COMMUTATIVE SUBRINGS OF CERTAIN NON-ASSOCIATIVE RINGS

BENEDICT H. GROSS AND WEE TECK GAN

The title of this paper was chosen more in homage to Drinfeld's famous note [D] than as a precise description of its contents. We will therefore describe the main results in the first three sections.

1. Embeddings into R

All of the non-associative rings that we will study arise from Coxeter's order R in the $\mathbb{Q}$-algebra of Cayley's octonions. Recall that the latter algebra has basis $\langle 1, e_1, e_2, ..., e_7\rangle$ over $\mathbb{Q}$ and multiplication rules:

$$e_i^2 = -1,$$

$$e_i(e_{i+1}e_{i+3}) = (e_ie_{i+1})e_{i+3} = -1,$$

where the indices are taken modulo 7. The order R is generated over $\mathbb{Z}$ by the e_i's, and the additional elements

$$\frac{1}{2}(1 + e_1 + e_2 + e_4),$$
$$\frac{1}{2}(1 + e_1 + e_5 + e_6),$$
$$\frac{1}{2}(1 + e_1 + e_3 + e_7),$$
$$\frac{1}{2}(e_1 + e_2 + e_3 + e_5).$$

Moreover, it has an anti-involution $x \mapsto \overline{x}$, defined by $\overline{e}_i = -e_i$. The trace $Tr(x) = x + \overline{x}$ and norm $\mathbb{N}(x) = x \cdot \overline{x} = \overline{x} \cdot x$ take integral values on R.

Let A be the ring of integers in an imaginary quadratic field k of discriminant D. We wish to count the number, denoted $N(A, R)$, of ways of embedding A as a commutative subring of the non-associative ring R. Note that by a commutative ring, we mean one which is both commutative and associative. Let

$$\varepsilon_A : (\mathbb{Z}/D\mathbb{Z})^\times \longrightarrow \langle \pm 1\rangle$$

be the odd quadratic Dirichlet character associated to k, and let $L(\varepsilon_A, s)$ be the corresponding Dirichlet L-function. Then the following result was obtained in [EG, Theorem 1].

1991 *Mathematics Subject Classification.* 11E25, 11E76.

Theorem 1.

$$N(A,R) = \frac{L(\varepsilon_A,-2)}{\zeta(-5)} = -252 \cdot L(\varepsilon_A,-2).$$

In this paper, we will give a more streamlined proof using results of [GG].

2. Embeddings into J_2

Next, let J_2 be the abelian group, under matrix addition, of all 2×2 Hermitian symmetric matrices with entries in R. An element of J_2 has the form

$$M = \begin{pmatrix} a & x \\ \overline{x} & b \end{pmatrix}$$

with $a, b \in \mathbb{Z}$ and $x \in R$. In particular, J_2 is free abelian of rank 10. The quadratic form

$$\begin{aligned} det : & J_2 \longrightarrow \mathbb{Z} \\ & M \mapsto \; ab - \mathbb{N}(x) \end{aligned}$$

has signature $(1,9)$ and discriminant -1. When we invert the prime 2, $J_2 \otimes \mathbb{Z}[1/2]$ is a non-associative Jordan algebra over $\mathbb{Z}[1/2]$, with multiplication law

$$M \circ N = \frac{MN + NM}{2}$$

and identity element I.

Let A be the ring of integral elements in an étale quadratic algebra k over $\mathbb{Q}$ with $k \otimes \mathbb{R} = \mathbb{R}^2$, and let D be the discriminant of A. Hence, A is either $\mathbb{Z} \oplus \mathbb{Z}$, or the ring of integers in a real quadratic field. Let

$$f : A \longrightarrow J_2$$

be a homomorphism of additive abelian groups which satisfies $f(1) = I$.

Lemma 1. *The following are equivalent:*

(1) for all $a \in A$, $det(f(a)) = \mathbb{N}(a)$;

(2) for all $a \in A$, $M = f(a)$ has the same characteristic polynomial as a;

(3) for all $a, b \in A$, $f(a \cdot b) = f(a) \circ f(b)$.

Proof. Clearly, (2) implies (1). On the other hand, the characteristic polynomial of a is $\mathbb{N}(\lambda \cdot 1 - a)$, and $f(\lambda \cdot 1 - a) = \lambda I - M$. Since $det(\lambda I - M)$ is the characteristic polynomial of M, (1) implies (2).

Now assume (2), and write $A = \mathbb{Z} + \mathbb{Z}a$, with

$$a = \frac{D + \sqrt{D}}{2}.$$

Then $M = f(a)$ satisfies

$$M^2 = DM + \frac{D - D^2}{4}$$

in J_2. Since $f(\alpha+\beta a) = \alpha I + \beta M$, we can check directly that f preserves multiplication, so that (3) holds. Conversely, if f preserves multiplication, then since $a^2 = Tr(a)\cdot a - \mathbb{N}(a)$, we have

$$M^2 = Tr(a)\cdot M - \mathbb{N}(a).$$

Hence the characteristic polynomial of M is equal to that of a. □

When the conditions of the lemma hold, we say that f is a ring embedding (even though J_2 is not a ring). We wish to count the number, denoted $N(A, J_2)$, of ring embeddings $f : A \to J_2$. Let

$$\varepsilon_A : (\mathbb{Z}/D\mathbb{Z})^\times \longrightarrow \langle \pm 1\rangle$$

be the even quadratic (or trivial, when $D = 1$) Dirichlet character associated to k, and let $L(\varepsilon_A, s)$ be the corresponding Dirichlet L-function. Then we have:

Theorem 2.

$$N(A, J_2) = \frac{L(\varepsilon_A, -3)}{\zeta(-7)} = 240\cdot L(\varepsilon_A, -3).$$

3. Embeddings into J_3

Finally, let J_3 be the abelian group, under matrix addition, of all 3×3 Hermitian symmetric matrices with entries in R. An element of J_3 has the form

$$M = \begin{pmatrix} a & z & \overline{y} \\ \overline{z} & b & x \\ y & \overline{x} & c \end{pmatrix}$$

with $a, b, c \in \mathbb{Z}$ and $x, y, z \in R$. In particular, J_3 is free abelian of rank 27. The cubic form

$$\begin{aligned} det : & J_3 \longrightarrow \mathbb{Z} \\ & M \mapsto abc + Tr(xyz) - a\mathbb{N}(x) - b\mathbb{N}(y) - c\mathbb{N}(z) \end{aligned}$$

is stabilized by a form of the simply-connected quasi-simple group of type E_6 over $\mathbb{Z}$ (see [G]).

