

# THE MASS OF UNIMODULAR LATTICES

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## 1. Introduction

An old but fundamental problem in the arithmetic theory of quadratic forms is the computation of the mass of a lattice  $L$  in a quadratic space  $(V, q)$  over a number field  $F$ . Among the pioneers of its study were Smith, Minkowski and Siegel. After the work of Kneser, Tamagawa and Weil, this problem can be neatly formulated in group theoretic terms. More precisely, if  $G$  denotes the special orthogonal group  $SO(V, q)$ , then  $\text{mass}(L)$  can be expressed as the volume of  $G(F)\backslash G(\mathbb{A})$  with respect to a volume form  $\mu_L$  associated naturally to  $L$ , and one can relate  $\mu_L$  to the Tamagawa measure of  $G(\mathbb{A})$  by virtue of certain local densities. Since the Tamagawa number of  $G$  is equal to 2, the computation of  $\text{mass}(L)$  is thus reduced to the computation of these local densities.

The computation of local densities is, however, not an entirely trivial task, especially at a 2-adic place of  $F$ . Though there has been much work in this direction, in the introduction to his recent paper [S], Shimura lamented the lack of exact formulas in the literature for  $\text{mass}(L)$ , even for certain special lattices. He then went on to obtain such an exact formula for the maximal lattice in  $(V, q)$ .

The purpose of this paper is to obtain the mass of a unimodular lattice from the point of view of Bruhat-Tits theory. This is achieved by relating the local stabilizer of the lattice to a maximal parahoric subgroup of the special orthogonal group, and appealing to an explicit mass formula for parahoric subgroups developed in [GHY]. This explicit formula is a consequence of the important work of Prasad [P] and its extension by Gross [Gr], and can be used to derive the results of [S] (as was done in [GHY]).

Of course, the exact mass formula for positive definite unimodular lattices is well-known (cf. for example [CS, Pg. 409]). Moreover, the exact formula for lattices of signature  $(n, 1)$  (which give rise to hyperbolic orbifolds) was obtained by Ratcliffe-Tschantz [RT], starting from the fundamental work of Siegel. Our approach works uniformly for unimodular lattices of arbitrary signature  $(r, s)$  and hopefully gives a more conceptual way of deriving the above known results. The final formulas are stated in Theorems 6.2 and 6.3.

## 2. Basic Notions

Let  $F$  be a number field with ring of integers  $A$ . For each place  $v$  of  $F$ , let  $F_v$  denote the corresponding local field with ring of integers  $A_v$  if  $v$  is finite. We shall let  $\mathbb{A}$  be the adèle ring of  $F$  and  $\mathbb{A}_f$  the subring of finite adèles.

Let  $(V, q)$  be a quadratic space over  $F$  of dimension  $d$ , and let  $B_q$  be the symmetric bilinear form defined by:

$$B_q(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y)), \quad x, y \in V.$$

Note that

$$B_q(x, x) = q(x).$$

The discriminant of  $(V, q)$  is defined as follows. One diagonalizes  $q$  using a suitable basis of  $V$ , say  $q(x_1, \dots, x_d) = \sum_i a_i x_i^2$ ; then one sets

$$\text{disc}(q) = (-1)^{d(d-1)/2} \cdot \prod_i a_i \in F^\times / F^{\times 2}.$$

The square class of  $\text{disc}(q)$  determines a quadratic character  $\chi_{\text{disc}(q)}$  of  $\text{Gal}(\overline{F}/F)$ , and we let  $E_{\text{disc}(q)}$  be the étale quadratic algebra determined by  $\chi_{\text{disc}(q)}$ . More precisely,

$$E_{\text{disc}(q)} = \begin{cases} F \times F, & \text{if } \text{disc}(q) \in F^{\times 2}; \\ F(\sqrt{\text{disc}(q)}), & \text{if } \text{disc}(q) \notin F^{\times 2}. \end{cases}$$

Let  $L \subset V$  be a lattice on which  $q$  takes integer values. We shall write  $L_v$  for the localization  $L \otimes A_v$ . Recall the following basic definition:

**Definition:** The *genus* of  $L$  is the set of isomorphism classes of lattices  $M$  in  $(V, q)$  such that

$$(L_v, q) \cong (M_v, q) \quad \text{for all finite places } v.$$

Let  $G = SO(V, q)$  be the special orthogonal group associated with  $(V, q)$ . To avoid having to work with non-semisimple groups, we assume henceforth that  $d = \dim(V) \geq 3$ . This is for uniformity of exposition and is not a serious assumption. If

$$K_L = \prod_{v < \infty} K_{L_v} \subset G(\mathbb{A}_f)$$

is the stabilizer of  $\widehat{L} = \prod_{v < \infty} L_v$ , then the genus of  $L$  can be indexed by the double coset space

$$G(F) \backslash G(\mathbb{A}_f) / K_L.$$

This is a finite set and for  $\alpha \in G(F) \backslash G(\mathbb{A}_f) / K_L$ , the corresponding lattice in  $(V, q)$  is given by

$$L_\alpha = V \cap \alpha \widehat{L},$$

with the intersection occurring in  $V \otimes \mathbb{A}_f$ . For each  $\alpha$ , let

$$\Gamma_\alpha = G(F) \cap \alpha K_L \alpha^{-1}.$$

It is the stabilizer in  $G(F)$  of  $L_\alpha$  and has finite covolume in  $G(F \otimes \mathbb{R})$ .

