THE SHIMURA CORRESPONDENCE À LA WALDSPURGER

WEE TECK GAN

1. Introduction

The purpose of this short instructional course is to describe the Shimura correspondence between half integral weight modular forms and integral weight modular forms, in the framework of automorphic representations. This representation theoretic treatment is largely due to Waldspurger in a series of papers [Wa1, Wa4]. If time permits, we will also cover certain applications to special values of L-functions and period integrals, which were treated by Waldspurger in [Wa2, Wa3].

As in any adelic treatment of such subjects, one first needs to understand the local picture, which in our case concerns the representation theory of the metaplectic group $\mathrm{Mp}_2(k)$ over a local field k of characteristic zero. This local study is discussed in Atsushi Ichino's lectures, but we will give a summary of the results in §2, and establish certain more refined properties of the local story which may not be covered in his lectures.

Our treatment of the Shimura correspondence does not follow Waldspurger's original approach completely. Rather, it is informed by 30 years of hindsight. Waldspurger's papers stimulated much research in the theory of theta correspondence and these subsequent developments have in turn led to the streamlining of his original approach. We have placed emphasis on subsequent developments which have turned out to be fundamental in the subject and which allow extension of Waldspurger's results to higher rank metaplectic groups. In particular,

- the local theta correspondence, which is used to establish the local Shimura correspondence, is studied from a more abstract and systematic viewpoint;
- we have made systematic use of the doubling see-saw identity and the standard L-functions defined using the doubling method, as developed by Piatetski-Shapiro and Rallis, which leads to the Rallis inner product formula via the Siegel-Weil formula;
- we have exploited an analytic theorem of Friedberg-Hoffstein on the nonvanishing of the central L-values of quadratic twist of cuspidal representations of GL_2; in Waldspurger's original treatment, this theorem was not available and Waldspurger bypassed this difficulty by exploiting results of Flicker on the Shimura lifting from GL_2 to $\widetilde{\mathrm{GL}}_2$ which was proved using the trace formula.
- in the study of torus periods, we have given a new proof of the local theorem of Tunnell-Saito, which is inspired by Wadlspurger's treatment of the global situation.

Finally, I thank Youngju Choie and Sug Woo Shin for the invitation to participate in this Theta Festival and the opportunity to lecture on this material, and the Pohang Mathematics Institute for providing the financial support for my visit.

2. **The Local Shimura Correspondence**

Let k be a local field of characteristic zero, and let $(W, \langle -,-\rangle)$ be a 2-dimensional symplectic vector space over k. The associated symplectic group $\mathrm{Sp}(W)$ is isomorphic to SL_2, and we may fix such an isomorphism by choosing a Witt basis $\{e, f\}$ of W, so that

$$\langle e, f\rangle = 1 \quad \text{and} \quad \langle e, e\rangle = \langle f, f\rangle = 0.$$

We have the Borel subgroup $B = T \cdot N$ of upper triangular matrices in $\mathrm{SL}_2(k)$, which is the stabilzier of the isotropic line $k \cdot e$.

2.1. **The metaplectic group.** When $k \neq \mathbb{C}$, the group $\mathrm{Sp}(W)$ has a unique two-fold central extension $\mathrm{Mp}(W)$, so that

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1.$$

As a set, we may write

$$\mathrm{Mp}(W) = \mathrm{Sp}(W) \times \{\pm 1\}$$

with group law given by

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \cdot c(g_1, g_2))$$

for some 2-cocycle c on $\mathrm{Sp}(W)$ valued in $\{\pm 1\}$. We record some values of this two-cocycle. For simplicity, let us write:

$$t(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for $a \in k^\times$ and $x \in k$. Then we have

$$c(t(a), t(b)) = (a, b)_k \quad \text{and} \quad c(n(x), n(y)) = 1.$$

For a subset $H \subset \mathrm{Sp}(W)$, we will let $\tilde{H}$ denote its preimage in $\mathrm{Mp}(W)$, which may sometimes be a split covering. For example,

- the metaplectic covering splits over N, so that $\tilde{N} = N \times \mu_2$,
- the metaplectic covering does not split over $\tilde{T}$, but note that $\tilde{T}$ is still abelian.
- If $Z \cong \mu_2$ is the center of $\mathrm{Sp}(W)$, $\tilde{Z}$ is a group of size 4 which is split over Z if and only if $(-1, -1)_k = 1$.
- when $p \neq 2$, the metaplectic cover splits over the maximal compact subgroup $\mathrm{SL}_2(\mathcal{O}_k)$.

Let $\mathrm{Irr}(\mathrm{Mp}(W))$ denote the set of genuine irreducible representations of $\mathrm{Mp}(W)$. Here, "genuine" means that the representations do not factor to $\mathrm{Sp}(W)$.

