GLOBAL ENDOSCOPIC LIFTS FROM PGL_3 TO G_2

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1. Introduction

This paper is part of a project devoted to the construction of endoscopic L-packets for the split exceptional group of type G_2 over a number field F. Recall that the dual group of G_2 is $G_2(\mathbb{C})$ and consider the natural inclusion

$$\gamma : SL_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

Let π be a cuspidal automorphic representation of $PGL_3(\mathbb{A})$ (where $\mathbb{A}$ is the adele ring of F). The functoriality conjecture of Langlands predicts that, via the inclusion γ of dual groups, there is a global L-packet $\mathcal{L}(\pi)$ of G_2 associated to π. Indeed, for each place v of F, there *should* be a local L-packet $\mathcal{L}(\pi_v)$ of $G_2(F_v)$. It is expected that $\mathcal{L}(\pi_v)$ contains a unique generic representation $\sigma(\pi_v)$. Hence, the global L-packet should contain a distinguished element $\sigma(\pi) = \otimes_v \sigma(\pi_v)$. It is the unique (abstractly) generic representation in $\mathcal{L}(\pi)$ and the conjecture of Langlands predicts that it is cuspidal automorphic. The lifting $\pi \mapsto \sigma(\pi)$ is the endoscopic lift of the title. In order to prove the existence of this lift, two major problems need to be solved. Firstly, the distinguished element $\sigma(\pi_v)$ of the local L-packet $\mathcal{L}(\pi_v)$ needs to be defined, and secondly the occurence of the representation $\sigma(\pi)$ in the space of cusp forms on G_2 needs to be proved.

Our approach to the lifting $\pi \mapsto \sigma(\pi)$ is via the theta correspondence arising from restricting the minimal representation Π of E_6 to the subgroup $PGL_3 \times G_2$. The whole task here can be also broken down into two parts. The first one is to determine the local correspondence, and to show that it is compatible with the local Langlands functoriality. The second, of course, is to prove the cuspidality and non-vanishing of the global theta lift.

For non-archimedean F_v, the first problem was solved in [GS]. The local theta correspondence arising from restricting the minimal representation Π_v of E_6 was determined essentially completely there. In particular, if π_v is generic, it was shown that there is a unique generic representation $\sigma(\pi_v)$ of $G_2(F_v)$ such that

$$Hom_{PGL_3 \times G_2}(\Pi_v, \pi_v \otimes \sigma(\pi_v)) \neq 0,$$

in which case the dimension of this space is 1. Moreover, if π_v is non-supercuspidal, then this $\sigma(\pi_v)$ is indeed equal to the functorial lift of π_v (generic non-supercuspidal representations of G_2 have been classified by Shahidi). When π_v is supercuspidal, so is $\sigma(\pi_v)$. We have thus specified a generic representation $\sigma(\pi_v)$ of $G_2(F_v)$ for each finite place v, and the function $\pi_v \mapsto \sigma(\pi_v)$ is functorial to the extent that parametrization of representations of $G_2(F_v)$ is known.

1991 *Mathematics Subject Classification.* Primary: 11F70, 22E35, 22E50.

Next, we turn our attention to an archimedean place v. In this case, the local parametrization is known, and the map $\pi_v \mapsto \sigma(\pi_v)$ is already well defined. Thus, we must show that it is compatible with the local theta correspondece. Unlike the case of classical theta correspondence, the analog of Howe's conjecture is not known here; so we do not know a priori that π_v has a unique theta lift. Nevertheless, in [HPS] (see also [L]), it was shown that the correspondence is functorial on the level of infinitesimal characters. More precisely, if π_v and σ_v are representation of $PGL_3(F_v)$ and $G_2(F_v)$ respectively, such that $\pi_v \otimes \sigma_v$ is a quotient of Π_v, then $d\gamma(\lambda) = \lambda'$ where $d\gamma$ is the differential of the inclusion γ of the dual groups, and λ and λ' are the infinitesimal characters of π_v and σ_v, respectively. For a given π_v, the correspondence of inifnitesimal characters severely limits what σ_v could be, but stops short of showing the compatibility of the two correspondences.

Our first main result of this paper is for $F_v = \mathbb{R}$. By controlling the minimal K-types of principal series representations in the theta correspondence, we show that the theta correspondence is functorial for spherical generic unitary representations, and for generic unitary representations with integral infinitesimal character. The failure to obtain a complete answer is due to the fact that some representations (such as non-spherical principal series) are not completely determined by its infinitesimal character and the minimal K-type.

Our next result is the non-vanishing of the global theta lifting of π. In particular, there exists a cuspidal automorphic representation σ of $G_2(\mathbb{A})$ such that $\sigma_v \cong \sigma(\pi_v)$ for every finite place v. At archimedean places, of course, we have the matching of infinitesimal characters. However, if we restrict ourselves to representation of real groups for which we have the compatibilty of the Langlands and theta correspondences, we have the following precise result:

Main Theorem . *Assume that F is totally real. Let π be an irreducible cuspidal representation of $PGL_3(\mathbb{A})$. Assume that for each real place ∞, π_∞ is either spherical or has integral infinitesimal character. Let $\sigma(\pi)$ be the generic representation of $G_2(\mathbb{A})$ defined above.*

(i) (Existence of functorial lift) The representation $\sigma(\pi)$ occurs in the space of globally generic cusp forms on G_2.

(ii) (Uniqueness of weak lifting) Let $\sigma = \otimes_v \sigma_v$ be a globally generic cuspidal representation of $G_2(\mathbb{A})$ which is nearly equivalent to $\sigma(\pi)$. Then σ is equal to $\sigma(\pi)$ as a subspace of the space of globally generic cusp forms.

The proof of the non-vanishing of the global theta lift in (i) is not difficult. However, as we point out in the remark at the end of Section 7, there is a common error at one point of the proof, which was perpetuated in many papers on this subject in the literature. The proof of (ii) will rely on a result of Ginzburg and Jiang [GJ], which says that a generic cuspidal representation σ of $G_2(\mathbb{A})$ is a global theta lift if and only if its partial standard L-function has a pole at $s = 1$.

Of course, we would like to construct the whole local packet $\mathcal{L}(\pi_v)$ and to show that the elements of the global packet occur in the cuspidal spectrum with the multiplicities predicted by Arthur. As remarked in [GS], the local theta correspondence for $PGL_3 \times G_2$ is not sufficient for constructing the whole local packet; one needs to consider the dual pair $PD^\times \times G_2$ with D a degree 3 division algebra. This dual pair was studied in [S] but the results there need to be sharpened. The global question requires a detailed study of the global theta correspondence which is usually very difficult. We hope to return to these questions in the future.

2. **Real correspondence**

In this section we shall state our main result on the theta correspondence in the real case. First, however, we need to describe the minimal representation of the split exceptional group $H(\mathbb{R})$ of type E_6. Let $\mathfrak{h}$ be the complexified Lie algebra of $H(\mathbb{R})$, and $K \cong Sp_8/\langle \pm 1 \rangle$ the complexified maximal compact subgroup of $H(\mathbb{R})$.

Proposition 1. *Let Π_∞ be the $(\mathfrak{h}, K)$-module of the minimal representation of $H(\mathbb{R})$. Then*

$$\Pi_\infty|_K = \bigoplus_{n=0}^{\infty} V_{n\omega_4}$$

where ω_4 is the fourth fundamental weight for Sp_8, equal to $(1,1,1,1)$ using the standard realization of the root system C_4, as in Bourbaki [Bo].

So far there has been practically no result on the continuous spectrum of exceptional dual pair correspondences. The best result (in that direction) is the so-called matching of infinitesimal characters. More precisely, if π and σ are two representations of $PGL_3(\mathbb{R})$ and $G_2(\mathbb{R})$ such that $\pi \otimes \sigma$ is a quotient of Π_∞, then we know that the infinitesimal character of π determines the infinitesimal character of σ as follows.

Recall that the root systems of SL_3 and G_2 can be realized in standard coordinates (t_1, t_2, t_3) with $t_1 + t_2 + t_3 = 0$. Let $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (0, 1, -1)$ be the simple roots for $SL(3)$, and let $\beta_1 = (1, -1, 0)$ and $\beta_2 = (-1, 2, -1)$ be the simple roots for G_2. This gives an identification γ of the root lattice Λ of SL_3 with the root lattice Λ' of G_2.