We can now introduce an important classical invariant for the genus of  $L$ ; it is called the mass of  $L$ . When  $F$  is totally real and  $q$  is totally definite, this is defined by

$$Mass(L) = \sum_{\alpha} \frac{1}{\#\Gamma_{\alpha}}.$$

Another way to define this is as follows. Let  $\mu_L$  be the Haar measure on  $G(\mathbb{A})$  which gives the open compact subgroup  $G(F \otimes \mathbb{R}) \times K_L$  volume 1. Then

$$Mass(L) = \int_{G(F) \backslash G(\mathbb{A})} \mu_L.$$

Using the above integral formula, one can extend the definition of  $Mass(L)$  to the indefinite case. However, to define the measure  $\mu_L$  in general, one needs to specify a Haar measure on  $G(F \otimes \mathbb{R})$ . Different authors made different choices for this; we shall now explain our choice.

There are two natural choices of Haar measure on  $G(F \otimes \mathbb{R})$ , at least from the group theoretic point of view. Any reductive algebraic group over  $\mathbb{R}$ , such as  $Res_{F/\mathbb{Q}} G \times \mathbb{R}$ , has a unique split form and a unique compact form and each of these has a natural Haar measure. For the split form, it is the measure induced by an invariant differential form of top degree on the canonical Chevalley model over  $\mathbb{Z}$ . For the compact form, it is the measure giving the group volume 1. Each of these two measures can be transferred to any other forms of the group in a standard way, as described in [Gr, §11]. In this way, we obtain two natural Haar measures  $\mu_{cpt}$  and  $\mu_{sp}$  on  $G(F \otimes \mathbb{R})$ . They are related by (cf. [Gr, §7])

$$\mu_{sp} = \gamma_G^{deg(F)} \cdot \mu_{cpt}$$

where  $deg(F)$  is the degree of  $F$  over  $\mathbb{Q}$  and

$$\gamma_G = \begin{cases} \frac{(2\pi)^{n(n+1)}}{\prod_{r=1}^n (2r-1)!}, & \text{if } d = 2n + 1; \\ \frac{(2\pi)^{n^2}}{(n-1)! \cdot \prod_{r=1}^{n-1} (2r-1)!}, & \text{if } d = 2n. \end{cases}$$

The measure we use for the definition of  $Mass(L)$  is  $\mu_{cpt}$ . Thus we set

$$\mu_L = \mu_{cpt} \times \prod_{v < \infty} \mu_{L_v}$$

where  $\mu_{L_v}$  is the Haar measure of  $G(F_v)$  giving  $K_{L_v}$  volume 1. Then  $Mass(L)$  is defined in general by:

$$Mass(L) = \int_{G(F) \backslash G(\mathbb{A})} \mu_L.$$

When  $q$  is totally definite, this agrees with the classical definition above. When  $q$  is indefinite, we have:

$$Mass(L) = \sum_{\alpha} \int_{\Gamma_{\alpha} \backslash G(F \otimes \mathbb{R})} \mu_{cpt}.$$

### 3. A Mass Formula

For a natural class of lattices  $L$  in  $(V, q)$ , an explicit formula for  $Mass(L)$  was given in [GHY, Proposition 2.13], based on the fundamental work [P] and its extension [Gr]. These are the lattices for which the local stabilizers  $K_{L_v}$  are parahoric subgroups of  $G(F_v)$  (in the sense of Bruhat-Tits theory). To state this formula, we need to introduce some more notations.

By Bruhat-Tits theory, there is a smooth affine group scheme  $\underline{G}_v$  over  $A_v$ , with generic fiber  $G \times_F F_v$ , such that

$$K_{L_v} = \underline{G}_v(A_v).$$

Let  $\overline{G}_v$  be the maximal reductive quotient of the special fiber of  $\underline{G}_v$ ; it is a (possibly disconnected) reductive algebraic group over the residue field of  $F_v$ . Let  $N(\overline{G}_v)$  be the number of positive roots of  $\overline{G}_v$  over the algebraic closure. Then the number

$$q_v^{-N(\overline{G}_v)} \cdot \#\overline{G}_v(A_v/\pi_v)$$

is a product of certain local  $L$ -factors and can be easily computed once one identifies  $\overline{G}_v$ .

Similarly, if  $G_{qs}$  is the quasi-split inner form of  $G$ , then we may consider the connected integral model  $\underline{G}_{qs,v}$  associated to the special maximal compact subgroup of  $G_{qs}(F_v)$  specified in [Gr, §4]. As above, we may define the number

$$q_v^{-N(\overline{G}_{qs,v})} \cdot \#\overline{G}_{qs,v}(A_v/\pi_v).$$

We can now state the formula of [GHY, Proposition 2.13]:

**Theorem 3.1.** *Let  $L$  be a lattice in  $(V, q)$  so that the local stabilizer  $K_{L_v}$  is a parahoric subgroup of  $G(F_v)$  for each finite place  $v$ . Then*

$$Mass(L) = \left( \tau(G) \cdot d_F^{\dim(G)/2} \cdot \gamma_G^{-\deg(F)} \cdot L(G) \right) \cdot \prod_v \lambda_{L_v},$$

where

- $\tau(G) = 2$  is the Tamagawa number of  $G$ ;
- $d_F$  is the discriminant of  $F/\mathbb{Q}$ ;
- $L(G)$  is the special value of an  $L$ -function associated to  $G$  and is given by:

$$L(G) = \begin{cases} \prod_{r=1}^n \zeta_F(2r) & \text{if } d = 2n + 1; \\ \prod_{r=1}^{n-1} \zeta_F(2r) \cdot L(n, \chi_{disc(q)}) \cdot d_{E_{disc(q)}}^{n-\frac{1}{2}}, & \text{if } d = 2n. \end{cases}$$