As before, $J_3 \otimes \mathbb{Z}[1/2]$ is a non-associative exceptional Jordan algebra over $\mathbb{Z}[1/2]$, with multiplication law

$$M \circ_I N = \frac{MN + NM}{2}$$

and identity element I. We will denote this algebra by $J_I \otimes \mathbb{Z}[1/2]$ to stress the choice of the identity element.

Let

$$\alpha = \frac{1}{2}(-1 + e_1 + e_2 + ... + e_7) \in R.$$

Then $\alpha^2+\alpha+2=0$, and the matrix

$$E = \begin{pmatrix} 2 & \alpha & \overline{\alpha} \\ \overline{\alpha} & 2 & \alpha \\ \alpha & \overline{\alpha} & 2 \end{pmatrix} \in J_3$$

satisfies $det(E) = 1$. There is a multiplication law on $J_3 \otimes \mathbb{Z}[1/2]$ with identity element E, which is denoted $M \circ_E N$, and described explicitly in [EG2, pg. 672]. We denote this algebra by $J_E \otimes \mathbb{Z}[1/2]$.

Note that it follows from the strong approximation theorem (applied to the anisotropic form of F_4 over $\mathbb{Q}$, with the set of primes $S = \{\infty, 2\}$) that the two algebras $J_I \otimes \mathbb{Z}[1/2]$ and $J_E \otimes \mathbb{Z}[1/2]$ are isomorphic. However, it follows from [EG2] that there is no isomorphism which preserves J_3. Hence we shall let J_I (respectively J_E) denote the lattice J_3, endowed with the "multiplication law" $J_I \otimes J_I \to \frac{1}{2}J_I$ (respectively $J_E \otimes J_E \to \frac{1}{2}J_E$) with "identity element" I (respectively E).

Let A be the ring of integral elements in an étale cubic algebra k over $\mathbb{Q}$ with $k \otimes \mathbb{R} = \mathbb{R}^3$, and let D be the discriminant of A. Then A is either $\mathbb{Z}^3$ (when $D = 1$), $\mathbb{Z} \oplus B$ with B the ring of integers in a real quadratic field, or the ring of integers in a totally real cubic field. Let

$$f : A \longrightarrow J_I$$

be a homomorphism of additive abelian groups which satisfies $f(1) = I$.

Lemma 2. *The following are equivalent:*

(1) for all $a \in A$, $det(f(a)) = \mathbb{N}(a)$;

(2) for all $a \in A$, $M = f(a)$ has the same characteristic polynomial as a;

(3) for all $a, b \in A$, $f(a \cdot b) = f(a) \circ_I f(b)$.

Proof. The equivalence of (1) and (2), as well as the fact that (3) implies (2), can be proved as in the previous lemma. Now assume that (2) holds. We have to show that f preserves multiplication. Without loss of generality, we can extend scalars to $\mathbb{R}$. Recall that $k \otimes \mathbb{R} \cong \mathbb{R}^3$. Let

$$M_1 = f(1,0,0),$$
$$M_2 = f(0,1,0),$$
$$M_3 = f(0,0,1),$$

with $M_1 + M_2 + M_3 = I$. Since f preserves characteristic polynomials, the characteristic polynomial of M_i is $x^3 - x^2$. Hence $M_i^3 = M_i^2$, and M_i has rank 1 or 2. We claim that M_i has rank 1, so that $M_i^2 = M_i$. Indeed, if M_i has rank 2, then its adjoint M_i^* [EG2, Pg. 667] would have rank 1 and trace zero, since its trace is the coefficient of x in the characteristic polynomial of M_i. But there is no such element in $J_3 \otimes \mathbb{R}$. Moreover, from

$$(I - M_1 - M_2)^2 = I - M_1 - M_2,$$

we deduce that $M_1 \circ M_2 = 0$. Hence, $M_i \circ M_j = 0$, for $i \neq j$, and the proof of the lemma is complete. □

When the conditions of the lemma hold, we say that f is a ring embedding into J_I and denote the number of such ring embeddings by $N(A, J_I)$. Similarly, we define a ring embedding $f : A \to J_E$ as an additive group homomorphism which satisfies

$$\begin{cases} f(1) = E, \\ det(f(a)) = \mathbb{N}(a), \text{ for all } a \in A. \end{cases}$$

The same argument as the proof of the lemma shows that, for all $a, b \in A$,

$$f(a \cdot b) = f(a) \circ_E f(b).$$

Let $N(A, J_E)$ be the number of ring embeddings of A into J_E.

Let V_A be the 2-dimensional representation of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ obtained by composing the homomorphism

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow S_3$$

associated to the étale cubic algebra k with the unique 2-dimensional irreducible representation V of S_3. Then V_A has Artin conductor D. Let $L(V_A, s)$ be the Artin L-function of V_A. Then we have:

Theorem 3.

$$91 \cdot N(A, J_I) + 600 \cdot N(A, J_E) = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13 \cdot L(V_A, -3).$$

In general, we do not have a formula for either $N(A, J_I)$ or $N(A, J_E)$ separately. However, in the case where A is not an integral domain, so that it is of the form $\mathbb{Z} \oplus B$, there is an element $a = (1, 0)$ in A which satisfies:

$$\begin{cases} a^2 = a, \\ Tr(a) = 1. \end{cases}$$

Since there are no such elements in J_E [EG2, Proposition 5.5], we have $N(A, J_E) = 0$. Hence, Theorem 3 implies that

$$N(A, J_I) = 2^7 \cdot 3^3 \cdot 5^2 \cdot L(V_A, -3).$$

On the other hand, we also have

$$L(V_A, s) = \zeta(s) \cdot L(\varepsilon_B, s),$$

from which we deduce that

$$\begin{aligned} N(A, J_I) &= 2^4 \cdot 3^2 \cdot 5 \cdot L(\varepsilon_B, -3) \\ &= 3 \cdot N(B, J_2). \end{aligned}$$

Of course, this reflects the fact that there are precisely three choices for $f(a)$ in J_I [EG2, Proposition 5.5], and serves as a nice check for the formulas in the theorems.

4. **Proof of Theorems**

In this section, we give an outline of the proof of Theorems 1-3, which combines the general method of Siegel and Weil [Se2], a local result on integral embeddings (Proposition 2 below), and a global result on the ratio of two adelic measures [GG, Proposition 10.7].