The infinitesimal character of π can be identified with a point λ in $\Lambda \otimes \mathbb{C}$. Let $\lambda' = \gamma(\lambda)$ be the corresponding point in $\Lambda' \otimes \mathbb{C}$. The matching theorem ([HPS] and [Li]) says that the infinitesimal character of σ is λ'. Of course, the matching of infinitesimal characters implies that for a given π, there could only be finitely many irreducible σ such that $\pi \otimes \sigma$ is a quotient of Π_∞, but it stops well short of determining σ in terms of π. However, we have the following important refinement:

Theorem 2. *Let π and σ be generic unitary representations of $PGL_3(\mathbb{R})$ and $G_2(\mathbb{R})$ respectively, such that $\pi \otimes \sigma$ is a quotient of Π_∞. Then (using the notation for representations introduced in the final section), we have:*

π	$\pi(\lambda, 1)$	$\pi_0(s,t)$	$\pi(k,t)$	$\pi(\lambda, \xi)$	$\pi_1(s,t), t \neq 0$	$\pi_1(s,0)$
σ	$\sigma(\lambda', 1)$	$\sigma_0(s,t)$	$\sigma(k,t)$	$\sigma(\lambda', \xi')$	$\sigma_1(s,t), t \neq 0$	*?*

where $\lambda' = \gamma(\lambda)$.

The first two columns deal with spherical unitary representations: tempered principal series and complementary series respectively. The third column contains tempered generalized principal series. In particular, we have proven that the theta correspondence of generic unitary representations is compatible with Langlands functoriality in these three cases.

The fourth and fifth columns deal with non-spherical representations: tempered principal series and complementary series respectively. If $\pi = \pi(\lambda, \xi)$, then $\sigma = \sigma(\lambda', \xi')$ for some non-trivial ξ'. Finally, if $\pi = \pi_1(s,t)$ and $t \neq 0$, then $\sigma = \sigma_1(s,t)$, and we have proven functoriality in this case as well. However, if $t = 0$, then $\sigma = \sigma_1(s,0)$ or possibly one other non-spherical complementary series representation with the same infinitesimal character.

To summarize, we can show that the correspondence for generic unitarizable representations is functorial except for non-spherical tempered representations, where we have not determined how ξ' depends on ξ, and for non-spherical complementary series in the case $t = 0$. We highlight an important special case:

Corollary 3. *Let π and σ be generic unitarizable representations of $PGL_3(\mathbb{R})$ and $G_2(\mathbb{R})$ respectively, such that $\pi \otimes \sigma$ is a quotient of Π_∞. Then π is spherical if and only if σ is so. Moreover, if λ and λ' are the corresponding infinitesimal characters, then $\lambda' = \gamma(\lambda)$.*

The proof of Theorem 2 is based on the matching of infinitesimal characters and the analysis of minimal K-types. The failure to completely describe the correspondence is due to the fact that a representation might not be determined by its infinitesimal character and a minimal K-type.

In the last section, we list all generic unitarizable representations of $PGL_3(\mathbb{R})$ according to their infinitesimal characters, and then all generic unitary representations of $G_2(\mathbb{R})$ with the matching infinitesimal characters. The matching of infinitesimal characters alone implies the third columns of Theorem 2, and shows that if π is a tempered principal series (resp. a complementary series) then σ must also be a tempered principal series (resp. a complementary series). Thus it remains to show that π is spherical if and only if σ is, which will be accomplished by exploiting minimal K-types.

Recall that the (complexified) maximal compact subgroups of $PGL_3(\mathbb{R})$ and $G_2(\mathbb{R})$ are $K_1 = SO(3)$ and $K_2 = SL_{2,l} \times SL_{2,s}/\langle \pm 1 \rangle$ respectively. Here, $SL_{2,l}$ (resp. $SL_{2,s}$) is generated by a pair of long (resp. short) roots. Let V_k be the irreducible representation of $SL_2(\mathbb{C})$ of highest weight k (and dimension $k+1$). The minimal K_1-type of a non-spherical principal series of $PGL_3(\mathbb{R})$ is V_2, and the minimal K_2-type of a non-spherical principal series of $G_2(\mathbb{R})$ is $V_0 \otimes V_2$. Clearly, to prove the theorem, it suffices to show that for every n, the restriction of $V_{n\omega_4}$ to $K_1 \times K_2$ does not contain $V_0 \otimes (V_0 \otimes V_2)$ and $V_2 \otimes (V_0 \otimes V_0)$.

This sort of branching result is very difficult, and we start by remarking that in both cases the type is trivial on $SL_{2,l}$. In particular our first task is to compute the $SL_{2,l}$-invariants in $V_{n\omega_4}$.

The inclusion of $K_1 \times K_2$ into K is given by a combination of

$$\begin{cases} SO(3) \times SL_{2,s} \subseteq Sp_6 \\ Sp_6 \times SL_{2,l} \subseteq Sp_8 \end{cases}$$

where the first is the usual Howe dual pair, and the second is given by decomposing an 8 dimensional symplectic space as a direct sum of two nondegenerate subspaces of dimension 6 and 2. We have the following special case of the branching from $Sp(2N)$ to $Sp(2N-2) \times Sp(2)$; see [C] and [Z].

Proposition 4.

$$V_{n\omega_4}|_{Sp_6\times SL_{2,l}} = \oplus_{k=0}^{n} V_{(n-k)\omega_2+k\omega_3}\otimes V_k$$

where V_k is the irreducible representation of SL_2 of highest weight k, and $\omega_2=(1,1,0)$ and $\omega_3=(1,1,1)$ are the second and third fundamental weight for Sp_6 respectively.

The proof of Theorem now reduces to the following proposition which will be proved in the next section.

Proposition 5. *Let ω_2 be the second fundamental weight for Sp_6.*

- *Let $X_n = Hom_{SO(3)}(V_0, V_{n\omega_2})$. Then the three-dimensional representation V_2 of $SL_{2,s}$ does not appear in the $SL_{2,s}$-module X_n.*
- *Let $Y_n = Hom_{SO(3)}(V_2, V_{n\omega_2})$. Then the trivial representation V_0 of $SL_{2,s}$ does not appear in the $SL_{2,s}$-module Y_n.*

3. **Branching rules**

The purpose of this section is to prove Proposition 5 which is not easy at all. Let GL_3 be the subgroup of Sp_6 containing $SO(3)$, and such that the center of GL_3 is equal to a maximal split torus of SL_2. We shall first calculate the restriction of $V_{n\omega_2}$ to GL_3, and then further restrict down to $SO(3)$ by exploiting the following fact:

Proposition 6. *Let $\mu=(x,y,z)$, with $x\geq y\geq z$ integers, be the typical highest weight for GL_3, and let V_μ denote the corresponding irreducible representation. Assume that $x+y+z$ is even. The restriction of V_μ to $SO(3)$ either contains V_0 or V_2 (but not both) with multiplicity one. It contains V_0 if and only if all integers x, y, and z are even. Otherwise, it contains V_2, and this happens when only one of the three integers is even.*

Proof. This is a simple reinterpretation of the following well known branching result:

Lemma 7. *Let $V_{p,q}$ be the irreducible representation of SL_3 with highest weight $p\omega_1+q\omega_2$, where ω_1 and ω_2 are the two fundamental weights of SL_3. Let $m_{p,q}(k)$ denote the multiplicity of V_k (the representation of $SO(3)$ with highest weight k) in $V_{p,q}$. Then*

$$m_{p,q}(0)+m_{p,q}(2)=1$$

and $m_{p,q}(0)=1$ iff p and q are both even.

□

The strategy of the proof of Proposition 5 will be based on the following simple observation.

Lemma 8. *Let V be a finite dimensional representation of SL_2. Let $V(k)\subseteq V$ be the weight k subspace. Then V_n does not appear in V iff $V(n)=V(n+2)$.*

Thus to prove the first part of Proposition 5, for example, we need to show that $dimX_n(2)$ is equal to $dimX_n(4)$ or equivalently to $dimX_n(-4)$.

How does one compute $dimX_n(2k)$ in general? Recall that the center of GL_3 is a maximal split torus of $SL_{2,s}$. Therefore, as a GL_3-module, $V_{n\omega_2}(2k)$ consists of representations V_μ with

$x+y+z=2k$. From proposition 6, these contribute to $X_n(2k)$ if and only if all three integers x, y and z are even. Let us denote by $X(2k)$ the set of all highest weights $\mu=(x,y,z)$ for GL_3 such that $x+y+z=2k$, with x, y and z all even, and write $m_n(\mu)$ for the multiplicity of V_μ in $V_{n\omega_2}$. Then

$$dim X_n(2k) = \sum_{\mu\in X(2k)} m_n(\mu).$$

Now for $\mu\in X(2)$, let $\mu_- = (x-2, y-2, z-2)$, which is an element of $X(-4)$. The map $\mu\mapsto\mu_-$ defines a bijection of $X(2)$ with $X(-4)$. We will show that the multiplicities $m_n(\mu)$ and $m_n(\mu_-)$ coincide. This implies that $dim X_n(2) = dim X_n(-4)$, and the first part of Proposition 5 follows. A similar, albeit more complicated, trick works for Y_n. We shall explain it later.