- for each finite place  $v$ ,

$$\lambda_{L_v} = \frac{q_v^{-N(\overline{G}_v)} \cdot \#\overline{G}_v(A_v/\pi_v)}{q_v^{-N(\overline{G}_{qs,v})} \cdot \#\overline{G}_{qs,v}(A_v/\pi_v)}.$$

**Remarks:** (i) The first factor in the mass formula depends only on the group  $G$  and should be regarded as the main term, whereas the  $\lambda$ -factors depend on the local stabilizers  $K_{L_v}$  and should be regarded as fudge factors. The point of the above formula is that the  $\lambda$ -factors are effectively computable from Bruhat-Tits theory. Note that the product of  $\lambda$ -factors is a finite product, since for almost all places  $v$ ,  $G_v$  is quasi-split and  $K_{L_v}$  is a hyperspecial maximal compact subgroup, in which case  $\lambda_{L_v} = 1$ .

(ii) In [GHY, Proposition 2.13], the formula was stated only under the assumption that  $F$  is totally real and  $q$  is totally definite. However, the derivation given in [GHY, §2] only relies on [GG, Proposition 9.3], and the latter holds without these restrictions, as long as the group  $G$  is semisimple.

(iii) The reader may notice that the formula in [GHY, Proposition 2.13] is cleaner than the one given above, and involves  $L$ -values at negative integers rather than positive integers. Not surprisingly, the two versions are related by the functional equation. If we had applied the functional equation to the formula of the Theorem, we expect to get the values  $\zeta_F(1 - 2r)$ . But for some number fields, these quantities are equal to 0; in this case, one has to use the leading term of the Taylor expansion of the zeta functions in place of the value. Because of this complication, we prefer to leave the formula as it is.

Suppose that  $L$  is an arbitrary lattice, so that  $K_{L_v}$  may not be parahoric. For each  $v$ , there will be a maximal parahoric subgroup  $K_v$  containing  $K_{L_v}$ , and clearly, if we know the index of  $K_{L_v}$  in  $K_v$ , we can determine  $Mass(L)$ . More precisely, if we set

$$\lambda_{L_v} = \#K_v/K_{L_v} \cdot \lambda_{K_v},$$

then  $\lambda_{L_v}$  is well-defined, i.e. independent of the choice of  $K_v$ , and the formula of the Theorem continues to hold for any lattice  $L$ .

In the following sections, we shall use this formula to obtain the mass of certain special lattices  $L$ .

#### 4. Maximal and Unimodular Lattices

We recall the following basic definitions:

**Definitions:**

- $L$  is a *maximal lattice* in  $(V, q)$  if  $q$  is not  $A$ -valued on any lattice strictly containing  $L$ .
- $L$  is a *unimodular lattice* in  $(V, q)$  if  $L$  is self-dual with respect to  $B_q$ , i.e.  $L^* = L$ , where

$$L^* = \{x \in V : B_q(x, L) \subset A\}.$$

Clearly, the analogous definitions can be made for the local lattices  $L_v$ . We note the following elementary remarks.

**Remarks:** (i)  $L$  is maximal if and only if  $L_v$  is maximal for all finite places  $v$  of  $F$ . Similarly,  $L$  is unimodular if and only if  $L_v$  is unimodular for all finite places  $v$ .

(ii) If  $L$  is a unimodular lattice, then of course  $L$  is a largest lattice on which  $B_q$  is  $A$ -valued. If  $F_v$  is a  $p$ -adic field with  $p \neq 2$ , then this implies that  $L_v$  is a maximal lattice. However, over the number field  $F$  or a 2-adic field  $F_v$ ,  $L$  need not be a maximal lattice.

For a maximal lattice  $L$ , it is a consequence of [BT] that the local stabilizers  $K_{L_v}$  are maximal parahoric subgroups. Exploiting this fact, the mass of  $L$  can be obtained using Theorem 3.1. This was carried out in [GHY], where the relevant  $\lambda$ -factors were tabulated.

In this paper, we shall explain how to obtain the mass of a unimodular lattice by exploiting Theorem 3.1, at least for a quadratic space over  $\mathbb{Q}$ . To do this, we need to relate the local stabilizer of a unimodular lattice to a parahoric subgroup. We treat this local question in the next section.

### 5. Local stabilizers of Unimodular Lattices

In this section, assume that  $F = \mathbb{Q}_p$ . We shall relate the stabilizers of unimodular lattices to maximal parahoric subgroups. Recall that we are assuming that  $d = \dim(V) \geq 3$ .

First recall the classification of quadratic spaces over  $\mathbb{Q}_p$ . We have already defined the discriminant of  $(V, q)$ . Another invariant of  $(V, q)$  is the Hasse-Witt invariant defined as follows. By choosing a suitable basis of  $V$ , we may diagonalize the form  $q$ , say

$$q(\underline{x}) = \sum_{i=1}^d a_i x_i^2.$$

Then the Hasse-Witt invariant is:

$$\epsilon_{HW}(V, q) = \prod_{i < j} (a_i, a_j) \in \{\pm 1\},$$

where  $(-, -)$  is the Hilbert symbol of  $\mathbb{Q}_p$ . The quadratic space  $(V, q)$  is then determined by the invariants

$$(\dim(V), \text{disc}(V, q), \epsilon_{HW}(V, q)).$$

Note that the definition of  $\epsilon_{HW}$  differs from the  $\epsilon$  in [GHY].