Let F be one of the following $\mathbb{Q}$-algebras:

$$\begin{cases} R \otimes \mathbb{Q}, \\ J_2 \otimes \mathbb{Q}, \\ J_I \otimes \mathbb{Q}. \end{cases}$$

Let G be the automorphism group of F, and let $\underline{G}$ be the group scheme over $\mathbb{Z}$ with generic fibre G, which stabilizes the lattice R, J_2 or J_3 in F respectively. Then $\underline{G}$ is a model for G over $\mathbb{Z}$ in the sense of [G], and $G(\mathbb{R})$ is compact. The groups which occur are listed below.

F	G	$\#\underline{G}(\mathbb{Z})$
$R \otimes \mathbb{Q}$	G_2	$2^6 \cdot 3^3 \cdot 7$
$J_2 \otimes \mathbb{Q}$	O_9	$2^{15} \cdot 3^5 \cdot 5^2 \cdot 7$
$J_I \otimes \mathbb{Q}$	F_4	$2^{15} \cdot 3^6 \cdot 5^2 \cdot 7$

Note that G is simply-connected except in the case $F = J_2 \otimes \mathbb{Q}$, where it is not even connected. Indeed, the element $-1 \in O_9$ acts on J_2 via:

$$\begin{pmatrix} a & x \\ \overline{x} & b \end{pmatrix} \mapsto \begin{pmatrix} b & -x \\ -\overline{x} & a \end{pmatrix}.$$

If we fix an isomorphism

$$i : J_E \otimes \mathbb{Z}[1/2] \xrightarrow{\sim} J_I \otimes \mathbb{Z}[1/2]$$

and let $\underline{G}'$ be the model of G which fixes the lattice $i(J_3)$, then we have

$$\#\underline{G}'(\mathbb{Z}) = 2^{12} \cdot 3^5 \cdot 7^2 \cdot 13.$$

In the first two cases, $\underline{G}$ is the unique model for G over $\mathbb{Z}$, whereas in the F_4 case, there are precisely two models $\underline{G}$ and $\underline{G}'$ over $\mathbb{Z}$ (see [G]). Now let $k = A \otimes \mathbb{Q}$, with A as described in the previous sections. Then we have:

Proposition 1. *There is an embedding of $\mathbb{Q}$-algebras $k \longrightarrow F$, and any two such embeddings are conjugate by $G(\mathbb{Q})$. The stabilizer of a fixed embedding is a quasi-simple algebraic subgroup $H \subset G$ over $\mathbb{Q}$ which is compact over $\mathbb{R}$ and quasi-split over $\mathbb{Q}_p$ for all p. The type of H is given by the following table.*

G	H
G_2	$SU_3(k)$
O_9	$O_8(k)$
F_4	$Spin_8(k)$

The proof of this Proposition will be given in Section 5. Since

$$O_9/O_8(k) \cong SO_9/SO_8(k),$$

we will replace the automorphism group of J_2 and the stabilizer of an embedding by their connected components. Hence, in all that follows, we will let

$$G = SO_9$$
$$H = SO_8(k)$$

in the case $F = J_2 \otimes \mathbb{Q}$.

Now let p be a finite prime, and consider the compact subset $C_p \subset G(\mathbb{Q}_p)/H(\mathbb{Q}_p)$ of algebra embeddings $f : k \otimes \mathbb{Q}_p \longrightarrow F \otimes \mathbb{Q}_p$ which map $A \otimes \mathbb{Z}_p$ into the lattice $R \otimes \mathbb{Z}_p$, $J_2 \otimes \mathbb{Z}_p$ or $J_3 \otimes \mathbb{Z}_p$ respectively. We call such an embedding integral. Then we have:

Proposition 2. *The set C_p is non-empty, and the hyperspecial maximal compact subgroup $\underline{G}(\mathbb{Z}_p)$ of $G(\mathbb{Q}_p)$ acts transitively on C_p. The stabilizer of a fixed embedding is a special maximal compact subgroup of the quasi-split group $H(\mathbb{Q}_p)$.*

We will give the proof of this Proposition in Section 6. Note that in the case when $H = SO_8(k)$ and $k \otimes \mathbb{Q}_p$ is ramified, the integral model $\underline{H}_x$ associated to a special vertex x in the building is disconnected, and a special maximal compact subgroup is, by definition, the group of points $\underline{H}^0_x(\mathbb{Z}_p)$ of the connected component. If -1 is the non-trivial element in the center of $H = SO_8(k)$, we have

$$\underline{H}_x(\mathbb{Z}_p) = \langle \pm 1 \rangle \times \underline{H}^0_x(\mathbb{Z}_p),$$

and the reductive quotient of $\underline{H}_x$ is the group O_7 over $\mathbb{Z}/p\mathbb{Z}$ in this case.

The following is an immediate consequence of Proposition 2:

Corollary 1. *Let $d\mu_G$ be the Haar measure on $G(\mathbb{Q}_p)$ giving any hyperspecial maximal compact subgroup volume 1, and let $d\mu_H$ be the Haar measure on $H(\mathbb{Q}_p)$ giving any special maximal compact subgroup volume 1. Then*

$$\int_{G(\mathbb{Q}_p)/H(\mathbb{Q}_p)} char(C_p) d\mu_G/d\mu_H = 1.$$

Assuming the two Propositions above, we can now prove the theorems. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$, and let $\varphi = \prod_v \varphi_v$ be the smooth function with compact support on $G(\mathbb{A})/H(\mathbb{A})$ which is defined by:

$$\begin{cases} \varphi_\infty = 1 \text{ on } G(\mathbb{R})/H(\mathbb{R}); \\ \varphi_p = char(C_p) \text{ on } G(\mathbb{Q}_p)/H(\mathbb{Q}_p). \end{cases}$$

Further, let $\{\alpha\}$ index the double cosets in $G(\mathbb{Q})\backslash G(\mathbb{A})/\left(G(\mathbb{R}) \times \underline{G}(\hat{\mathbb{Z}})\right)$, and let w_α be the order of the finite group

$$\underline{G}_\alpha(\mathbb{Z}) = G(\mathbb{Q}) \cap g_\alpha \left(G(\mathbb{R}) \times \underline{G}(\hat{\mathbb{Z}})\right) g_\alpha^{-1},$$

where $g_\alpha \in G(\mathbb{A})$ is a representative of the double coset indexed by α. Each α corresponds to one of our lattices in F. Indeed, in the cases when $G = G_2$ or SO_9, there is a unique double coset, and the group $\underline{G}_\alpha(\mathbb{Z})$ is exactly $\underline{G}(\mathbb{Z})$, whereas in the case $G = F_4$, the double coset space has two elements and the two groups $\underline{G}_\alpha(\mathbb{Z})$ are exactly $\underline{G}(\mathbb{Z})$ and $\underline{G}'(\mathbb{Z})$ (see [G]).