Thus, we need to calculate the multiplicity $m_n(\mu)$ of V_μ in $V_{n\omega_2}$. Since $V_{n\omega_2}$ is self dual, and $V_\mu^*\cong V_\lambda$, where $\lambda=(-z,-y,-x)$, we have $m_n(\mu)=m_n(\lambda)$. So, from now on, we shall assume that $y\geq 0$. The multiplicity $m_n(\mu)$ will be calculated using the Weyl character formula.

3.1. **Partition function.** Let e_1, e_2 and e_3 be the standard basis of $\mathbb{R}^3$. Then e_i+e_j (we allow $i=j$) are positive roots of Sp_6 which are not in GL_3. Define the partition function $p(a,b,c)$ to be the number of ways the vector (a,b,c) in $\mathbb{Z}^3$ can be written as a linear combination of e_i+e_j with non-negative integer coefficients. We shall now gather several elementary facts about this partition function.

Lemma 9. *(i) $p(a,b,c)=0$ unless a,b,c are non-negative and $a+b+c$ is even.*

(ii) If $a+b+c$ is even, then (a,b,c) is a combination of e_i+e_j (with $i\neq j$) with non-negative integral coefficients iff

$$\begin{cases} a\leq b+c \\ b\leq c+a \\ c\leq a+b \end{cases}$$

in which case the combination is unique, since e_i+e_j with $(i\neq j)$ are linearly independent.

Define $q(a,b,c)$ to be 1 or 0 depending on whether (a,b,c) is a combination of e_i+e_j $(i\neq j)$ with non-negative integer coefficients or not. Clearly, we have the following simple lemma.

Lemma 10.

$$p(a,b,c)=\sum q(a-i,b-j,c-k)$$

where the sum is taken over all even, non-negative integers i, j and k.

3.2. **Branching law continued.** Let $\mu=(x,y,z)$ with $y\geq 0$. The Weyl character formula implies that the multiplicity $m_n(\mu)$ of V_μ in $V_{n\omega_2}$ is

$$m_n(\mu)=\sum_{w\in W} e(w)p(w(n\omega_2+\rho)-\rho-\mu)$$

where the sum is over the Weyl group for Sp_6, and $\rho = (3, 2, 1)$. Since $p = 0$ unless the argument has non-negative entries, one easily checks, due to $y \geq 0$ and the specific nature of ω_2, that the sum reduces to

$$p((n, n, 0) - \mu) - p((n, n, -2) - \mu) - p((n-1, n+1, 0) - \mu) + p((n-1, n+1, -2) - \mu).$$

Moreover, none of these terms appear if $z > 0$ or $x + y + z$ is odd. So we shall assume that $z \leq 0$ and $x + y + z$ is even.

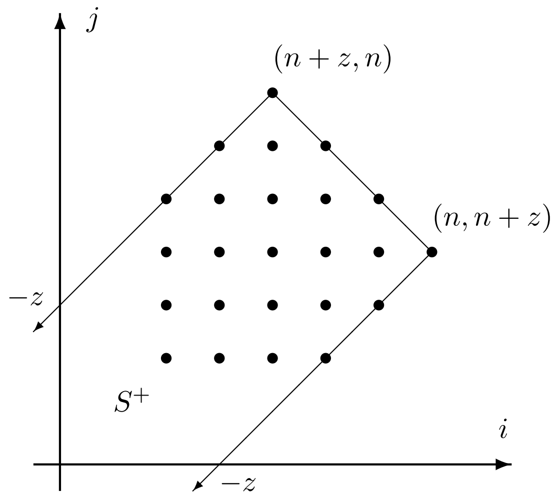
After a slight change of variables, Lemma 10 implies that

$$p(n - x, n - y, -z) - p(n - x, n - y, -z - 2) = \sum q(n - i, n - j, -z)$$

where the sum is taken over all $(i, j) \equiv (x, y) \pmod 2$ such that $x \leq i$ and $y \leq j$. From the second part of Lemma 9, we see that $q(n-i, n-j, -z) = 0$ unless (i, j) satisfies the equations

$$\begin{cases} i + j \leq 2n + z \\ i \leq j - z \\ j \leq i - z. \end{cases}$$

This is a half-strip S^+ with vertices at $(n+z, n)$ and $(n, n+z)$, and in the direction $(-1, -1)$. We illustrate it by the following figure, modeled after $\mu = (4, 4, -6)$, and $n = 14$.



The black dots are the points (i, j) in S^+ such that $x \leq i$, $y \leq j$, and $(i, j) \equiv (x, y) \pmod 2$. It follows that the difference of the first two terms in the formula for $m_n(\mu)$ counts the black points in the above picture.

Similarly, the difference

$$p(n - x, n - y, -z) - p(n - x, n - y, -z - 2)$$

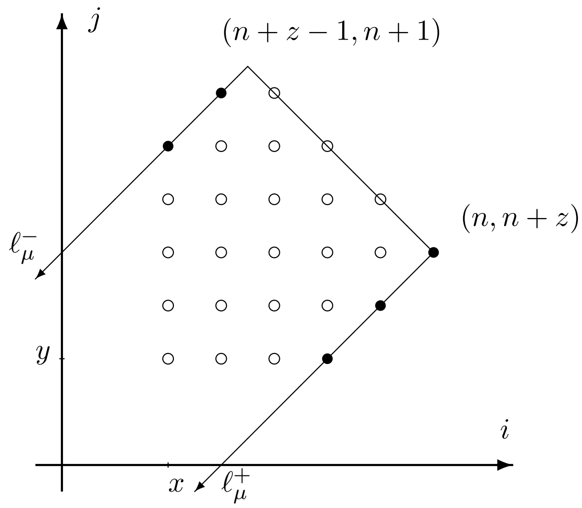
leads us to a half strip S^-, which is a translate of S^+ by $(-1, 1)$. Its vertices are at $(n + z - 1, n + 1)$ and $(n - 1, n + z + 1)$. In particular, we have obtained the following:

Theorem 11. *Let $\mu = (x, y, z)$ with $y \geq 0$. Then $m_n(\mu)$ (the multiplicity of V_μ in $V_{n\omega_2}$) is zero unless $z \leq 0$ and $x + y + z$ even. When these hold, let ℓ_μ^+ be the ray $(n, n+z) - t(1,1)$ with $t \geq 0$, and ℓ_μ^- be the ray $(n+z-1, n+1) - t(1,1)$ with $t \geq 0$.*

Let $n^+(\mu)$ and $n^-(\mu)$ be the number of points $(i, j) \equiv (x, y) \pmod 2$ with $x \leq i$ and $y \leq j$ on ℓ_μ^+ and ℓ_μ^- respectively. Then

$$m_n(\mu) = n^+(\mu) - n^-(\mu).$$

We can illustrate this theorem with the following figure, again modeled after $\mu = (4, 4, -6)$, and $n = 14$. In particular, it follows that the multiplicity of $V_{(4,4,-6)}$ in $V_{14\cdot\omega_2}$ is $3 - 2 = 1$.



Now we can finish the proof of the first part of Proposition 5. Let $\mu = (x, y, z)$ such that $x + y + z = 2$. Assume first that $y > 0$, so $y \geq 2$, since y is even. Thus both multipicities $m_n(\mu)$ and $m_n(\mu_-)$ can be calculated using the above theorem. The rays $\ell_{\mu_-}^+$ and $\ell_{\mu_-}^-$ are obtained by translating the rays ℓ_μ^+ and ℓ_μ^- down and left by 2 respectively. Next, note that $x + y + z = 2$ and $y \geq 2$ imply that the point (x, y) is stricitly between the rays ℓ_μ^+ and ℓ_μ^-. It follows that

$$\begin{cases} n^+(\mu) = n^+(\mu_-) \\ n^-(\mu) = n^-(\mu_-) \end{cases}$$

so that $m_n(\mu) = m_n(\mu_-)$. If $y \leq 0$, then we can apply similar considerations to the dual representations V_μ^* and $V_{\mu_-}^*$. The first part of Proposition 5 is proved.

Now we consider the second part of Proposition 5. We need to show that $dimY_n(0) = dimY_n(2)$. Recall that for an even integer $2k$, we have defined $X(2k)$ to be the set of all highest weights (x, y, z) for GL_3 such that $x + y + z = 2k$, with x, y, z all even. Let $Y(2k)$ be similarly defined, except that only one of the three integers is even. Now, if (x, y, z) is in $Y(2k)$, then we can add (or subtract) 1 to each of the two odd entries to obtain, after permuting the entries if necessary, an element in $X(2k+2)$ (or in $X(2k-2)$). This gives us a partition of $Y(2k)$ into subsets $Y_\mu(2k)$ parametrized by elements of $X(2k+2)$ (or $X(2k-2)$).