As we mentioned in the previous section, if  $p \neq 2$ , a unimodular lattice in  $(V, q)$  is necessarily a maximal lattice. These were enumerated in [GHY], with their stabilizers identified and their  $\lambda$ -factors tabulated. Hence the main local problem is to understand the stabilizers of unimodular lattices over  $\mathbb{Q}_2$ .

*We shall assume for the rest of the section that  $p = 2$ .*

In this case, one distinguishes between two types of unimodular lattices.

**Definitions:** A unimodular lattice  $L$  in  $(V, q)$  is said to be *even* if  $q(L) \subset 2\mathbb{Z}_2$ ; it is said to be *odd* otherwise.

**Proposition 5.1.** *Fix a quadratic space  $(V, q)$ . Then there is at most one isomorphism class of odd (resp. even) unimodular lattices in  $(V, q)$ .*

*Proof.* This follows from [OM, Theorem 93.16, Pg. 259] and the discussion in [OM, §93G].  $\square$

To decide which quadratic spaces actually possess a unimodular lattice, we first begin with the even case. We have:

**Proposition 5.2.** *(i)  $L$  is an even unimodular lattice in  $(V, q)$  if and only if  $L$  is a maximal lattice in  $(V, \frac{1}{2}q)$ . This occurs only if  $d = 2n$  is even and  $\text{disc}(q)$  can be represented by an element of  $\mathbb{Z}_2^\times$ .*

(ii)  $(V, q)$  contains an even unimodular lattice if and only if  $E_{disc(q)}$  is not a ramified quadratic extension and  $\epsilon_{HW}(q) = (-1)^{n(n-1)/2}$ .

(iii) More explicitly, the possible quadratic spaces are:

- $(V, q) \cong \mathbb{H}^n$ , where  $\mathbb{H} = \langle e, f \rangle$  is the hyperbolic plane. An even unimodular lattice is

$$L_{even} = \langle e_1, \dots, e_n, 2f_1, \dots, 2f_n \rangle.$$

- $(V, q) = (E, 2\mathbb{N}_E) \oplus \mathbb{H}^{n-1}$ , where  $E$  is the unramified quadratic extension of  $\mathbb{Q}_2$  with norm map  $\mathbb{N}_E$ . An even unimodular lattice is

$$L_{even} = A_E \oplus \langle e_i, \dots, e_{n-1}, 2f_1, \dots, 2f_{n-1} \rangle.$$

*Proof.* The first assertion of (i) is clear. The fact that  $d$  is even was shown in [OM, 93.15, Pg. 258] and the statement about  $disc(q)$  is obvious.

By (i), to obtain an explicit list of  $(V, q)$  which contains even unimodular lattices, it suffices to examine a maximal lattice in  $(V, \frac{1}{2}q)$  and see if it is self-dual with respect to  $B_q$ . Using the enumeration of maximal lattices in [GHY], a short check gives the list in (iii) and it is easy to see that these two quadratic spaces have the discriminant and Hasse-Witt invariant stated in (ii).  $\square$

**Corollary 5.3.** *If  $L$  is even unimodular, then  $K_L$  is the stabilizer in  $G(\mathbb{Q}_2) = SO(V, q)(\mathbb{Q}_2) = SO(V, \frac{1}{2}q)(\mathbb{Q}_2)$  of a maximal lattice in  $(V, \frac{1}{2}q)$ . In particular, it is a maximal parahoric subgroup and  $\lambda_L = 1$ .*

Now we come to the odd unimodular lattices; the situation here is more interesting. It is not difficult to enumerate the odd unimodular lattices in the spirit of the previous proposition. However, we shall refrain from doing so at the moment, since we would like to avoid case-by-case analysis as much as possible. We begin by noting:

**Lemma 5.4.**  *$(V, q)$  contains an odd unimodular lattice if and only if  $disc(V, q)$  can be represented by an element of  $\mathbb{Z}_2^\times$ .*

*Proof.* The “only if” part is clear. Conversely, for given  $\delta \in \mathbb{Z}_2^\times$  and  $\epsilon = \pm 1$ , consider the lattice  $L_{\delta, \epsilon}$  defined by the following quadratic form on  $\mathbb{Z}_2^d$ :

$$q_{\delta, \epsilon}(x) = (-1)^{[d/2]} \delta x_1^2 + \epsilon x_2^2 + \epsilon x_3^2 + \sum_{x \geq 4} x_i^2.$$

This defines a unimodular lattice and a quick check shows that

$$disc(q_{\delta, \epsilon}) = \delta \quad \text{and} \quad \epsilon_{HW}(q_{\delta, \epsilon}) = \epsilon.$$

This proves the reverse implication.  $\square$

Let  $L$  be an odd unimodular lattice in  $(V, q)$ . The rest of the section is devoted to the determination of  $K_L$  and the computation of  $\lambda_L$ . A simple but crucial observation is that the induced map

$$\bar{q} : L \rightarrow \mathbb{Z}/2\mathbb{Z}$$

is a group homomorphism, which is surjective since  $L$  is odd. Let  $\Lambda$  be the kernel of  $\bar{q}$ ; it is a sublattice with index 2 in  $L$ .

We want to relate  $K_L$  to  $K_\Lambda$ . From the definition of  $\Lambda$ , the following lemma is clear.

