_i,c}$ given by the composite of the first projection to $\mathrm{GL}_{r_i}(\mathbb{C})$ and the determinant map of $\mathrm{GL}_{r_i}(\mathbb{C})$.

In the metaplectic tensor product, one starts with a tuple $(\pi_1, ..., \pi_k, \chi)$ satisfying the compatibility condition (1.1). Let us imagine for a moment that LLC holds and we have fixed a nontrivial additive character ψ of F, which determines distinguished splittings of each ${}^L\overline{G}_{r_i,c}$, ${}^L\overline{Z(H_{r,c})}$ and ${}^L\overline{M}_{r,c}$. Let

$$\phi_i : WD_F \longrightarrow \overline{G}_{r_i,c}^{\vee} \quad \text{and} \quad \phi_\chi : W_F \longrightarrow \mathbb{C}^\times$$

be the associated L-parameters of π_i and χ. Hence, we have

$$\phi_1 \times ... \times \phi_k \times \phi_\chi : WD_F \longrightarrow \left(\prod_{i=1}^{k} \overline{G}_{r_i,c}^{\vee}\right) \times \mathbb{C}^\times.$$

How is the compatibility condition (1.1) expressed in terms of L-parameters?

Lemma 3.1. *The compatibility condition (1.1) is equivalent to*

$$\prod_{i=1}^{k} \det \phi_i = \phi_\chi^d.$$

Proof. We need to work out

$$Z(\overline{G}_{r,c}) \cap p^{-1}\left(\prod_{i=1}^{k} G_{r_i,c}^{(n)}\right).$$

The projection of this to $\mathrm{GL}_r(F)$ consists of scalar matrices $a^{n/d} \cdot I_r$ satisfying

$$a^{nr_i/d} \in F^{\times n} \quad \text{for each } i = 1, ..., k.$$

This is equivalent to

$$a \in (F^\times)^{d/(d,r_i)} \quad \text{for each } i. \tag{3.2}$$

Now observe that

$$(d, r_1, r_2, ..., r_k) = (n, (2c+1)\cdot(\sum_{i=1}^{k} r_i) - 1, r_1, ..., r_k) = 1.$$

Hence

$$\mathrm{LCM}(d/(d,r_1), ..., d/(d,r_k)) = d.$$

Hence the condition in (3.2) is equivalent to $a \in F^{\times d}$. In particular,

$$p\left(Z(\overline{G}_{r,c}) \cap p^{-1}\left(\prod_{i=1}^{k} G^{(n)}_{r_i,c}\right)\right) = \{a^n I_r : a \in F^\times\} = Z(G_r)^n.$$

Since

$$i(A^{d_i}_{r_i,c}) = T^n_{r_i} \quad \text{and} \quad i(A^d_{r,c}) = T^n_r,$$

it follows that (1.1) is equivalent to the identity of L-parameters in the lemma. □

The above lemma implies that the parameter $\phi_1 \times ... \times \phi_k \times \phi_\chi$ factors through the subgroup

$$\mathcal{M}^\heartsuit \subset \left(\prod_{i=1}^{k} \overline{G}^\vee_{r_i,c}\right) \times \mathrm{GL}_1(\mathbb{C})$$

consisting of those elements

$$\left(\prod_{i=1}^{k} (g_i, \lambda_i), \lambda\right)$$

satisfying

$$\lambda^d = \prod_{i=1}^{k} \det(g_i) = \prod_{i=1}^{k} \lambda_i^{d_i}.$$

3.2. **The conjecture.** Observe that one may define a map

$$f : \mathcal{M}^\heartsuit \longrightarrow \overline{M}^\vee_{r,c}$$

by

$$\left(\prod_{i=1}^{k} (g_i, \lambda_i), \lambda\right) \mapsto (g_1, ..., g_k, \lambda).$$

Note that the kernel of f is

$$\mu_{d_i} \times ...... \times \mu_{d_k},$$

consisting of elements $\left(\prod_{i=1}^{k} (g_i, \lambda_i), \lambda\right)$ with $g_i = 1$, $\lambda = 1$ and $\lambda_i \in \mu_{d_i}$. The above discussion motivates the following conjecture:

Conjecture

The metaplectic tensor product $\tilde{\otimes}$ defined in (1.2) is the Langlands functorial lift associated to the map $f : \mathcal{M}^\heartsuit \longrightarrow \overline{M}^\vee_{r,c}$ defined above.