Then it is not difficult to see that

$$\begin{cases} dimY_n(0) = \sum_{\mu\in X(-2)} \sum_{\lambda\in Y_\mu(0)} m_n(\lambda) \\ dimY_n(2) = \sum_{\mu\in X(4)} \sum_{\lambda\in Y_\mu(2)} m_n(\lambda). \end{cases}$$

Now if $\mu = (x, y, z) \in X(4)$, then $\mu_- = (x-2, y-2, z-2) \in X(-2)$ and this sets up a bijection of $X(4)$ with $X(-2)$. Thus the second part of Proposition 5 will follow after we show:

Lemma 12. *For each* $\mu = (x, y, z)$ *in* $X(4)$*,*

$$\sum_{\lambda\in Y_\mu(2)} m_n(\lambda) = \sum_{\lambda\in Y_{\mu_-}(0)} m_n(\lambda)$$

Proof. Assume first that $y > 0$, so that $y \geq 2$ since y is even. Since $x + y + z = 4$, it follows that $z \leq 0$. Since the case $x = y$ can be similarly dealt with, we shall assume that $x > y$ as well. Then

$$Y_\mu(2) = \{\lambda_1 = (x-1, y-1, z), \lambda_2 = (x, y-1, z-1), \lambda_3 = (x-1, y, z-1)\}$$

and

$$Y_{\mu_-}(0) = \{\lambda'_1 = (x-1, y-1, z-2), \lambda'_2 = (x-2, y-1, z-1), \lambda'_3 = (x-1, y-2, z-1)\}.$$

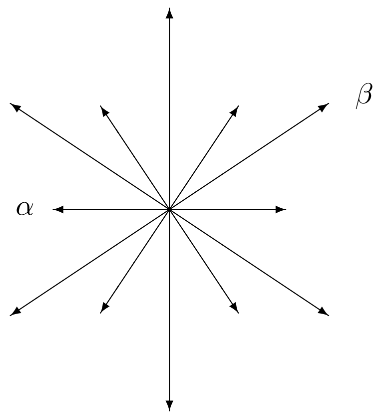
Now one easily checks that we have the following sequence of equalities

$$\begin{cases} n^+(\lambda_1) = n^+(\lambda'_3) \\ n^-(\lambda'_3) = n^-(\lambda_3) \\ n^+(\lambda_3) = n^+(\lambda'_1) \\ n^-(\lambda'_1) = n^-(\lambda_2) \\ n^+(\lambda_2) = n^+(\lambda'_2) \\ n^-(\lambda'_2) = n^-(\lambda_1) \end{cases}$$

Summing up these equalities gives the desired conclusion. Of course, if $y \leq 0$, then we can compute the multiplicites by passing to dual representations. We leave the details to the reader. □

4. **Structure of Groups**

In this section, we establish some notations to be used in the rest of the paper. Let F be any field of characteristic zero. We first describe the maximal parabolic subgroups of G_2. Fix a maximal split torus M of G_2 contained in a Borel subgroup $P = MN$. This establishes a system of simple roots $\{\alpha, \beta\}$ for G_2 with α short:



Let $P_1 = M_1N_1$ and $P_2 = M_2N_2$ be the two non-conjugate maximal parabolic subgroups of G_2 so that $P = P_1 \cap P_2$ and $M = M_1 \cap M_2$. Both M_1 and M_2 are isomorphic to GL_2. Further, we stipulate that P_2 is the Heisenberg parabolic so that N_2 is a 5-dimensional Heisenberg group. Then N_1 is a 3-step nilpotent group.

Let $\bar{P}_2 = M_2\bar{N}_2$ be the opposite parabolic of P_2. Then using the Killing form and the exponential map, $Hom(N_2, \mathbb{G}_a)$ can be identified with $V_2 = Lie(\bar{N}_2^{ab})$ where $\bar{N}_2^{ab} = \bar{N}_2/[\bar{N}_2, \bar{N}_2]$. Fix a Chevalley basis of $Lie(G_2)$. For every root δ, this gives an identification of the corresponding root subgroup N_δ with the additive group $\mathbb{G}_a$. The Lie algebra of N_δ is the line spanned by the corresponding Chevalley basis element and will be denoted by F_δ. We now have

$$V_2 = F_{-\beta} \oplus F_{-\alpha-\beta} \oplus F_{-2\alpha-\beta} \oplus F_{-3\alpha-\beta}.$$

Moreover, for an appropriate identification of M_2 with GL_2, the representation of M_2 on V_2 is isomorphic to $Sym^3(F^2) \otimes det^{-1}$.

If F is a local field and ψ is a unitary character of F, then composition with ψ gives an identification of V_2 with the group of unitary characters of $N_2(F)$. Hence a unitary character of $N_2(F)$ can be represented by a 4-tuple $(a, b, c, d) \in V_2(F)$. Similarly, if F is a global field and ψ a unitary character of $F\backslash\mathbb{A}$, then composition with ψ gives an identification of $V_2(F)$ with the group of characters of $N_2(\mathbb{A})$ trivial on $N_2(F)$.

Now let H be the split adjoint group of type E_6. Then $PGL_3 \times G_2$ forms a dual pair in H (cf. [MS] and [GRS2]). If we consider the adjoint action of the maximal torus $M \subset G_2$ on $Lie(H)$, the zero weight space of course contains $Lie(PGL_3)$, whereas the non-zero weights form a root system of type G_2 (cf. [GS2]). While the long root spaces are still of dimension 1, the short root spaces have dimension 9 and can naturally be given the structure of a 3×3 matrix algebra J. Hence, morally speaking, $H \cong G_2(J)$.

In any case, for each root δ of G_2, we write $\mathbf{N}_\delta$ for the corresponding root subgroup of H, so that

$$\mathbf{N}_\delta = \begin{cases} \mathbb{G}_a \text{ if } \delta \text{ is long;} \\ J \text{ if } \delta \text{ is short,} \end{cases}$$

and the inclusion of N_δ into $\mathbf{N}_\delta$ is given by the inclusion of scalars. The action of PGL_3 on $\mathbf{N}_\delta$ is trivial if δ is long and is given by conjugation of 3×3 matrices if δ is short.

Using this restricted root system of type G_2, we can define a maximal parabolic subgroup $\mathbf{P}_2 = \mathbf{M}_2\mathbf{N}_2$ of H such that its unipotent radical $\mathbf{N}_2$ is a Heisenberg group which is supported on the same root subgroups as N_2 (using $\mathbf{N}_\delta$ in place of N_δ). In particular,

$$\mathbf{P}_2 \cap (PGL_3 \times G_2) = PGL_3 \times P_2.$$

As before, using the Killing form and the exponential map, we can identify $Hom(\mathbf{N}_2, \mathbb{G}_a)$ with $\mathbf{V}_2 = Lie(\bar{\mathbf{N}}_2^{ab})$. Here,

$$\mathbf{V}_2 = F_{-\beta} \oplus J_{-\alpha-\beta} \oplus J_{-2\alpha-\beta} \oplus F_{-3\alpha-\beta}$$

and we shall represent an element in $\mathbf{V}_2$ as (a, x, y, d) with $a, d \in F$ and $x, y \in J$. As before, elements of $\mathbf{V}_2(F)$ can be identified with a suitable group of unitary characters when F is local or global.

The Levi subgroup $\mathbf{M}_2$ is isomorphic to $\mathbb{G}_m \times_{\mu_2} (SL_6/\mu_3)$. The center of $\mathbf{M}_2$ is equal to the center of M_2, and thus is isomorphic to $\mathbb{G}_m$ by the coroot $(3\alpha+2\beta)^\vee$. The quotient of $\mathbf{M}_2$ by its center is isomorphic to PGL_6. Moreover, there is a basis element of $Hom_F(\mathbf{M}_2, \mathbb{G}_m)$ whose restriction to the maximal split torus M of G_2 is equal to the highest root $\alpha_0 = 3\alpha+2\beta$. We denote this rational character by α_0 as well.