**Lemma 5.5.** *We have:  $K_L \subset K_\Lambda$ .*

Now the quadratic form  $\frac{1}{2}q$  takes integer value on  $\Lambda$ . So we may ask if  $\Lambda$  is a maximal lattice in  $(V, \frac{1}{2}q)$ . We have:

**Proposition 5.6.** *Let  $L$  be an odd unimodular lattice in  $(V, q)$  and let  $\Lambda$  be defined as above.*

(i) *If  $(V, q)$  does not contain an even unimodular lattice, then  $\Lambda$  is a maximal lattice in  $(V, \frac{1}{2}q)$ .*

(ii) *If  $(V, q)$  contains an even unimodular lattice, then  $\Lambda$  is not a maximal lattice in  $(V, \frac{1}{2}q)$ . There is an even unimodular lattice  $L_{\text{even}}$  of  $(V, q)$  such that  $\Lambda = L \cap L_{\text{even}}$ .*

*Proof.* (i) Suppose that  $\Lambda' \supset \Lambda$  and  $\frac{1}{2}q$  is integer-valued on  $\Lambda'$ . Then the symmetric bilinear form  $B_q$  is integer valued on  $\Lambda'$ . Since  $\Lambda$  is contained in a self-dual lattice with index 2, this forces  $\Lambda'$  to be self-dual with respect to  $B_q$  as well. So  $\Lambda'$  is an even unimodular lattice in  $(V, q)$ . But  $(V, q)$  does not contain such a lattice by assumption, and so (i) is proved.

(ii) The two quadratic spaces listed in Proposition 5.2(iii) do contain odd unimodular lattices. Indeed, an odd unimodular lattice in  $\mathbb{H}^n$  is:

$$L_{\text{odd}} = \langle e_1, \dots, e_{n-1}, 2f_1, \dots, 2f_{n-1} \rangle \oplus \langle e_n + f_n, e_n - f_n \rangle$$

and one for  $(E, 2\mathbb{N}_E) \oplus \mathbb{H}^{n-1}$  is

$$L_{\text{odd}} = A_E \oplus \langle e_2, \dots, e_{n-1}, 2f_2, \dots, 2f_{n-1} \rangle \oplus \langle e_n + f_n, e_n - f_n \rangle.$$

If  $L_{\text{even}}$  is the lattice defined in Proposition 5.2(iii), then one sees easily that  $\Lambda = L_{\text{even}} \cap L_{\text{odd}}$ . This proves (ii).  $\square$

To determine the index of  $K_L$  in  $K_\Lambda$ , we observe that

$$\Lambda \subset_2 L \subset_2 \Lambda^*$$

where  $\Lambda^*$  is dual of  $\Lambda$  with respect to  $B_q$ . The following lemma determines the order 4 group  $\Lambda^*/\Lambda$ :

**Lemma 5.7.**

$$\Lambda^*/\Lambda = \begin{cases} \mathbb{Z}/4\mathbb{Z}, & \text{if } d \text{ is odd;} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{if } d \text{ is even.} \end{cases}$$

*Proof.* Let  $L_{\delta, \epsilon}$  be as given in the proof of Lemma 5.4, so that  $L_{\delta, \epsilon} = \mathbb{Z}_2^d$  and  $q$  has the form:

$$q(x) = \sum_i a_i x_i^2,$$

where each  $a_i$  is a unit in  $\mathbb{Z}_2$ . Since  $a_i \equiv 1 \pmod{2}$ , we see that

$$\Lambda = \{x \in L : \sum_i x_i \equiv 0 \pmod{2}\}$$

and

$$\Lambda^* = \langle L, x_0 = \frac{1}{2} \sum_i e_i \rangle.$$

Now  $\Lambda^*/\Lambda \cong \mathbb{Z}/4\mathbb{Z}$  if and only if  $x_0$  has order 4 in  $\Lambda^*/\Lambda$ , and this occurs if and only if  $2x_0$  does not lie in  $\Lambda$ . Clearly, this holds if and only if  $d$  is odd.  $\square$

Now when  $d$  is odd,  $L/\Lambda$  can be characterized as the unique order 2 subgroup of  $\Lambda^*/\Lambda \cong \mathbb{Z}/4\mathbb{Z}$ . This implies that  $K_\Lambda \subset K_L$ , and together with Lemma 5.5, we deduce that  $K_L = K_\Lambda$ , which is a maximal parahoric subgroup of  $G(F_v)$ . Hence,  $\lambda_L = \lambda_\Lambda$  can be read off from the appropriate table in [GHY]. We have shown:

**Theorem 5.8.** *Let  $L$  be an odd unimodular lattice in  $(V, q)$ . If  $d = 2n + 1$  is odd, then  $K_L$  is the stabilizer of a maximal lattice in  $(V, \frac{1}{2}q)$ . Further, the value of  $\lambda_L$  is given by the following table.*

$(V, q)$	<i>split</i>	<i>non-split</i>
$\lambda_L$	$(2^n + 1)/2$	$(2^n - 1)/2$

Assume henceforth that  $d = 2n$  is even. In this case, Lemma 5.7 is not good enough to pinpoint  $K_L$ ; we need more structures on  $\Lambda^*/\Lambda$ . Indeed, the quadratic form  $q$  induces a quadratic form on the  $\mathbb{F}_2$ -vector space  $\Lambda^*/\Lambda$  and we want to identify this quadratic space. The non-trivial elements of  $\Lambda^*/\Lambda$  can be represented by

$$x_0 = \frac{1}{2} \sum_i e_i, \quad y_0 = e_1 \quad \text{and} \quad z_0 = x_0 - y_0.$$