Now, by the results of the two proposition, we can apply [Se2] to obtain the global formula:

$$\left(\sum_\alpha \frac{N_\alpha}{w_\alpha}\right) / \left(\sum_\alpha \frac{1}{w_\alpha}\right) = \frac{\tau(H)}{\tau(G)} \int_{G(\mathbb{A})/H(\mathbb{A})} \varphi \; dg/dh$$

where N_α denotes the number of ring embeddings of A into the lattice (i.e. R, J_2, J_I or J_E) corresponding to α, dg and dh are the Tamagawa measures on $G(\mathbb{A})$ and $H(\mathbb{A})$, and $\tau(G)$ and $\tau(H)$ are the Tamagawa numbers of G and H.

To evaluate the integral on the right hand side, we let $d\mu_G$ be the product measure on $G(\mathbb{A})$ which gives the open compact subgroup $G(\mathbb{R}) \times \underline{G}(\hat{\mathbb{Z}})$ volume 1, and let $d\mu_H$ be the similarly defined product measure on $H(\mathbb{A})$. The local components of these measures were introduced in Corollary 1, and for them, we can evaluate the integral (which is equal to 1). On the other hand, by [GG, Proposition 10.7], we have:

$$d\mu_G = \frac{1}{2^l} L(M_G) dg;$$
$$d\mu_H = \frac{1}{2^l} L(M_H) dh,$$

where $l = rank(G(\mathbb{C})) = rank(H(\mathbb{C}))$, M_G and M_H are the Artin-Tate motives attached to G and H in [G2], and $L(M_G)$ and $L(M_H)$ are the values of their L-functions at $s = 0$. Since $\tau(G) = \tau(H)$ in all cases [K], we deduce that:

$$\left(\sum_\alpha \frac{N_\alpha}{w_\alpha}\right) / \left(\sum_\alpha \frac{1}{w_\alpha}\right) = \frac{L(M_H)}{L(M_G)}.$$

Hence, if there is a single class α, we have

$$N_\alpha = \frac{L(M_H)}{L(M_G)}.$$

This gives Theorem 1, where

$$L(M_H) = \zeta(-1) \cdot L(\varepsilon_A, -3),$$
$$L(M_G) = \zeta(-1) \cdot \zeta(-5);$$

and Theorem 2, where

$$L(M_H) = \zeta(-1) \cdot \zeta(-3) \cdot L(\varepsilon_A, -3) \cdot \zeta(-5),$$
$$L(M_G) = \zeta(-1) \cdot \zeta(-3) \cdot \zeta(-5) \cdot \zeta(-7).$$

If there are two classes α and β, we have

$$w_\beta \cdot N_\alpha + w_\alpha \cdot N_\beta = (w_\alpha + w_\beta) \cdot \frac{L(M_H)}{L(M_G)}.$$

This gives Theorem 3, where

$$L(M_H) = \zeta(-1) \cdot L(V_A, -3) \cdot \zeta(-5),$$
$$L(M_G) = \zeta(-1) \cdot \zeta(-5) \cdot \zeta(-7) \cdot \zeta(-11),$$

and

$$w_\alpha = \#\underline{G}(\mathbb{Z}) = 2^{15} \cdot 3^6 \cdot 5^2 \cdot 7,$$
$$w_\beta = \#\underline{G}'(\mathbb{Z}) = 2^{12} \cdot 3^5 \cdot 7^2 \cdot 13.$$

This completes the outline of the proof.

5. **Proof of Proposition 1**

In this section, we give the proof of Proposition 1. We first show that embeddings exist by an explicit construction, using the fact that any positive rational number is the norm of an octonion of trace zero. This fact is in turn a consequence of the Hasse-Minkowski theorem.

If $k = \mathbb{Q} + \mathbb{Q}a$ is imaginary quadratic, with $a^2 = D < 0$, we define

$$f(\alpha + \beta a) = \alpha + \beta x$$

where $Tr(x) = 0$, and $\mathbb{N}(x) = -D > 0$. Similarly, if $k = \mathbb{Q} + \mathbb{Q}a$ is real quadratic, with $a^2 = D > 0$, we define

$$f(\alpha + \beta a) = \begin{pmatrix} \alpha & \beta x \\ \beta \overline{x} & \alpha \end{pmatrix}$$

where $\mathbb{N}(x) = D > 0$.

When $k = \mathbb{Q}(a)$ is a totally real cubic algebra, we can suppose without loss of generality that the characteristic polynomial of a is

$$P(x) = x^3 + px - q$$

with $\mathbb{N}(a) = q < 0$. Let b be a positive rational number where $P(b) < 0$, which exists by our assumptions on P. Let y and z be octonions with

$$\mathbb{N}(y) = -q/b,$$
$$\mathbb{N}(z) = -P(b)/b.$$

Then the element

$$M = \begin{pmatrix} -b & z & \overline{y} \\ \overline{z} & b & 0 \\ y & 0 & 0 \end{pmatrix} \in J_I \otimes \mathbb{Q}$$

has the same characteristic polynomial as a, and the map

$$f(\alpha + \beta a + \gamma a^2) = \alpha \cdot I + \beta \cdot M + \gamma \cdot M^2$$

is the desired embedding.

The stabilizer H of an embedding $k \to R \otimes \mathbb{Q}$ was shown to be $SU_3(k)$ by Jacobson [J, Theorem 3]. On the other hand, the stabilizer of an embedding $k \to J_2 \otimes \mathbb{Q}$ acts on the orthogonal complement $k^\perp$, which is a definite quadratic space of discriminant D and rank 8. This identifies the stabilizer H in SO_9 with the group $SO_8(k)$, and the full stabilizer with $O_8(k)$. Finally, the stabilizer H of an embedding $k \to J_I \otimes \mathbb{Q}$ was calculated to be the group $Spin_8(k)$ by Soda [So].

To show that G acts transitively on embeddings, we invoke a basic result of Jacobson, which gives the result in the split case (see [J, Section 3, Pg. 62] and [J2, Theorem 10, Pg. 389]; actually, when $G = G_2$, [J, Section 3] gives the result in the non-split case as well). Hence the embeddings over $\overline{\mathbb{Q}}$ form a single orbit under $G(\overline{\mathbb{Q}})$. A standard cohomological argument [PR, Section 1.3, Pg. 22] shows that there is a single $G(\mathbb{Q})$-orbit of embeddings over $\mathbb{Q}$ if and only if the natural map

$$H^1(\mathbb{Q}, H) \longrightarrow H^1(\mathbb{Q}, G)$$

of pointed sets has trivial kernel.