The group $\mathbf{M}_2 \cap (PGL_3 \times G_2) = PGL_3 \times M_2$ acts naturally on $\mathbf{V}_2$. The action of PGL_3 is clear from our description above. For later purposes, it will be necessary to know the action of $\mathbf{N}_\alpha \subset \mathbf{M}_2$ more explicitly. It is easiest to describe this action on the level of Lie algebras; one then uses the exponential map to obtain the action of a group element. More generally, for $z \in Lie(\mathbf{N}_\alpha)$, we have:

$$z : (a, x, y, d) \mapsto (Tr(xz), y \times z, dz, 0).$$

Here, Tr is the trace form on J and $y \times z = (y+z)^\# - y^\# - z^\#$ where $y \mapsto y^\#$ is the quadratic map such that $y \cdot y^\# = det(y)$.

The inclusion $N_2 \hookrightarrow \mathbf{N}_2$ induces a projection map

$$Hom(\mathbf{N}_2, \mathbb{G}_a) \to Hom(N_2, \mathbb{G}_a)$$

which is $PGL_3 \times M_2$-equivariant. The resulting projection $\mathbf{V}_2 \to V_2$ is given by

$$(a, x, y, d) \mapsto (a, Tr(x), Tr(y), d).$$

The restricted root system of type G_2 also allows us to define another maximal parabolic subgroup $\mathbf{P}_1$ whose unipotent radical is supported on the same root subgroups as N_1, using $\mathbf{N}_\delta$ instead of N_δ. In particular, $\mathbf{M}_2 \cap \mathbf{P}_1$ is a maximal parobolic subgroup of $\mathbf{M}_2$. Its unipotent radical is equal to $\mathbf{N}_\alpha$ and a Levi subgroup is $\mathbf{M} = \mathbf{M}_1 \cap \mathbf{M}_2$.

5. Global correspondence

Now we consider the global setting, so that F is a number field. Let $\Pi = \otimes_v \Pi_v$ be the global minimal representation of $H(\mathbb{A})$. In [GRS1], an $H(\mathbb{A})$-equivariant embedding

$$\Pi \hookrightarrow \mathcal{A}_2(H)$$

of Π into the space of square-integrable automorphic forms on H was constructed. We shall identify Π with its image in $\mathcal{A}_2(H)$. There are several important properties of this automorphic theta module, especially concerning its Fourier expansion along $\mathbf{N}_2$, which will be discussed at the end of this section.

Now let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $PGL_3(\mathbb{A})$. By the multiplicity one theorem, we know that π has a canonical realization in the space of cusp forms and we shall identify π with this realization. For $f \in \pi$ and $\varphi \in \Pi$, define an automorphic form on $G_2(\mathbb{A})$ by:

$$\Theta(\varphi, f)(g) = \int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} \varphi(gh) \cdot f(h) dh.$$

We shall denote the linear span of all such automorphic forms on G_2 by $\Theta(\pi)$; it is a $G_2(\mathbb{A})$-submodule of $\mathcal{A}(G_2)$.

Given an automorphic form on G_2, one may define its global Whittaker function (associated to a given generic character of $N(F)\backslash N(\mathbb{A})$). This is defined by the usual integral. We say that an automorphic form is *globally degenerate* if its global Whittaker function is zero (as a function on $G_2(\mathbb{A})$). Let $\mathcal{A}_{cusp,d}$ be the subspace of degenerate cusp forms, and let $\mathcal{A}_{cusp,gen}$ be its orthogonal complement in the space of cusp forms. An irreducible summand of $\mathcal{A}_{cusp,gen}$ is said to be *globally generic.*

The main result of this section is:

Theorem 13. *(i) $\Theta(\pi)$ is contained in the space of cusp forms on G_2.*

(ii) Let $\Theta_{gen}(\pi)$ be the projection of $\Theta(\pi)$ onto $\mathcal{A}_{cusp,gen}$. Then $\Theta_{gen}(\pi)$ is non-zero.

The proof of this theorem is largely computational and will be given in the following two sections. It is based on the expansion of φ along $\mathbf{N}_2$. More precisely, let $\mathbf{Z}$ be the center of $\mathbf{N}_2$, and consider the constant term of φ along

$$\varphi_{\mathbf{Z}}(g) = \int_{\mathbf{Z}(F)\backslash \mathbf{Z}(\mathbb{A})} \varphi(zg)\ dz.$$

Recall from the previous section that any character of $\mathbf{N}_2(F)\backslash \mathbf{N}_2(\mathbb{A})$ is ψ_X for some X in $\mathbf{V}_2(F)$. Let $\Omega \subseteq \mathbf{V}_2$ be the unique non-trivial $\mathbf{M}_2$-orbit in $\mathbf{V}_2$. It is the orbit of the highest weight vector. Assume that $X \neq 0$. By [MS], for any finite place v, we know that

$$dim\ Hom_{\mathbf{N}_2(F_v)}(\Pi_v, \psi_X) = \begin{cases} 1 \text{ if } x \in \Omega; \\ 0 \text{ otherwise.} \end{cases}$$

Now let φ_X be the Fourier coefficient of φ defined by the character ψ_X. We have:

Proposition 14. *For any $\varphi \in \Pi$, one has the Fourier expansion*

$$\varphi_{\mathbf{Z}}(g) = \varphi_{\mathbf{N}_2}(g) + \sum_{X \in \Omega(F)} \varphi_X(g).$$

Moreover, it follows by the weak approximation theorem that for any $m \in [\mathbf{M}_2(\mathbb{A}), \mathbf{M}_2(\mathbb{A})]$ which fixes X, we have

$$\varphi_X(mg) = \varphi_X(g).$$

We also need some facts about the constant term $\varphi_{\mathbf{N}_2}$. From the construction of the embedding $\Pi \hookrightarrow \mathcal{A}_2(H)$ in [GRS], one can show that the subspace of $\mathcal{A}(\mathbf{M}_2)$ consisting of the functions $\varphi_{\mathbf{N}_2}|_{\mathbf{M}_2}$ is isomorphic to $|\alpha_0|^2 \oplus |\alpha_0|^{3/2} \cdot \Pi(PGL_6)$. Here, $\Pi(PGL_6)$ is the irreducibie representation of $PGL_6(\mathbb{A})$ unitarily induced from the trivial character of the maximal parabolic subgroup of PGL_6 with Levi factor GL_5. The embedding of $\Pi(PGL_6)$ into $\mathcal{A}(\mathbf{M}_2)$ is by the formation of Eisenstein series.

In some sense, $\Pi(PGL_6)$ can be considered the minimal representation of $\mathbf{M}_2(\mathbb{A})$. More precisely, if we consider the Fourier expansion of $\varphi \in \Pi(PGL_6)$ along $\mathbf{N}_\alpha \cong J$ (the unipotent radical of $\mathbf{P}_1 \cap \mathbf{P}_2 = \mathbf{M} \cdot \mathbf{N}_\alpha$), then we have the following analog of Prop. 14:

Proposition 15. *For any $\varphi \in \Pi(PGL_6)$, one has the Fourier expansion*

$$\varphi(g) = \varphi_{\mathbf{N}_\alpha}(g) + \sum_{X \in \Omega(F)} \varphi_X(g), \quad g \in \mathbf{M}_2(\mathbb{A})$$

where $\Omega(F)$ is the set of $X \in \mathbf{N}_\alpha(F) \cong J(F)$ which has rank one. Moreover, if $m \in [\mathbf{M}(\mathbb{A}), \mathbf{M}(\mathbb{A})]$ fixes X, then $\varphi_X(mg) = \varphi_X(g)$.

Finally, we need the following lemma concerning the constant term $\varphi_{\mathbf{N}_\alpha}$.

Lemma 16. *For any $\varphi \in \Pi(PGL_6)$, the restriction of $\varphi_{\mathbf{N}_\alpha}$ to the subgroup $PGL_3 \subset \mathbf{M}$ is an automorphic form with cuspidal support along the Borel subgroup of PGL_3. In particular, $\varphi_{\mathbf{N}_\alpha}|_{PGL_3}$ is orthogonal to the space of cusp forms.*

6. **Cuspidality**

In this section, we prove Theorem 13(i). We need to show that the constant terms of $\Theta(\varphi, f)$ along N_1 and N_2 are both identically zero.

We first compute the constant term $\Theta(\varphi, f)_{N_2}$ of $\Theta(\varphi, f)$ along N_2. Since $\mathbf{Z}$ is also the center of N_2, we can replace φ by $\varphi_{\mathbf{Z}}$ in $\Theta(\varphi, f)_{N_2}$. Using the Fourier expansion of $\varphi_{\mathbf{Z}}$ along $\mathbf{N}_2$, we deduce that

$$\Theta(\varphi, f)_{N_2}(1) = \int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} [\varphi_{\mathbf{N}_2}(h) + \sum_{X \in \Omega_0} \varphi_X(h)] f(h) dh$$

where Ω_0 is the subset of Ω consisting of all X such that the restriction of φ_X to N_2 is trivial.