Further,

$$q(x_0) = \frac{1}{4} \sum_i a_i = q(z_0) \quad \text{and} \quad q(y_0) = a_1.$$

Working with  $q = q_{\delta, \epsilon}$ , we have

$$\sum_i a_i = (-1)^{[d/2]} \delta + 2\epsilon(q) + d - 3$$

where  $\delta \in \mathbb{Z}_2^\times$  is a representative of  $\text{disc}(q)$ . Clearly, the isomorphism class of the quadratic space  $\Lambda^*/\Lambda$  depends on the valuation of the element

$$\kappa(q) = (-1)^{[d/2]} \delta + 2\epsilon(q) + d - 3.$$

We have the following 3 cases:

- Case A:**  $\text{ord}(\kappa(q)) = 1$ . In this case,  $\frac{1}{2}q$  induces a quadratic form on  $\Lambda^*/\Lambda$ . With respect to the basis  $\{x_0, z_0\}$ , it is given by  $x^2 + z^2$ .
- Case B:**  $\text{ord}(\kappa(q)) = 2$ . In this case,  $q$  induces a quadratic form on  $\Lambda^*/\Lambda$  isomorphic to  $(\mathbb{F}_4, \mathbb{N}_{\mathbb{F}_4})$ .
- Case C:**  $\text{ord}(\kappa(q)) \geq 3$ . In this case,  $q$  induces a quadratic form on  $\Lambda^*/\Lambda$ . With respect to the basis  $\{x_0, z_0\}$ , it is given by  $xz$ . In other words,  $\Lambda^*/\Lambda$  is a hyperbolic plane.

**Proposition 5.9.** (i) *Cases A occurs if and only if  $E_{disc(q)}$  is a ramified quadratic extension of  $\mathbb{Q}_2$ . In this case,  $K_L = K_\Lambda$  and so  $\lambda_L = \lambda_\Lambda$ .*

(ii) *Case B occurs if and only if  $E_{disc(q)}$  is not ramified and  $\epsilon_{HW}(q) = -(-1)^{n(n-1)/2}$ . In this case,  $K_L$  has index 3 in  $K_\Lambda$  and so  $\lambda_L = 3\lambda_\Lambda$ .*

(iii) *Case C occurs if and only if  $E_{disc(q)}$  is not ramified and  $\epsilon_{HW}(q) = (-1)^{n(n-1)/2}$ , i.e.  $(V, q)$  contains even unimodular lattices. In this case,  $K_L = K_\Lambda$  and so  $\lambda_L = \lambda_\Lambda$ .*

*Proof.* The characterization of the various cases in terms of discriminant and Hasse-Witt invariant is a straightforward check; we omit the details.

In Case A,  $L/\Lambda$  is the unique isotropic line in the quadratic space  $\Lambda^*/\Lambda$ ; so any element of  $K_\Lambda$  has to fix  $L/\Lambda$ . In Case C,  $L/\Lambda$  is the unique non-isotropic line in  $\Lambda^*/\Lambda$ , and so again  $K_\Lambda \subset K_L$ .

In Case B, each of the 3 lines in  $\Lambda^*/\Lambda$  is non-isotropic and gives rise to an odd unimodular lattice. We need to show that  $K_\Lambda$  acts transitively on these lines. If  $L_1/\Lambda$  and  $L_2/\Lambda$  are 2 such lines, then by Proposition 5.1, there is an element  $g \in G(\mathbb{Q}_2)$  such that  $g(L_1) = L_2$ . But  $\Lambda$  can be characterized as the subset of  $L_1$  (resp.  $L_2$ ) on which  $q$  takes even-integer values. So we must have  $g(\Lambda) = \Lambda$ . In other words, we have found an element of  $K_\Lambda$  which takes  $L_1$  to  $L_2$ .  $\square$

The proposition allows us to compute  $\lambda_L$  in Cases A and B, since  $\Lambda$  is a maximal lattice in  $(V, \frac{1}{2}q)$  in these cases. The values of  $\lambda_L$  are tabulated at the end of this section. In Case C, knowing that  $\lambda_L = \lambda_\Lambda$  does not help us since  $\Lambda$  is not a maximal lattice; we need to do some more work.

Assume hence that  $(V, q)$  contains even unimodular lattices. As we saw in Proposition 5.6(ii), there is an even unimodular lattice  $L_{even}$  such that

$$\Lambda = L_{even} \cap L.$$

Now  $\frac{1}{2}q$  induces a quadratic form on  $L_{even}/2L_{even}$ , and using the two  $L_{even}$ 's given in Proposition 5.2(iii), it is easy to check that

$$L_{even}/2L_{even} \cong \begin{cases} \mathbb{H}^n, & \text{if } (V, q) \text{ is split;} \\ \mathbb{F}_4 \oplus \mathbb{H}^{n-1}, & \text{otherwise.} \end{cases}$$

Hence,

$$SO(L_{even}/2L_{even}) \cong \begin{cases} SO_{2n}, & \text{if } (V, q) \text{ is split;} \\ {}^2SO_{2n}, & \text{otherwise.} \end{cases}$$

Moreover, the action of  $K_{L_{even}}$  on  $L_{even}/2L_{even}$  gives a surjection

$$r : K_{L_{even}} \longrightarrow SO(L_{even}/2L_{even}).$$

Now we note:

**Proposition 5.10.** (i)  *$K_\Lambda \cap K_{L_{even}}$  has index 2 in  $K_\Lambda$ .*

(ii)  $K_\Lambda \cap K_{L_{\text{even}}}$  is the subgroup of  $K_{L_{\text{even}}}$  stabilizing an isotropic line in  $L_{\text{even}}/2L_{\text{even}}$ . Its index in  $K_{L_{\text{even}}}$  is given by:

$$\begin{cases} (2^{n-1} + 1)(2^n - 1), & \text{if } (V, q) \text{ is split;} \\ (2^{n-1} - 1)(2^n + 1), & \text{otherwise.} \end{cases}$$

*Proof.* (i) This is because  $L_{\text{even}}/\Lambda$  is one of the two isotropic lines in  $\Lambda^*/\Lambda$ , and  $K_\Lambda$  acts transitively on the set of isotropic lines.