In the cases when $G = G_2$ or F_4, both the groups G and H are simply-connected, so that

$$H^1(\mathbb{Q}, H) = H^1(\mathbb{R}, H),$$
$$H^1(\mathbb{Q}, G) = H^1(\mathbb{R}, G).$$

Since $H(\mathbb{R})$ and $G(\mathbb{R})$ are both compact, by [Se, Chapter 3, Section 4.5], $H^1(\mathbb{R}, H)$ (respectively $H^1(\mathbb{R}, G)$) can be identified as the set of conjugacy classes of elements in $H(\mathbb{R})$ (respectively $G(\mathbb{R})$) of order ≤ 2. Hence, from this description, the result is clear in these cases. Finally, we show that the map

$$H^1(\mathbb{Q}, SO_8(k)) \longrightarrow H^1(\mathbb{Q}, SO_9)$$

is a bijection of infinite sets. Indeed, the map takes the isomorphism class of a quadratic space W of rank 8 and determinant D to the isomorphism class of the quadratic space $V = W \oplus \langle D \rangle$ of rank 9 and determinant 1.

It is injective by Witt's cancellation theorem: if $W \oplus \langle D \rangle \cong W' \oplus \langle D \rangle$, then $W \cong W'$. Moreover, it is surjective by the Hasse-Minkowski theorem, which shows that every V of rank 9 and determinant 1 represents D. This completes the proof of Proposition 1.

6. **Proof of Proposition 2**

We now give the proof of Proposition 2 on integral embeddings. The proof that C_p is non-empty is similar to that for the existence of rational embeddings, and relies on the fact that $(R \otimes \mathbb{Z}_p)_{Tr=0}$ is a split, non-degenerate quadratic space of rank 7 over $\mathbb{Z}_p$. This implies that any p-adic integer is the norm of an element x in $R \otimes \mathbb{Z}_p$ with $Tr(x) = 0$. We sketch the proof for embeddings into $J_2 \otimes \mathbb{Z}_p$ and $J_3 \otimes \mathbb{Z}_p$, and leave the case of embeddings into $R \otimes \mathbb{Z}_p$ to the reader.

Suppose first that k is quadratic. Then $A \otimes \mathbb{Z}_p = \mathbb{Z}_p + \mathbb{Z}_p a$ for some element a with characteristic polynomial $P(x) = x^2 - \alpha x + \beta$. To define an integral embedding f, it suffices to set:

$$f(a) = \begin{pmatrix} \alpha & x \\ \overline{x} & 0 \end{pmatrix}$$

where $x \in R \otimes \mathbb{Z}_p$ satisfies $\mathbb{N}(x) = -\beta$.

Now consider the case when k is cubic. If $A \otimes \mathbb{Z}_p$ is not an integral domain, then the question of integral embeddings reduces to the case of embeddings into $J_2 \otimes \mathbb{Z}_p$. Hence assume that $A \otimes \mathbb{Z}_p$ is an integral domain, so that $A \otimes \mathbb{Z}_p$ can be simply generated over $\mathbb{Z}_p$ by an element a. Except in the case when $p = 3$ and $A \otimes \mathbb{Z}_3$ is unramified over $\mathbb{Z}_3$, a can be chosen to satisfy $Tr(a) = 0$. Suppose that the characteristic polynomial of a is $P(x) = x^3 + \alpha x - \beta$. Then we define an integral embedding f of $A \otimes \mathbb{Z}_p$ into $J_3 \otimes \mathbb{Z}_p$ by:

$$f(a) = \begin{pmatrix} -1 & z & \overline{y} \\ \overline{z} & 1 & 0 \\ y & 0 & 0 \end{pmatrix}$$

where y and z are elements of $R \otimes \mathbb{Z}_p$ satisfying $\mathbb{N}(y) = -\beta$ and $\mathbb{N}(z) = -P(1)$.

In the case when $p = 3$ and $A \otimes \mathbb{Z}_3$ is unramified over $\mathbb{Z}_3$, suppose the characteristic polynomial of a is $P(x) = x^3 - \alpha x^2 + \beta x - \gamma$, with α a unit in $\mathbb{Z}_3$. Then we define an integral embedding f by setting:

$$f(a) = \begin{pmatrix} \alpha & 0 & \overline{y} \\ 0 & 0 & x \\ y & \overline{x} & 0 \end{pmatrix}$$

where x and y are elements of $R \otimes \mathbb{Z}_3$ satisfying $\mathbb{N}(x) = -\gamma/\alpha \in \mathbb{Z}_3$, and $\mathbb{N}(y) = -\beta + \gamma/\alpha \in \mathbb{Z}_3$. This completes our sketch of the proof that integral embeddings exist.

Now let f be an integral embedding of $A\otimes\mathbb{Z}_p$, and let $\underline{H}$ be the subgroup scheme of $\underline{G}$ over $\mathbb{Z}_p$ stabilizing f. Then $\underline{H}(\mathbb{Z}_p)$ is a compact subgroup of $H(\mathbb{Q}_p)$. To show that it is (the connected component of) a special maximal compact subgroup, and that $\underline{G}(\mathbb{Z}_p)$ acts transitively on C_p, we count the number of embeddings modulo p^n.

Consider the case when G is of type G_2. We wish to count the number $N_n(A,R)$ of embeddings

$$f : A/p^nA \longrightarrow R/p^nR,$$

which are defined in the same way as in characteristic zero. Note that even though we continue to call f an embedding, f need not be injective now. For $n=1$, it follows from a result of Jacobson [J] that:

$$N_1(A,R) = \begin{cases} p^6+p^3, & \text{if } p \text{ splits in } A; \\ p^6-p^3, & \text{if } p \text{ is inert in } A; \\ p^6, & \text{if } p \text{ is ramified in } A. \end{cases}$$

For $n\geq 2$, we find, using Hensel's lemma, that:

$$N_n(A,R) = \begin{cases} (p^6+p^3)p^{6(n-1)}, & \text{if } p \text{ splits in } A; \\ (p^6-p^3)p^{6(n-1)}, & \text{if } p \text{ is inert in } A; \\ (p^6-1)p^{6(n-1)}, & \text{if } p \text{ is ramified in } A. \end{cases}$$

Indeed, in the ramified case, the embedding $f(\pi)=0(mod\ p)$, with π a uniformizing element in $A\otimes\mathbb{Z}_p$, does not lift $(mod\ p^2)$, but all other embeddings lift with no obstruction.