We shall now show that the integral of the sum over Ω_0 is zero. Using the projection $\mathbf{V}_2 \to V_2$, it follows that

$$\Omega_0 = \{(a, x, y, d) \in \Omega \mid a = Tr(x) = Tr(y) = d = 0\}.$$

The orbits of $PGL_3(F)$ on Ω_0 are decribed in [MS, Prop 7.4]. More precisely, let X be in Ω_0, and $\mathcal{O}(X)$ its $PGL_3(F)$-orbit. If S_X is the stabilizer in $PGL_3(F)$ of X, then S_X contains (as a normal subgroup) the unipotent radical U_X of a parabolic subgroup of PGL_3. It follows that

$$\int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} \sum_{Y\in\mathcal{O}(X)} \varphi_Y(h)f(h)dh = \int_{S_X(F)\backslash PGL_3(\mathbb{A})} \varphi_X(h)f(h)dh.$$

Since φ_X is left invariant under $U_X(\mathbb{A})$ by Prop. 14, the cuspidality of f now implies that the integral on the right hand side is zero.

It remains to show that the integral against $\varphi_{\mathbf{N}_2}$ is also zero. But $PGL_3 \times M_2$ is a dual pair in $\mathbf{M}_2$ and the integral

$$\int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} \varphi_{\mathbf{N}_2}(hg)\cdot f(h)dh, \quad \text{for } g\in M_2(\mathbb{A})$$

is simply the theta lift (using $\Pi(PGL_6)$) of $f\in\pi$ to $M_2 \cong GL_2$. Hence, to show that the above integral vanishes, we need to show that in the theta correspondence furnished by the dual pair $PGL_2 \times PGL_3 \subset PGL_6$, a cuspidal representation of PGL_3 always lifts to zero.

Using Prop. 15 and Lemma 16, it is easy to show that the theta lift of π to $M_2 \cong GL_2$ is always cuspidal; we omit the details here. To show that it is zero, one needs to examine a non-constant Fourier coefficient of the theta lift along $N_\alpha \subset M_2$. After a short computation, using Prop. 15, one finds that such a Fourier coefficient is zero if and only if the following holds:

$$\int_{GL_2(F)\backslash GL_2(\mathbb{A})} f(h)dh = 0 \quad \text{for all } f\in\pi,$$

where GL_2 is the Levi subgroup of a maximal parabolic subgroup of PGL_3. But the above period integral vanishes by the results of Friedberg-Jacquet [FJ]. This shows that the constant term $\Theta(\varphi,f)_{N_2}$ is identically zero.

We now consider the other constant term $\Theta(\varphi,f)_{N_1}$. Consider this as a function on $M_1 \cong GL_2$. Then, because of the vanishing of $\Theta(\varphi,f)_{N_2}$, this function on M_1 is a cusp form. To show that it is zero, we examine a non-constant Fourier coefficient of $\Theta(\varphi,f)_{N_1}$ along $N_\beta \subset M_1$. In other words, if ψ_β is a character of $N(F)\backslash N(\mathbb{A})$ which is trivial on N_α but non-trivial on N_β, then we want to show that $\Theta(\varphi,f)_{N,\psi_\beta} = 0$. Let us give this computation in detail, as a similar computation appears in the next section. Before that, we need to introduce some notations.

Without loss of generality, we may assume that the restriction of ψ_β to $N_2(\mathbb{A})$ is represented by $(1,0,0,0)\in V_2$ in the notation of Section 4. Let Ω_β be the set of all X in Ω such that the restriction of ψ_X to $N_2(\mathbb{A})$ is ψ_β. Then

$$\Theta(\varphi,f)_{N_2,\psi_\beta}(1) = \int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} f(h)\cdot [\sum_{X\in\Omega_\beta} \varphi_X(h)]\ dh$$

To describe Ω_β, we proceed as follows. By [MS, Lemma 7.5] (see also [GrS, Pg. 214]), if $X=(a,x,y,d)$ lies in Ω with $a=1$, then

$$X = (1, x, x^{\#}, det(x)),$$

for some x in $J(F)$. If further $X \in \Omega_\beta$, then $Tr(x) = Tr(x^\#) = det(x) = 0$ and thus the charactersitic polynomial of x is $x^3 = 0$. It follows that x is a nilpotent matrix. In particular, $PGL_3(F)$ has three orbits on Ω_β, parametrized by the rank of x. If $x = 0$, then the centralizer of x is $PGL_3(F)$, and if x has rank 1, then its centralizer contains the unipotent radical of a minimal parabolic subgroup. In particular, the cuspidality of f implies that the corresponding contributions in the above expression vanish. It remains to deal with the orbit parametrized by rank 2 nilpotent matrices. Fix a representative $X = (1, x_0, x_0^\#, 0)$ with

$$x_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

If U denotes the group of unipotent upper triangular matrices in PGL_3, then the stabilizer of ψ_X is a codimension one subgroup $U_0 \subset U$. It follows that

$$\Theta(\varphi, f)_{N_2,\psi_\beta}(1) = \int_{U(F)\backslash PGL_3(\mathbb{A})} f(h) \sum_{\gamma \in U_0(F)\backslash U(F)} \varphi_X(\gamma h)\ dh.$$

Next, consider a homomorphism $\mathbf{n} : U \to N_\alpha \cong \mathbb{G}_a$ defined by $\mathbf{n}(u) = a - b$ where

$$u = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 17. *Let $X = (1, x_0, x_0^\#, 0)$ as above. Under the action of $U \subset \mathbf{M}_2$ on $\mathbf{V}_2$, we have*

$$u \cdot X = \mathbf{n}(u) \cdot X$$

for every $u \in U$.

Proof. Recall the action of $Lie(\mathbf{N}_\alpha)$ on $\mathbf{V}_2$ given by the formula in Section 4. Exponentiating, we can obtain the corresponding action of $\mathbf{N}_\alpha$. In particular, if n is in $N_\alpha \cong \mathbb{G}_a$, then

$$n : (1, x_0, x_0^\#, 0) \mapsto (1, x_0 + nx_0^*, 0)$$

This clearly coincides with the action of U, via the map $\mathbf{n}$. The lemma is proved. □

Note that the kernel of $\mathbf{n}$ is precisely U_0. It follows that the expression for $\Theta(\varphi, f)_{N_2,\psi_\beta}(1)$ can be rewritten as

$$\Theta(\varphi, f)_{N_2,\psi_\beta}(1) = \int_{U(F)\backslash PGL_3(\mathbb{A})} f(h) \sum_{\delta \in N_\alpha(F)} \varphi_X(\delta h)\ dh,$$

where $\delta = \mathbf{n}(\gamma)$. Integrating over $N_\alpha(F)\backslash N_\alpha(\mathbb{A})$, it follows that

$$\Theta(\varphi, f)_{N,\psi_\beta}(1) = \int_{U(F)\backslash PGL_3(\mathbb{A})} f(h) \cdot [\int_{N_\alpha(\mathbb{A})} \varphi_X(nh)\ dn]\ dh.$$

The inner integral is now a left $U(\mathbb{A})$-invariant function of h. The vanishing of $\Theta(\varphi, f)_{N_2,\psi_\beta}$ now follows from the cuspidality of f. This proves Theorem 13(i).

7. Non-Vanishing

In this section, we prove Theorem 13(ii). For this, we take a generic character ψ_N of $N(\mathbb{A})$. Without loss of generality, we may assume that the restriction of ψ_N to N_2 is represented by the element $(1,0,0,0) \in V_2$ and its restriction to N_α is a non-trivial character ψ^{-1}. Now we need to show that $\Theta(\varphi, f)_{N,\psi_N}$ is non-zero for some f and φ.