The above conjecture is not a statement which can be proved at this moment, since it is conditional upon the LLC for covering groups. We make a couple of remarks as a sort of consistency check:

- the metaplectic tensor product construction does not depend on the choice of distinguished characters of $Z(\overline{T}_{r,c})$, $Z(\overline{G}_{r,c})$ or $Z(\overline{T}_{r_i,c})$, but the map f only induces a lifting of L-parameters if one fixes distinguished characters on these groups. So it will be pertinent to check that in fact, the induced lifting of L-parameters is independent of the choice of such distinguished characters.

  To see this, note that for each i, it follows by Lemma 2.4 that two distinguished characters of $Z(\overline{T}_{r_i,c})$, regarded as characters of $\overline{A}_{r_i,c}$, differs by twisting by a character μ of $Z(H_{r_i,c})$ with $\mu^{d_i} = 1$. Their L-parameters differ by a homomorphism $\phi_\mu : W_F \hookrightarrow Z(H_{r_i,c})^\vee = \mathbb{C}^\times$ with $\phi_\mu^{d_i} = 1$. Hence $f \circ \phi_\mu$ is trivial, so that the choice of the distinguished character of $Z(\overline{T}_{r_i,c})$ is not important.

  On the other hand, having chosen and fixed a distinguished character χ on $Z(\overline{T}_{r,c})$, we inherit one on $Z(\overline{G}_{r,c})$ by restriction and hence one on $\overline{Z(H_{r,c})}$ by pullback. One checks that as long as one uses distinguished splittings of ${}^L\overline{G}_{r,c}$ and ${}^L\overline{Z(H_{r,c})}$ related in this way, the lifting of L-parameters induced by f is independent of the choice of distinguished splittings.

- the lifting of L-parameters induced by f is not injective: $(\phi_i, ..., \phi_k, \chi)$ and $(\phi'_1, ..., \phi'_k, \chi')$ have same image if and only if $\chi = \chi'$ and for each i, ϕ_i and ϕ'_i differs by a homomorphism
$$\mu_i : W_F \longrightarrow \{1\} \times \mu_{d_i} \subset \{1\} \times \mathrm{GL}_1(\mathbb{C}).$$
  This agrees exactly with the non-injectivity of the metaplectic tensor product construction, as we now explain. By the discussion at the end of §1.3, the metaplectic tensor product does not change if and only if we replace the representation π_i by $\pi_i \otimes (\chi_i \circ \det)$ with $\chi_i^n = 1$. By the desiderata (Twisting) in §2.4, this replaces the L-parameter ϕ_i by $\phi_i \otimes (\delta \circ \phi_{\chi_i})$, where ϕ_{χ_i} is the L-parameter of χ_i and δ is defined in §2.4. But for $w \in W_F$,
$$\delta \circ \phi_{\chi_i}(w) = (\chi_i(w)^n, \chi_i(w)^{r_i \cdot \frac{n}{d_i}}) = (1, \chi_i^{r_i n/d_i}(w)).$$
  The character $\mu_i := \chi_i^{r_i n/d_i}$ satisfies $\mu_i^{d_i} = 1$; its order is the same as that of χ_i^{n/d_i} since $(r_i, d_i) = 1$.

3.3. **Case of principal series.** As we have explained in the previous section, the LLC is known for principal series representations induced from a Borel subgroup. The main result of this note is the demonstration of the above conjecture for such principal series representations.

Proposition 3.3. *The above conjecture holds when each* π_i *belongs to* $\mathrm{Irr}_{\epsilon,ps}(\overline{G}_{r_i,c})$.

Proof. The metaplectic tensor product of principal series representations was determined by Cai [C, Theorem 3.26] and the point is to interpret the result on the dual side. We give an independent treatment here.