(ii) The first statement follows since  $2\Lambda^*/2L_{\text{even}}$  is an isotropic line in the quadratic space  $L_{\text{even}}/2L_{\text{even}}$ . Further,  $SO(L_{\text{even}}/2L_{\text{even}})$  acts transitively on the set of isotropic lines, and the stabilizer of one such line is a maximal parabolic subgroup  $P$  with Levi factor

$$\begin{cases} GL_1 \times SO_{2n-2}, & \text{if } (V, q) \text{ is split;} \\ GL_1 \times {}^2SO_{2n-2}, & \text{otherwise.} \end{cases}$$

Hence  $K_\Lambda \cap K_{L_{\text{even}}}$  is equal to the inverse image of  $P(\mathbb{F}_2)$  under the projection  $r$  (which is a non-maximal parahoric subgroup). Thus

$$\#K_{L_{\text{even}}}/K_\Lambda \cap K_{L_{\text{even}}} = \#SO(L_{\text{even}}/2L_{\text{even}})/P(\mathbb{F}_2),$$

and one obtains the values listed in (ii).  $\square$

The proposition allows us to compute volume of  $K_L = K_\Lambda$  given the volume of  $K_{L_{\text{even}}}$ . Thus we can compute the value of  $\lambda_L = \lambda_\Lambda$  from the value of  $\lambda_{L_{\text{even}}}$ . The latter is known from [GHY] since  $L_{\text{even}}$  is a maximal lattice in  $(V, \frac{1}{2}q)$ ; in fact,  $\lambda_{L_{\text{even}}} = 1$ .

**Remarks:** Using the results of [BT], one can check that  $K_\Lambda$  is a maximal compact subgroup. It has an associated integral group scheme  $\underline{G}_\Lambda$  whose special fiber has maximal reductive quotient

$$\overline{G}_\Lambda = \begin{cases} S(O_2 \times O_{2n-2}) & \text{if } (V, q) \text{ is split;} \\ S(O_2 \times {}^2O_{2n-2}) & \text{otherwise.} \end{cases}$$

To conclude this section, we tabulate the values of  $\lambda_L$  when  $d = 2n$  is even; the values for odd  $d$  were given in Theorem 5.8.

$\text{disc}(q)$	$\epsilon_{HW}(q)$	$\lambda_L$
1	$(-1)^{n(n-1)/2}$	$(2^{n-1} + 1)(2^n - 1)/2$
1	$-(-1)^{n(n-1)/2}$	$(2^{n-1} - 1)(2^n - 1)/2$
$E_{\text{disc}(q)}$ unramified	$(-1)^{n(n-1)/2}$	$(2^{n-1} - 1)(2^n + 1)/2$
$E_{\text{disc}(q)}$ unramified	$-(-1)^{n(n-1)/2}$	$(2^{n-1} + 1)(2^n + 1)/2$
$E_{\text{disc}(q)}$ ramified	$\pm 1$	$1/2$

## 6. Mass of Unimodular Lattices

Assembling the results of the previous section, and using Theorem 3.1, one can give explicit formulas for the mass of unimodular lattices over  $\mathbb{Z}$ . We shall only consider the odd unimodular lattices here, since the even case is easy. For the following proposition, see [Se].

**Proposition 6.1.** *Let  $r \geq s \geq 0$  with  $r + s = d$ . Let  $(V_{r,s}, q_{r,s})$  be the quadratic space  $\mathbb{Q}^d$  defined by the quadratic form*

$$q_{r,s}(x) = \sum_{i=1}^r x_i^2 - \sum_{j=r+1}^{r+s} x_j^2.$$

Let  $L_{r,s} = \mathbb{Z}^d$ .

(i) *Any odd unimodular lattice is in the genus of some  $L_{r,s}$ .*

(ii) *If  $s > 0$ , so that  $V_{r,s}$  is indefinite over  $\mathbb{R}$ , then any odd unimodular lattice is isomorphic to some  $L_{r,s}$ .*

The proposition shows that there is no loss of generality in working with the lattice  $L = L_{r,s}$  above. Over a finite place  $p$ , the basic invariants of  $V_{r,s}$  are

$$\text{disc}(q_{r,s}) = (-1)^{[d/2]+s} \quad \text{and} \quad \epsilon_{HW}(q_{r,s}) = (-1, -1)^{[s/2]}.$$

If  $p \neq 2$ , then  $L_{r,s} \otimes \mathbb{Z}_p$  is a maximal lattice in the relevant quadratic space and a quick check shows that the  $\lambda$ -factor is 1. On the other hand, when  $p = 2$ , we can read off the value of  $\lambda_{L \otimes \mathbb{Z}_2}$  from the two tables of the previous section. We state the results for odd and even dimensional spaces separately:

**Theorem 6.2.** *Assume that  $d = 2n + 1$  is odd. Then*

$$\text{Mass}(L_{r,s}) = \lambda_2 \cdot \prod_{k=1}^n \frac{(2k-1)! \cdot \zeta(2k)}{(2\pi)^{2k}} \cdot \tau(G)$$

where  $\tau(G) = 2$  and

$$\lambda_2 = \begin{cases} (2^n + 1)/2, & \text{if } r - s \equiv \pm 1 \pmod{8}; \\ (2^n - 1)/2, & \text{if } r - s \equiv \pm 3 \pmod{8}. \end{cases}$$

**Theorem 6.3.** *Assume that  $d = 2n$  is even. Then*

$$\text{Mass}(L_{r,s}) = \lambda_2 \cdot \frac{(n-1)! \cdot L(n, \chi_{(-1)^{(r-s)/4}})}{(2\pi)^n} \cdot \prod_{k=1}^{n-1} \frac{(2k-1)! \cdot \zeta(2k)}{(2\pi)^{2k}} \cdot d_{\mathbb{Q}((-1)^{(r-s)/4})}^{n-1/2} \cdot \tau(G)$$

where  $\tau(G) = 2$ ,

$$d_{\mathbb{Q}((-1)^{(r-s)/4})} = \begin{cases} 1 & \text{if } r - s \equiv \pm 2 \pmod{8}; \\ 4 & \text{if } r - s \equiv 0 \text{ or } 4 \pmod{8}, \end{cases}$$

and

$$\lambda_2 = \begin{cases} 1/2, & \text{if } r - s \equiv \pm 2 \pmod{8}; \\ (2^{n-1} + 1)(2^n - 1)/2, & \text{if } r - s \equiv 0 \pmod{8}; \\ (2^{n-1} - 1)(2^n - 1)/2, & \text{if } r - s \equiv 4 \pmod{8}. \end{cases}$$

In the following tables, we give the values of masses for small  $n$ .

1.  $r + s = 2n + 1$  is odd:

$n$	$r - s \equiv \pm 1(8)$	$r - s \equiv \pm 3(8)$
1	$\frac{1}{8}$	$\frac{1}{24}$
2	$\frac{1}{1152}$	$\frac{1}{1920}$
3	$\frac{1}{322560}$	$\frac{1}{414720}$
4	$\frac{17}{1393459200}$	$\frac{1}{92897280}$
5	$\frac{1}{11147673600}$	$\frac{31}{367873228800}$
6	$\frac{691}{370816214630400}$	$\frac{691}{382588157952000}$
7	$\frac{29713}{192824431607808000}$	$\frac{87757}{578473294823424000}$
8	$\frac{642332179}{9440684171518279680000}$	$\frac{2499347}{37022290868699136000}$
9	$\frac{109638854849}{528678313605023662080000}$	$\frac{8003636403977}{38744567839911019806720000}$
10	$\frac{784910153445588299}{143199922736311129205637120000}$	$\frac{593468652605200909}{108484789951750855458816000000}$

2.  $r + s = 2n$  is even:

$n$	$r - s \equiv 0(8)$	$r - s \equiv \pm 2(8)$	$r - s \equiv 4(8)$
2	$\frac{1}{64}$	$\frac{L(2)}{12 \pi^2}$	$\frac{1}{192}$
3	$\frac{7 \zeta(3)}{4608 \pi^3}$	$\frac{1}{23040}$	$\frac{7 \zeta(3)}{7680 \pi^3}$
4	$\frac{1}{5160960}$	$\frac{L(4)}{60480 \pi^4}$	$\frac{1}{6635520}$
5	$\frac{527 \zeta(5)}{1857945600 \pi^5}$	$\frac{1}{1114767360}$	$\frac{31 \zeta(5)}{123863040 \pi^5}$
6	$\frac{1}{89181388800}$	$\frac{L(6)}{95800320 \pi^6}$	$\frac{31}{2942985830400}$
7	$\frac{87757 \zeta(7)}{65922882600960 \pi^7}$	$\frac{42151}{96412215803904000}$	$\frac{87757 \zeta(7)}{68015672524800 \pi^7}$
8	$\frac{505121}{6170381811449856000}$	$\frac{691 L(8)}{896690995200 \pi^8}$	$\frac{1491869}{18511145434349568000}$
9	$\frac{46890249067 \zeta(9)}{17125957680758784000 \pi^9}$	$\frac{692319119}{7552547337214623744000}$	$\frac{182452331 \zeta(9)}{67160618355916800 \pi^9}$
10	$\frac{3398804500319}{4229426508840189296640000}$	$\frac{109638854849 L(10)}{1459741204905984000 \pi^{10}}$	$\frac{248112728523287}{309956542719288158453760000}$

**Comments:** The values of masses of the odd unimodular lattices first decrease when the dimension of the space grows and attain their minima  $\sim 10^{-14}$  at  $n = 8$  for each of the cases, then the mass starts to grow exponentially. All the values in the first table are rational numbers which is related to the fact that the Euler-Poincaré characteristic for symmetric spaces of the corresponding orthogonal groups does not vanish. In the second table one can see many irrationalities some of which are of a particular interest. Thus, for the type (3, 1)

which corresponds to the hyperbolic 3-space, we find that the value of the mass is a rational multiple of Catalan's constant

$$C = L(2) = 1 - \frac{1}{9} + \frac{1}{25} - \dots$$

divided by  $\pi^2$ . It is conjectured but not known that both  $C$  and  $C/\pi^2$  are irrational.

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