This integral is similar to that in the previous section and the same computation gives:

$$\Theta(\varphi, f)_{N,\psi_N}(1) = \int_{U(F)\backslash PGL_3(\mathbb{A})} f(h)[\int_{N_\alpha(\mathbb{A})} \psi(n)\varphi_X(nh)dn]\ dh,$$

where $X = (1, x_0, x_0^\#, 0)$, for the fixed nilpotent rank 2 matrix x_0 as in the previous section. This time, however, the inner integral is not left $U(\mathbb{A})$-invariant, but it transforms according to the generic character ψ_U^{-1} with:

$$\psi_U(u) = \psi(\mathbf{n}(u)).$$

In particular, if f_{U,ψ_U} is the Whittaker coefficient of f relative to ψ_U, then

$$\Theta(\varphi, f)_{N,\psi_N}(1) = \int_{U(\mathbb{A})\backslash PGL_3(\mathbb{A})} f_{U,\psi_U}(h) \cdot [\int_{N_\alpha(\mathbb{A})} \psi(n)\varphi_X(nh)dn]\ dh.$$

We want to show that $\Theta(\varphi, f)_{N,\psi_N}(1) \neq 0$ for some choices of φ and f. More precisely, if $f_{U,\psi_U}(1) \cdot \varphi_X(1) \neq 0$ (which clearly can be assumed), we shall show that the Whittaker coefficient does not vanish if we replace φ by

$$\Phi * \varphi(g) = \int_{\mathbf{N}_{\alpha+\beta}(\mathbb{A})} \Phi(y)\varphi(gy)\ dy,$$

for some Schwarz function Φ on $\mathbf{N}_{\alpha+\beta}(\mathbb{A})$. Indeed, after commuting nh and y, we get

$$\varphi_X(nhy) = \psi(Tr(h^{-1}n(x_0)h \cdot y))\varphi_X(nh)$$

where $n(x_0) = x_0 + nx_0^\#$. Let $\hat{\Phi}$ denote the Fourier transform of Φ. It follows that

$$(\Phi * \varphi)_X(nh) = \hat{\Phi}(h^{-1}n(x_0)h) \cdot \varphi_X(nh)$$

and

$$\Theta(\Phi * \varphi, f)_{N,\psi_N}(1) = \int_{U(\mathbb{A})\backslash PGL_3(\mathbb{A})} f_{U,\psi_U}(h)[\int_{N_\alpha(\mathbb{A})} \psi(n)\hat{\Phi}(h^{-1}n(x_0)h))\varphi_X(nh)dn]\ dh.$$

For any finite place v, let A_v be the maximal order in F_v. Let S be the set of places, including the archimedean ones, such that for every place $v \notin S$, our functions f, φ and the character ψ are unramified. We shall choose Φ of the form

$$\Phi = \Phi_S \otimes (\otimes_{v \notin S} \Phi_v)$$

where Φ_v is the charactersitic function of $\mathbf{N}_{\alpha+\beta}(A_v)$ for all $v \notin S$ and Φ_S is to be chosen later.

Next, using the Iwasawa decomposition $PGL_3 = UTK$, we can write $h = utk$ where t is represented by a 3×3 matrix

$$t = \begin{pmatrix} a & & \\ & b & \\ & & 1 \end{pmatrix}.$$

A well-known fact is that the Whittaker coefficient $f_{U\psi_U}(utk)$ could be non-zero only if for all $v \notin S$

$$ord_v(a) \geq ord_v(b) \geq 0.$$

On the other hand, $t^{-1}n(x_0)t$ is equal to

$$\begin{pmatrix} 0 & b/a & n/a \\ 0 & 0 & 1/b \\ 0 & 0 & 0 \end{pmatrix}.$$

If $v \notin S$, then $\hat{\Phi}_v = \Phi_v$. Since Φ_v is supported on $\mathbf{N}_{\alpha+\beta}(A_v)$ we also have the inequalities

$$ord_v(a) \leq ord_v(b) \leq 0 \quad \text{and} \quad ord_v(n) \geq ord_v(a).$$

In particular, our integrand is zero unless t_v is in $T(A_v)$, and n_v in $N_\alpha(A_v)$, for all $v \notin S$. It follows that our integral reduces to integration over groups with coefficients in $\mathbb{A}_S$.

Let $\mathcal{O}$ be the set of rank 2 matrices in $\mathbf{N}_{\alpha+\beta} \cong J$. Let ω be an element in $\mathcal{O}(\mathbb{A}_S)$, and write $\omega = h^{-1}n(x_0)h$ for some h in $PGL_3(\mathbb{A}_S)$ and n in $N_\alpha(\mathbb{A}_S)$. Then Lemma 17 implies that

$$g(\omega) = f_{U,\psi_U}(h)\psi(n)\varphi_X(nh)$$

is a well-defined continuous function on $\mathcal{O}(\mathbb{A}_S)$. It follows that

$$\Theta(\Phi * \varphi, f)_{N,\psi_N}(1) = \int_{\mathcal{O}(\mathbb{A}_S)} g(\omega)\hat{\Phi}_S(\omega)\ d\omega.$$

By our assumption, $g(x_0) \neq 0$. Because $\mathcal{O}(\mathbb{A}_S)$ is a locally closed subset of $\mathbf{N}_{\alpha+\beta}(\mathbb{A}_S)$, we can find a Schwarz function Φ_S such that its Fourier transform is non-negative, and supported in a sufficiently small neigborhood of x_0 to assure the non-vanishing of the integral. The non-vanishing of the global theta lifting is now proved.

Remarks: In the proof of [GJ, Thm. 3.3], the above argument was used directly over $\mathbb{A}$ (rather than $\mathbb{A}_S$). Translating to our setting, it was claimed there that "(the restriction of) $\widehat{\Phi}$ can be an arbitrary Schwarz function on $\mathcal{O}(\mathbb{A})$". This claim would be true if $\mathcal{O}(\mathbb{A})$ is actually a closed subset of $\mathbf{N}_{\alpha+\beta}(\mathbb{A})$. It is not, and as a result, the restriction is never a Schwarz function. The situation here is analogous to restricting a Schwarz function on $\mathbb{A}$ to the subset of ideles. This same error has been perpetuated in many papers.

8. **Proof of the Main Theorem**

We can now give the proof of the main theorem. For (i), we have shown that $\Theta_{gen}(\pi)$ is non-zero and is contained in the space of cusp forms. Hence, it remains to show that $\Theta_{gen}(\pi) \cong \sigma(\pi)$. If $\sigma = \otimes_v \sigma_v$ is any irreducible summand of $\Theta_{gen}(\pi)$, then $\pi_v^\vee \otimes \sigma_v$ is a quotient of Π_v and thus $\sigma_v = \sigma(\pi_v)$ by the local results of [GS] and Theorem 2. Thus any irreducible summand of $\Theta_{gen}(\pi)$ is isomorphic to $\sigma(\pi)$. By the local uniqueness of Whittaker functionals, any irreducible summand of $\mathcal{A}_{cusp,gen}$ occurs with multiplicity one and thus we deduce that $\Theta_{gen}(\pi)$ is irreducible and isomorphic to $\sigma(\pi)$. In particular, (i) is proved.

To prove (ii), we shall use the following result of Ginzburg-Jiang [GJ].

Theorem 18. *Let $\sigma = \otimes_v \sigma_v$ be an irreducible submodule of $\mathcal{A}_{cusp,gen}$. Suppose that the partial (degree 7) standard L-function of σ has a pole at $s = 1$. Then there exists a cuspidal representation π of PGL_3 such that $\sigma \subset \Theta_{gen}(\pi)$.*

Now suppose that σ' is nearly equivalent to $\sigma(\pi)$. Then the partial standard L-function of σ' has a pole at $s = 1$ and by the theorem of Ginzburg-Jiang, one concludes that $\sigma' \subset \Theta_{gen}(\pi')$ for some cuspidal representation π' of PGL_3. It follows that $\sigma(\pi_v)$ is isomorphic to $\sigma(\pi'_v)$ for almost all v, and we deduce by [GS] that outside a finite set of places,

$$\pi'_v \cong \pi_v \text{ or } \pi_v^\vee .$$

Now we note:

Lemma 19. *If π and π' are two cuspidal representation of $GL(n)$ such that $\pi'_v \cong \pi_v$ or $\pi_v^\vee$ for almost all v, then $\pi' \cong \pi$ or $\pi^\vee$.*

Proof. Indeed, consider the induced representations $\pi \boxplus \pi^\vee$ and $\pi' \boxplus \pi'^\vee$ of $GL(2n)$. Any constituents of these are automorphic and nearly equivalent to each other. Hence, the lemma follows by the generalized strong multiplicity one theorem of Jacquet-Shalika [JS]. □

With the lemma, we now know that $\pi' \cong \pi$ or $\pi^\vee$, so that $\sigma' = \Theta_{gen}(\pi') = \sigma(\pi')$. Since $\sigma(\pi) \cong \sigma(\pi^\vee)$, we see that $\sigma(\pi) = \sigma(\pi') = \sigma'$, as desired. The main theorem is proved.

9. Generic unitarizable representations

In this section, we list the generic unitarizable representations of $PGL_3(\mathbb{R})$ and $G_2(\mathbb{R})$ which were quoted in Theorem 2.

9.1. Generic unitary representations of $PGL_3(\mathbb{R})$.

It will be convenient to think of representations of $PGL_3(\mathbb{R})$ as representations of $GL_3(\mathbb{R})$ with trivial central character. In particular, every representation π is a submodule of principal series $Ind_B(\chi_1, \chi_2, \chi_3)$ for some three characters of $\mathbb{R}^\times$ such that $\chi_1 \cdot \chi_2 \cdot \chi_3 = 1$. Since π is generic, it is completely determined by the character $\chi = \chi_1 \times \chi_2 \times \chi_3$.