Now note that the special maximal compact subgroups have maximum volume among all compact subgroups of $H(\mathbb{Q}_p)$. Hence, after a brief computation, we find that for $n\geq 2$,

$$\frac{\#\underline{G}(\mathbb{Z}/p^n\mathbb{Z})}{\#\underline{H}(\mathbb{Z}/p^n\mathbb{Z})} \geq N_n(A,R)$$

with equality holding if and only if $\underline{H}(\mathbb{Z}_p)$ is a special maximal compact subgroup. On the other hand,

$$\frac{\#\underline{G}(\mathbb{Z}/p^n\mathbb{Z})}{\#\underline{H}(\mathbb{Z}/p^n\mathbb{Z})} \leq N_n(A,R)$$

with equality holding if and only if $\underline{G}(\mathbb{Z}_p)$ acts transitively on embeddings modulo p^n. Thus equality must hold in both cases, and the proposition is proved in this case. Moreover, we can identify the special maximal compact subgroup $\underline{H}(\mathbb{Z}_p)$, as there are two possibilities, up to conjugacy by $H^{ad}(\mathbb{Q}_p)$, when p is ramified in A. Indeed, we see from the above that $\underline{H}(\mathbb{Z}/p\mathbb{Z})$ fixes a non-zero element $x\in R/pR$ with $x^2=0$, and hence its reductive part is SL_2, the derived group of the Levi factor of the maximal parabolic subgroup of G_2 fixing the isotropic line $\langle x\rangle$, rather than SO_3.

The argument is entirely similar in the J_2 and J_3 cases. We simply give the formula for the number of embeddings modulo p^n. For $n \geq 2$, we have:

$$N_n(A, J_2) = \begin{cases} (p^8 + p^4)p^{8(n-1)}, & \text{if } p \text{ splits in } A; \\ (p^8 - p^4)p^{8(n-1)}, & \text{if } p \text{ is inert in } A; \\ (p^8 - 1)p^{8(n-1)}, & \text{if } p \text{ is ramified in } A. \end{cases}$$

As before, the above formula holds when $n = 1$, except in the ramified case, where $N_1(A, J_2) = p^8$. Again, this is due to the fact that the embedding $f(\pi) = 0(mod\ p)$ does not lift $(mod\ p^2)$.

Note that to obtain the first inequality as above, which compares the volume of $\underline{H}(\mathbb{Z}_p)$ with that of a special maximal compact subgroup, we need to show that in the ramified case, $\underline{H}(\mathbb{Z}_p)$ does not contain the central element -1 of $H = SO_8(k)$. For simplicity, suppose that $p \neq 2$, so that $A \otimes \mathbb{Z}_p$ is tamely ramified over $\mathbb{Z}_p$. Then $J_2 \otimes \mathbb{Z}_p$ contains the sub-lattice

$$L_p = (A \otimes \mathbb{Z}_p) \oplus (A \otimes \mathbb{Z}_p)^{\perp}$$

with index p. Hence we need to show that the element $-1 \in H$, which acts trivially on $A \otimes \mathbb{Z}_p$ and as -1 on $(A \otimes \mathbb{Z}_p)^{\perp}$, does not preserve the lattice $J_2 \otimes \mathbb{Z}_p$. But if L_p^* denotes the dual lattice of L_p, then the image of $J_2 \otimes \mathbb{Z}_p$ in the rank 2 quadratic space L_p^*/L_p over $\mathbb{Z}/p\mathbb{Z}$ is an isotropic line, and the element $-1 \in H$ sends it to the other isotropic line. This gives the first inequality above and also shows that the group scheme $\underline{H}$ over $\mathbb{Z}_p$ is connected.

Finally, we have, for $n \geq 3$,

$$N_n(A, J_3) = \begin{cases} (p^{24} + 2p^{20} + 2p^{16} + p^{12})p^{24(n-1)}, & \text{if } p = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3; \\ (p^{24} - p^{12})p^{24(n-1)}, & \text{if } p = \mathfrak{p}_1\mathfrak{p}_2; \\ (p^{24} - p^{20} - p^{16} + p^{12})p^{24(n-1)}, & \text{if } p = \mathfrak{p}; \\ (p^{24} + p^{20} - p^{12} - p^8)p^{24(n-1)}, & \text{if } p = \mathfrak{p}_1^2\mathfrak{p}_2; \\ (p^{24} - p^{16} - p^{12} + p^4)p^{24(n-1)}, & \text{if } p = \mathfrak{p}^3. \end{cases}$$

The above formula holds for $n = 1, 2$ as well, except in the following cases:

$$N_1(A, J_3) = \begin{cases} p^{24} + p^{20} + p^{16}, \text{ if } p = \mathfrak{p}_1^2\mathfrak{p}_2; \\ p^{24}, \text{ if } p = \mathfrak{p}^3; \end{cases}$$

$$N_2(A, J_3) = (p^{24} - 1)p^{24}, \text{ if } p = \mathfrak{p}^3.$$

This completes the proof of Proposition 2, and hence of Theorems 1-3.

7. The lattice $A^{\perp}$

In this section, we study the lattice $A^{\perp}$, as a representation of the group $\underline{H}(\mathbb{Z})$ stabilizing a ring embedding of A. For simplicity, we shall only consider those cases where all the ring embeddings are conjugate under $\underline{G}(\mathbb{Z})$, so that $A^{\perp}$ depends only on A and not on the choice of embedding.

When $f : A \to R$, the lattice $A^\perp$ has rank 6 and discriminant $-D$. In the table below, we tabulate the cases where all embeddings are $\underline{G}(\mathbb{Z})$-conjugate, giving the number of roots in $A^\perp$, and describing it as a root lattice whenever possible.

$-D$	$\#\underline{H}(\mathbb{Z})$	#roots in $A^\perp$	$A^\perp$
3	216	72	E_6
4	96	60	D_6
7	21	42	A_6
8	16	42	$A_1 \times D_5$
11	8	40	
15	3	26	$A_2 \times A_4$
23	1	22	

This corrects an error in [EG1], where it was claimed that for $D = -7$, $A^\perp$ is the lattice of the Klein quartic, which has discriminant 7^3. Indeed, by our description of $\underline{H}(\mathbb{Z}_p)$ at the ramified primes, it follows that $A^\perp$ is not a Hermitian space over A. Otherwise, the reductive part of $\underline{H}(\mathbb{Z}/p\mathbb{Z})$ would be SO_3, rather than SL_2.