We first consider the case when $M_r = T_r$ is the maximal split torus; this is the key case to understand. Thus, we are assuming that $r_i = 1$ for all i and $k = r$, so that

$$d_0 := d_i = \mathrm{GCD}(n, 2c).$$

Then

$$H_{1,c} = A_{1,c} = (\mathrm{GL}_1(F) \times \mathrm{GL}_1(F))/\{(t, t^{-d_0}) : t \in F^\times\} \cong \mathrm{GL}_1(F)$$

by the second projection, so that

$$H_{1,c}^\vee = A_{1,c}^\vee \cong \{(g, \lambda) \in \mathrm{GL}_1(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C}) : g = \lambda^{d_0}\} \cong \mathbb{C}^\times$$

via the second projection. The isogeny $i : A_{1,c} \longrightarrow T_1$ is the map $\lambda \mapsto \lambda^{n/d_0}$.

The input into the metaplectic tensor product is then a tuple $(\pi_1, ..., \pi_r, \chi)$ where each π_i is a genuine representation $\tau(\chi_\psi \chi_i)$ of $\overline{T}_{1,c}$, where χ_ψ is a distinguished character of $Z(\overline{T}_{1,c}) = p^{-1}(T_1^{n/d_0})$ and χ_i is a character of $A_{1,c} \cong F^\times$ which is trivial on $\mu_{n/d_0}(F)$. The compatibility condition (1.1) is given by

$$(\prod_{i=1}^{k} \chi_i)^{d_0} = \chi^d.$$

Moreover, one has

$$\phi_{\chi_1} \times ... \times \phi_{\chi_r} \times \phi_\chi : W_F \longrightarrow \mathcal{M}^\heartsuit \subset \mathrm{GL}_1(\mathbb{C})^r \times \mathrm{GL}_1(\mathbb{C}),$$

so that

$$f \circ \phi_{\chi_1} \times ... \times \phi_{\chi_r} \times \phi_\chi = (\phi_{\chi_1}^{d_0} \times ... \times \phi_{\chi_r}^{d_0}) \times \phi_\chi : W_F \longrightarrow \overline{T}_{r,c}^\vee \subset \mathrm{GL}_1(\mathbb{C})^r \times \mathrm{GL}_1(\mathbb{C}).$$

Consider now the construction of the metaplectic tensor product. We first restrict each π_i to

$$T_1^n \times \mu_n(F) = \overline{T}_{1,c}^{(n)} = i\left(\overline{A}_{1,c}^{d_0}\right).$$

Since T_1^n is contained in the center of $\overline{T}_{1,c}$ (this center is $\overline{T}_{1,v}^{(n/d_0)}$), the restriction of π_i to T_1^n is simply the isotypic sum of its central character χ_i restricted to T_1^n (here we have used Lemma 2.4 which says that the distinguished character χ_ψ is trivial on T_1^n). We then consider the character

$$\chi_\psi \cdot (\chi \boxtimes (\boxtimes_{i=1}^r \chi_i)) \text{ on the subgroup } Z(\overline{G}_{r,c}))(T_1^n \times .... \times T_1^n),$$

where now χ_ψ denotes a distinguished character of $Z(\overline{T}_{r,c})$ restricted to $Z(\overline{G}_{r,c})$. But this subgroup is precisely the center $Z(\overline{T}_{r,c})$ of $\overline{T}_{r,c}$, and so this character determines an irreducible genuine representation of $\overline{T}_{r,c}$. Explicitly, the character

$$\chi \boxtimes (\boxtimes_{i=1}^r \chi_i) \text{ of } p(Z(\overline{G}_{r,c}))(T_1^n \times .... \times T_1^n) = p(Z(\overline{T}_{r,c}))$$

is given by:

$$\begin{pmatrix} a_1^n \lambda^{n/d} & & & \\ & a_2^n \lambda^{n/d} & & \\ & & \cdots & \\ & & & a_r^n \lambda^{n/d} \end{pmatrix} \mapsto \chi_1(a_1)^{d_0} \cdot ... \cdot \chi_r(a_r)^{d_0} \cdot \chi(\lambda).$$

By our construction of the LLC for $\overline{T}_{r,c}$, we see that the L-parameter of the genuine representation of $\overline{T}_{r,c}$ with this central character is precisely

$$(\phi_{\chi_1}^{d_0} \times ... \times \phi_{\chi_r}^{d_0}) \times \phi_\chi : W_F \longrightarrow T_r^\vee \times \mathrm{GL}_1(\mathbb{C}) \longrightarrow \mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_1(\mathbb{C})$$

which is equal to $f \circ (\phi_{\chi_1} \times .... \times \phi_{\chi_r} \times \phi_\chi)$, as desired.