Note that every character χ_i can be written as $\chi_i(x) = \epsilon(x)^{\delta_i}|x|^{t_i}$ where ϵ is the sign character, δ_i is an integer considered modulo 2, and t_i is a complex number. As usual, write the maximal torus as a product $T(\mathbb{R}) = MA$, where $M \cong \langle \pm 1 \rangle \times \langle \pm 1 \rangle$ is the maximal compact subgroup of $T(\mathbb{R})$. The restriction of χ to A determines the infinitesimal character of π, which can be identfied with

$$\lambda = (t_1, t_2, t_3) \in \Lambda \otimes \mathbb{C},$$

where Λ is a root lattice of $SL_3(\mathbb{C})$. On the other hand, let ξ be the restriction of χ to M, the maximal compact subgroup of $T(\mathbb{R})$. Then ξ is given by a triple of integers

$$\xi = (\delta_1, \delta_2, \delta_3),$$

such that $\delta_1 + \delta_2 + \delta_3 \equiv 0 \pmod 2$. Note that the characters of M correspond to elements of order 2 in the torus $\hat{T}(\mathbb{C}) = \Lambda \otimes \mathbb{C}^\times$ of the dual group $SL_3(\mathbb{C})$.

The following is a list of all generic unitary representations. (Parts (2) and (3) list representations up to contragredient.)

(1) *Tempered principal series* $\pi(\lambda, \xi)$. The infinitesimal character $\lambda = (it_1, it_2, it_3)$ is purely imaginary. Assume that $t_1 \geq t_2 \geq t_3$. Then $\pi(\lambda, \xi)$ is the fully induced representation

$$Ind_B(\epsilon^{\delta_1}|\cdot|^{it_1}, \epsilon^{\delta_2}|\cdot|^{it_2}, \epsilon^{\delta_3}|\cdot|^{it_3}).$$

Note that $\pi(\lambda, \xi)$ is spherical if and only if ξ is trivial. Furthermore, $\pi(\lambda, \xi) \cong \pi(\lambda, \xi^w)$ for every w in $W(\lambda)$ (the stabilizer of λ in the Weyl group). In particular, there are up to 4 generic representations with the same infinitesimal character λ.

(2) *Generalized principal series* $\pi(k, t)$. Here $k = 1, 2, 3, ...$ is a positive integer integer, and t in $\mathbb{R}$. The infinitesimal character is

$$\lambda = (k/2 + it, -k/2 + it, -2it).$$

The representation $\pi(k, t)$ is a submodule of

$$Ind_B(\epsilon^{k+1}|\cdot|^{k/2+it}, |\cdot|^{-k/2+it}, \epsilon^{k+1}|\cdot|^{-2it}).$$

As a generalized principal series, $\pi(k, t)$ is obtained by inducing $\pi(k) \otimes |det|^{it}$ from a maximal parabolic subgroup, where $\pi(k)$ a discrete series of $GL(2)$ with minimal K-type $k+1$. For a given pair (k, t), $\pi(k, t)$ is the unique generic unitary representation with infinitesimal character λ.

(3) *Complementary series* $\pi_\delta(s, t)$. Here s is in $(0, 1/2)$, t is in $\mathbb{R}$, and $\delta = 0$ or 1. The infinitesimal character is

$$\lambda = (s + it, -s + it, -2it).$$

The representation $\pi_\delta(s, t)$ is the fully induced representation

$$Ind_B(\epsilon^{\delta}|\cdot|^{s+it}, \epsilon^{\delta}|\cdot|^{-s+it}, |\cdot|^{-2it}).$$

This representation is spherical if and only if $\delta = 0$. For a given pair (s, t), there are two generic unitary representations with infinitesimal character λ.

9.2. **Generic unitary representations of $G_2(\mathbb{R})$.** We restrict ourselves only to representations with the infinitesimal character $\lambda' = \gamma(\lambda)$ where λ is infinitesimal character of a unitary generic representation of $PGL_3(\mathbb{R})$.

(1) *Tempered principal series* $\sigma(\lambda, \xi)$. Here ξ can be identified with an element of order two in $\hat{T}(\mathbb{C})$. Again, $\sigma(\lambda, \xi)$ is spherical if and only if ξ is trivial, and there are up to 4 generic representations with the same infinitesimal character λ.

(2) *Generalized principal series* $\sigma(k, t)$. Here $k = 1, 2, 3, ...$ is a positive integer, and t in $\mathbb{R}$. The infinitesimal character is

$$\lambda' = (k/2 + it, -k/2 + it, -2it).$$

The representation $\sigma(k,t)$ is given in parts b), c) and d) of Theorem 10.9 in [V]. It is a unitary generalized principal series, obtained by inducing $\pi(k) \otimes |det|^{it}$ from a short-root maximal parabolic subgroup (i.e. the Heisenberg parabolic). Here $\pi(k)$ a discrete series of $GL(2)$ with minimal K-type $k+1$. The induced representation is irreducible unless $t=0$ and k is even, in which case $\sigma(k,0)$ is taken to be the unique generic summand. For a given pair (k,t), $\sigma(k,t)$ is the unique generic unitary representation with infinitesimal character λ.

(3) *Complementary series* $\sigma_\delta(s,t)$. Here s is in $(0,1/2)$, t is in $\mathbb{R}$, and δ is 0 or 1. The infinitesimal character is

$$\lambda' = (s+it, -s+it, -2it).$$

Then $\sigma_0(s,t)$ is the spherical representation in part b) of Theorem 12.1 in [V] (the short root case), and $\sigma_1(s,t)$ is the non-spherical representation in Theorem 13.2 part b) in [V]. If $t \neq 0$, these are the unique generic unitary representations with infinitesimal character λ. If $t=0$, there is one additional non-spherical unitary representation with the same infinitesimal character; see part d) of Theorem 13.2 in [V].

10. Acknowledgments

The authors are partially supported by NSF grants DMS-0202989 and DMS-0138604 respectively.

References

[Bo] N. Bourbaki, *Groupes et algebres des Lie.*

[C] M. Cerkaski, *Branching rules for* sp$(2N)$ *algebra reduction on the chain* sp$(2N-2) \times$ sp(2), J. Math. Phys. **28** (1987), no. 5, 989–990.

[FJ] S. Friedberg and H. Jacquet, *Linear periods*, Journal Reine Angew. Math. **443** (1993), 91-139.

[GJ] D. Ginzburg and D. H. Jiang, *Periods and Liftings: from G_2 to C_3*, Israel Math. Journal **123** (2001), 29-59.

[GRS] D. Ginzburg, S. Rallis and D. Soudry, *On the automorphic theta representations for simply-laced groups*, Israel Math. Journal **100** (1997), 61-116.

[GRS2] D. Ginzburg, S. Rallis and D. Soudry, *A tower of theta correspondences for G_2*, Duke Math. Journal **88** (1997), 537-624.

[GS] W. T. Gan and G. Savin, *Endoscopic lifts from PGL_3 to G_2*, to appear in Compositio Math.

[GS2] W. T. Gan and G. Savin, *Minimal representations: definitions and properties*, preprint (2002).

[GrS] B. Gross and G. Savin, *Motives with galois group G_2: an exceptional theta correspondence*, Compositio Math. **114** (1998), 153-217.

[HPS] J.-S. Huang, P. Pandžić and G. Savin, *New dual pair correspondences*, Duke Math. Journal **82** (1996), 447-471.

[JS] H. Jacquet and J. Shalika, *Euler products and classification of automorphic forms I & II*, American Journal of Math. **103** (1981), 499-558 & 777-855.

[J] A. Joseph, *The minimal orbit in a simple Lie algebra and its associated maximal ideal*, Ann. Sci. Ecole Norm Sup. **9** (1976), No. 1, 1-29.

[Li] J. S. Li, *The correspondences of infinitesimal characters for reductive dual pairs in simple Lie groups*, Duke Math. Journal **97** (1999), 347-377.

[MS] K. Magaard and G. Savin, *Exceptional Θ-correspondences*, Compositio Math. **107** (1997), 89-123.

[S] G. Savin, *A class of supercuspidal representations of $G_2(k)$*, Canadian Math. Bulletin **42** (1999), 393-400.

[V] D. Vogan, *The unitary dual of G_2*, Invent. Math. **116** (1994), 677-791.

[Z] D. P. Zhelobenko, *An analogue of Gelfand-Zetlin basis for symplectic Lie algebras*, Uspekhi Mat. Nauk **42** (1987), no. 6 (258), 193-194.

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