When $f : A \to J_2$, the lattice $A^\perp$ has rank 8 and discriminant D. The only two cases where all embeddings are $\underline{G}(\mathbb{Z})$-conjugate are given below:

D	$\#\underline{H}(\mathbb{Z})$	$A^\perp$
1	$2^{13} \cdot 3^5 \cdot 5^2 \cdot 7 = \#SO(E_8)$	E_8
5	$2^9 \cdot 3^4 \cdot 5 \cdot 7 = \#SO(E_7)$	$E_7 + \langle 10 \rangle \subset_2 A^\perp$

where in the case $D = 5$, $A^\perp$ contains $E_7 + \langle 10 \rangle$ with index 2.

When $f : A \to J_I$ or $f : A \to J_E$, the lattice $A^\perp$ has rank 24 and discriminant D. When A is the ring of integers in a cyclic cubic extension field with $D = d^2$, there are interesting Niemeier (i.e. even and unimodular) lattices

$$A^\perp \subset L \subset (A^\perp)^*$$

which are stable under $\underline{H}(\mathbb{Z})$. For example, assume that $D = p^2$, so that $p = \mathfrak{p}^3$ is totally ramified in A, and $p \equiv 1 (mod\ 3)$. Then, as rank 2 quadratic spaces,

$$(A^\perp)^*/A^\perp \cong A^*/A \cong \mathfrak{p}^{-2}A/A.$$

This contains the self-dual subgroup $\mathfrak{p}^{-1}A/A$, which therefore determines a Niemeier lattice L. Since $(A^\perp)^*/A^\perp$ is split over $\mathbb{Z}/p\mathbb{Z}$, there is another Niemeier lattice L' between $A^\perp$ and $(A^\perp)^*$. Both L and L' are stable under $\underline{H}(\mathbb{Z})$. In the next section, we will give an explicit example in the case when $D = 7^2$.

8. **Fourier Coefficients of Modular Forms**

We end with some remarks on the Fourier coefficients of modular forms, which count the number of embeddings in the style of Siegel.

In [EG1], it was shown that $N(A_D, R)$ (where D is the discriminant of A_D) is the $|D|$-th Fourier coefficient of the theta function of the lattice $L = 2 \cdot E_7^*$, of rank 7 and discriminant 2^{13}. Moreover, this theta function is equal to Cohen's Eisenstein series H_3 of weight 7/2 on $\Gamma_0(4)$, and constant term 1.

Similarly, the integer $N(A_D, J_2)$ is the D-th Fourier coefficient of the theta function of the lattice $L = 2E_8 + \langle 2 \rangle$ of rank 9 and discriminant 2^{17}. Indeed, since

$$A = \mathbb{Z} + \mathbb{Z}\left(\frac{D+\sqrt{D}}{2}\right)$$

and $M := \langle I \rangle^{\perp} = E_8 + \langle 2 \rangle$ in J_2, to give an embedding $f : A \to J_2$ is equivalent to giving a vector $v = f(\sqrt{D})$ in M satisfying

$$\begin{cases} \langle v, v \rangle = 2D; \\ v \equiv D \ mod \ 2J_2. \end{cases}$$

The latter condition implies that $v \in 2M^* = 2E_8 + \langle 2 \rangle$. Again, since the Kohnen space of forms of weight 9/2 on $\Gamma_0(4)$ is 1-dimensional [Ko], this theta function is equal to Cohen's Eisenstein series H_4, with constant term 1 [C]. This gives another independent proof of Theorem 2:

$$N(A_D, J_2) = 240 \cdot L(\varepsilon_D, -3).$$

Using Cohen's tables [C], we tabulate some values for small D.

D	$N(A_D, J_2)$
1	2
5	$480 = 2^5 \cdot 3 \cdot 5$
8	$2640 = 2^4 \cdot 3 \cdot 5 \cdot 11$
12	$11040 = 2^5 \cdot 3 \cdot 5 \cdot 23$
13	$13920 = 2^5 \cdot 3 \cdot 5 \cdot 29$
17	$39360 = 2^6 \cdot 3 \cdot 5 \cdot 41$
21	$73920 = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 11$
24	$125280 = 2^5 \cdot 3^3 \cdot 5 \cdot 29$

Now we come to the last, and perhaps the most interesting, case: when A is the ring of integers of a real cubic field. Theorem 3 does not give us the value of $N(A, J_I)$ or $N(A, J_E)$, but only their weighted sum. For the first three real cyclic cubic fields, we have:

D	$91 \cdot N(A, J_I) + 600 \cdot N(A, J_E)$
7^2	$2^9 \cdot 3^3 \cdot 5^2 \cdot 13 \cdot 79$
9^2	$2^9 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 199$
13^2	$2^9 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 103 \cdot 109$

In the case when $D = 7^2$, so that $A = \mathbb{Z}[\mu_7]^+$, the ring of integers in the totally real cubic subfield of the cyclotomic field $\mathbb{Q}(\mu_7)$, we can actually work out the values of $N(A, J_I)$ and $N(A, J_E)$. Here, there is a unique embedding $A \to J_I$ up to $\underline{G}(\mathbb{Z})$-conjugacy, and a unique embedding $A \to J_E$ up to $\underline{G}'(\mathbb{Z})$-conjugacy. The stabilizer of an embedding $A \to J_I$ is $\underline{H}(\mathbb{Z}) = G_2(2)$, which has order $2^6 \cdot 3^3 \cdot 7$, and the lattices L and L' of the last section are both E_8^3. On the other hand, the stabilizer of an embedding $A \to J_E$ has stabilizer $\underline{H}'(\mathbb{Z}) = 7^2 \cdot 2A_4$, which has order $2^3 \cdot 3 \cdot 7^2$, and the lattices L and L' are the Leech lattice and the Niemeier lattice A_6^4 respectively. Hence, we have:

$$N(A, J_I) = 2^9 \cdot 3^3 \cdot 5^2,$$
$$N(A, J_E) = 2^9 \cdot 3^4 \cdot 13.$$

One can ask if the numbers

$$\begin{aligned} N_A := & \frac{N(A, J_I)}{\#\underline{G}(\mathbb{Z})} + \frac{N(A, J_E)}{\#\underline{G}'(\mathbb{Z})} \\ = & \frac{91 \cdot N(A, J_I) + 600 \cdot N(A, J_E)}{2^{15} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13} \end{aligned}$$

can be interpreted as the Fourier coefficients of a suitable modular form. The results of [Ga2] suggest that such an interpretation does exist. Indeed, the numbers N_A are the Fourier coefficients of an automorphic form on the split group over $\mathbb{Q}$ of type G_2. To explain what we mean, we need to recall some structural results about G_2. Recall that G_2 has a Heisenberg maximal parabolic subgroup $P = L{\cdot}U$, whose Levi factor $L \cong GL_2$ is generated by the short simple root, and whose unipotent radical U is a Heisenberg group of dimension 5, with a 1-dimensional center Z. The adjoint action of L on U/Z is isomorphic to the representation $Sym^3(\mathbb{Q}^2) \otimes det^{-1}$. It is known [Wr] that the non-trivial $L(\mathbb{Q})$-orbits in this representation are parametrized naturally by cubic algebras over $\mathbb{Q}$, with the étale cubic algebras corresponding to the generic orbits.