Now we consider the case of general $M_r = \mathrm{GL}_{r_1} \times ... \times \mathrm{GL}_{r_k}$. Given irreducible genuine principal series representations π_i of $\overline{G}_{r_i,c}$ associated to genuine characters χ_i of $p(Z(\overline{T}_{r_i,c}))$, we are to restrict them to $\overline{G}^{(n)}_{r_i,c}$, take an irreducible summand σ_i and then consider the representation

$$\boxtimes_{i=1}^k \sigma_i \boxtimes \chi \text{ on } Z(\overline{G}_{r,c}) \cdot \left(\overline{G}^{(n)}_{r_1,c} \times_{\mu_n} \times ..... \times_{\mu_n} \overline{G}^{(n)}_{r_k,c}\right).$$

The resulting metaplectic tensor product representation is undoubtedly a principal series representation of $\overline{M}_{r,c}$ (as shown in [C, Theorem 3.26]), and hence is determined by a character of $p(Z(\overline{T}_{r,c}))$. Now the main point is that

$$Z(\overline{T}_{r,c}) = Z(\overline{G}_{r,c}) \cdot T_r^n = Z(\overline{G}_{r,c}) \cdot (T_{r_1}^n \times .... \times T_{r_k}^n) \subset Z(\overline{G}_{r,c}) \cdot \left(\overline{G}^{(n)}_{r_1,c} \times_{\mu_n} \times ..... \times_{\mu_n} \overline{G}^{(n)}_{r_k,c}\right).$$

Hence, the resulting metaplectic tensor product representation is determined by the behaviour of $(\boxtimes_{i=1}^k \pi_i) \boxtimes \chi$ on $Z(\overline{T}_{r,c})$. Because of the commutativity in (2.2) and (2.3), we are basically reduced to a question on covering tori, which is a slight generalization of the case when $M_r = T_r$ treated above. Arguing as in that special case, one sees that the metaplectic tensor product on $\overline{G}_{r,c}$ is constructed from the character

$$\chi \boxtimes (\boxtimes_{i=1}^k \chi_i) \quad \text{ of } p(Z(\overline{G}_{r,c})) \cdot T_r^n,$$

and this gives the parameter

$$\phi_{\chi_1}^{d_1} \times ... \times \phi_{\chi_k}^{d_k} \times \phi_\chi = f \circ (\phi_{\chi_1} \times ... \times \phi_{\chi_k} \times \phi_\chi))$$

as desired. □

As we mentioned earlier, Takeda [T1, T2] has developed the notion of metaplectic tensor product in the global setting of automorphic representations. The proposition thus allows one to interpret his construction as an instance of weak Langlands functorial lifting relative to the homomorphism $f : \mathcal{M}^\heartsuit \longrightarrow \overline{M}_{r,c}^\vee$ of dual groups.

Acknowledgments: I thank Shuichiro Takeda for his suggestion to write up this note and for helpful discussions during the course of this work. This note was initially meant to be an appendix to [T2], but it has become too lengthy for this purpose. I also thank Yuanqing Cai for bringing Theorem 3.26 of his paper [C] to my attention and Fan Gao for his comments on the first draft of this note.

References

[C] Y.Q. Cai, *Fourier coefficients for theta representaitons on covers of general linear groups*, preprint, available at http://arxiv.org/abs/1602.06614.

[GG] W.T. Gan and F. Gao, *The Langlands-Weissman program for Brylinski-Deligne extensions*, preprint, available at http://arxiv.org/pdf/1409.4039v1.pdf

[M] P. Mezo, *Metaplectic tensor products for irreducible representations*, Pacific J. of Math. Vol. 215, No 1, 85-96 (2004).

[KP] D. A. Kazhdan and S. J. Patterson, *Metaplectic forms*, I.H.E.S. Publ. Math. **59** (1984), 35-142.

[T1] S. Takeda, *Metaplectic tensor products for automorphic representations of* $\mathrm{GL}(r)$, Canad. J. Math 68 (2016), 179-240.

[T2] S. Takeda, *Remarks on metaplectic tensor products for covers of* $\mathrm{GL}(r)$, preprint (2016).

[We] M. Weissman, *L-groups and parameters for covering groups*, preprint, available at arXiv:1507.01042.

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road Singapore 119076

E-mail address: matgwt@nus.edu.sg