Now if φ is an automorphic form on G_2, then we can consider its Fourier expansion along U/Z. To be precise, suppose that ψ is a character of the compact abelian group $Z(\mathbb{A})U(\mathbb{Q})\backslash U(\mathbb{A})$. Then the ψ-Fourier coefficient of φ is the function

$$\varphi_\psi(g) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \psi(u)^{-1} \cdot \varphi(ug)du$$

on $G_2(\mathbb{A})$. The characters of $Z(\mathbb{A})U(\mathbb{Q})\backslash U(\mathbb{A})$ can be parametrized by $Z(\mathbb{Q})\backslash U(\mathbb{Q})$. For each A, let us fix a character ψ_A in the $L(\mathbb{Q})$-orbit corresponding to the cubic algebra $A \otimes \mathbb{Q}$.

In [Ga2], it was shown that there exists a non-zero vector subspace $\mathbb{V}_\infty$ of the space of automorphic forms on G_2, which affords a certain discrete series representation of $G_2(\mathbb{R})$, and whose elements have non-zero ψ_A-Fourier coefficient only if A is totally real. Moreover, if A is totally real, then there exists $g_A \in L(\mathbb{A})$ such that for any $\varphi \in \mathbb{V}_\infty$,

$$\varphi_{\psi_A}(g_A) = |det(g_A)|^5 \cdot N_A \cdot c(\varphi).$$

Here the non-zero linear map

$$c : \mathbb{V}_\infty \longrightarrow \mathbb{C}$$

is a generalized Whittaker functional with $U(\mathbb{R})$ transforming via a fixed character ψ in the $L(\mathbb{R})$-orbit corresponding to $\mathbb{R}^3$.

The space $\mathbb{V}_\infty$ above is constructed using theta series lifting from the group G (the anisotropic form of F_4 over $\mathbb{Q}$). Indeed, $G_2 \times G$ forms a dual reductive pair in the quaternionic form of E_8 over $\mathbb{Q}$ [Ga], and $\mathbb{V}_\infty$ is spanned by certain theta lifts of the constant function on $G(\mathbb{A})$. For more details, as well as the proofs of the above statements, we refer the reader to [Ga2]. For a better formulation, we need an analogue of the q-expansion of holomorphic modular forms on GL_2 for those automorphic forms on G_2 whose infinite component lies in the quaternionic discrete series. We also need a determination of the $GL_2(\mathbb{Z})$-orbits on the characters of $Z(\mathbb{R})U(\mathbb{Z})\backslash U(\mathbb{R})$.

Finally, we end this paper with the speculation that there should be a distinguished automorphic form on G_2 whose Fourier coefficients are given by the difference $M_A := N(A, J_I) - N(A, J_E)$. Indeed, since

$$\#G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R}) \cdot \underline{G}(\hat{\mathbb{Z}}) = 2,$$

there is a non-constant automorphic form on G which is right-invariant under $G(\mathbb{R}) \cdot \underline{G}(\hat{\mathbb{Z}})$ and orthogonal to the constant functions. As shown in [EG2], this automorphic form on G is lifted from the Ramanujan Δ-function on the upper half plane. It is natural to conjecture that a suitable theta lift of this form to G_2 is what we are looking for.

References

[C] H. Cohen, *Sums Involving the Values at Negative Integers of L-functions of Quadratic Characters*, Math. Annalen 217 (1975), Pg. 271-285.

[D] V. Drinfeld, *Commutative Subrings of Certain Non-Commutative Rings*, Functional Anal. Appl. 11 (English Translations), No. 1 (1977), Pg. 9-12.

[EG] N. Elkies and B. H. Gross, *Embeddings into Integral Octonions*, Pacific Journal of Math Vol. 181, No. 3 (1997), Pg. 147-158.

[EG2] N. Elkies and B. H. Gross, *The Exceptional Cone and the Leech Lattice*, International Math. Research Notices No. 14 (1996), Pg. 665-698.

[G] B. H. Gross, *Groups over $\mathbb{Z}$*, Invent. Math. 124 (1996), Pg.263-278.
[G2] B. H. Gross, *On the Motive of a Reductive Group*, Invent. Math. 130, No. 2 (1997), Pg. 287-313.
[Ga] W. T. Gan, *An Automorphic Theta Module for Quaternionic Exceptional Groups*, preprint (1998).
[Ga2] W. T. Gan, *Towards a Siegel-Weil Formula in Exceptional Groups*, preprint (1998).
[GG] B. H. Gross and W. T. Gan, *Haar Measure and the Artin Conductor*, to appear in Transactions of American Math. Society.
[J] N. Jacobson, *Composition Algebras and Their Automorphisms*, Rend. Palermo (1958).
[J2] N. Jacobson, *Structure and Representations of Jordan Algebras*, American Math. Society, Providence, RI (1968).
[K] R. Kottwitz, *Tamagawa Numbers*, Annals of Math. 127 (1988), Pg. 629-646.
[Ko] W. Kohnen, *Modular Forms of Half-Integral Weight on* $\Gamma_0(4)$, Math. Annalen 248 (1980), Pg. 249-266.
[PR] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press (1994).
[So] D. Soda, *Some Groups of Type D_4 defined by Jordan Algebras*, Journal Reine Angew. Math. 223 (1966), Pg. 150-163.
[Se] J. P. Serre, *Galois Cohomology*, Springer (1997).
[Se2] J. P. Serre, *Résumé des Cours des 1982-83*, Collected Works Vol. 3, Springer (1985).
[Wr] D. Wright, *The Adelic Zeta Function Associated to the Space of Binary Cubic Forms*, Math. Annalen 270 (1985), Pg. 503-534.

Mathematics Department, Harvard University, Cambridge, MA 02138, USA.

Mathematics Department, Princeton University, Princeton, NJ 08544, USA.
E-mail address: gross@math.harvard.edu
E-mail address: wtgan@math.princeton.edu