Goal of the local Shimura correspondence: Parametrize the set $\mathrm{Irr}(\mathrm{Mp}(W))$.

One may ask: what is the parametrizing set? It turns out to be the set of irreducible representations of $\mathrm{PGL}_2(k)$ and its variant.

2.2. **Quadratic spaces of rank 3.** Let V denote a rank 3 quadratic space of discriminant 1. Then there are two such V's; we denote V^+ to be the split quadratic space and V^- the non-split one. Let $\mathrm{O}(V)$ denote the associated orthogonal group, which is disconnected with identity component $\mathrm{SO}(V)$. These quadratic spaces and groups can be explicitly described as follows.

Let B denote a central simple k-algebra of dimension 4, so that B is either the split algebra of 2×2 matrices over k or the quaternion divison algebra D over k. Let B_0 denote the subspace of trace zero elements in B and lert N_B denote the norm form on B, which is a quadratic form. Then the 3-dimensional quadratic space $V_B = (B_0, -N_B)$ has discriminant 1 and V_B is equal to V^+ or V^- according to whether B is spllt or not. The group $B^\times$ acts on $(B_0, -N_B)$ by its conjugation action on B_0, thus giving rise to an isomorphism

$$\mathrm{SO}(V_B) \cong PB^\times.$$

An element in $\mathrm{O}(V_B) \smallsetminus \mathrm{SO}(V_B)$ is given by the conjugation action, which simply acts on B_0 by $x \mapsto -x$. Thus,

$$\mathrm{O}(V_B) = \mathrm{SO}(V_B) \times \langle \pm 1 \rangle.$$

2.3. **Local Shimura correspondence.** Fix a nontrivial additive character ψ of k. The local Shimura correspondence is a bijection

$$Sh_\psi : \mathrm{Irr}(\mathrm{Mp}(W)) \longleftrightarrow \mathrm{Irr}(\mathrm{SO}(V^+)) \sqcup \mathrm{Irr}(\mathrm{SO}(V^-)).$$

Note that this bijection depends on the choice of ψ. In the rest of this section, we shall discuss various aspects of this bijection, such as its definition, making it explicit and clarifying some of its properties such as how it varies with the choice of ψ. We shall first describe certain basic constructs in the representation theory of $\mathrm{Mp}(W)$ and $\mathrm{SO}(V)$, before introducing certain explicit elements of the sets $\mathrm{Irr}(\mathrm{Mp}(W))$ and $\mathrm{Irr}(\mathrm{SO}(V^\pm))$.

2.4. **A genuine character of $\tilde{T}$.** Fix a nontrivial additive character ψ of k. Then associated to ψ is a genuine character χ_ψ of the abelian group $\tilde{T}$ defined by:

$$(t(a), \epsilon) \mapsto \epsilon \cdot \gamma(a, \psi)^{-1}$$

with

$$\gamma(a, \psi) = \gamma(\psi_a)/\gamma(\psi)$$

and the Weil index $\gamma(\psi)$ is an 8-th root of unity associated to ψ by Weil. Using this character χ_ψ, one obtains an identification of the representations of T with those of $\tilde{T}$ by

$$\mu \mapsto \mu \cdot \chi_\psi$$

for μ a character of $T \cong k^\times$. Note however that this identification depends on ψ. If $\psi_a(x) = \psi(ax)$ for $a \in k^\times$, then

$$\chi_{\psi_a} = \chi_\psi \cdot \chi_a$$

where χ_a is the quadratic character associated to a.

2.5. **Central sign.** Since $\tilde{Z} \subset \tilde{T}$, we may restrict χ_ψ to $\tilde{Z}$. If σ is an irreducible genuine representation of $\mathrm{Mp}(W)$, then its central character ω_σ is either equal to χ_ψ or $\chi_\psi \cdot \mathrm{sgn}$ where sgn is the non-trivial character of $\mathbb{Z}$. We define the central sign of σ to be $+1$ or -1 depending on whether $\omega_\sigma = \chi_\psi$ or not.

2.6. **Whittaker functionals.** Since $N(k)$ has been identified with k, we may regard ψ as a character of $N(k)$. A ψ-Whittaker functional on a representation σ of $\mathrm{Mp}(W)$ is a linear form

$$\ell : \sigma \longrightarrow \mathbb{C} \quad \text{such that} \quad \ell(n(x)v) = \psi(x) \cdot \ell(v).$$

Thus, σ has nonzero ψ-Whittaker functional if and only if $\sigma_{N,\psi} \neq 0$. It is clear that any genuine representation σ possesses a ψ_a-Whittaker funcitonal for some $a \in k^\times$. Moreover, one has the uniqeness of Whittaker functionals:

$$\dim \mathrm{Hom}\sigma_{N,\psi_a} \leq 1$$

if σ is irreducible.

2.7. **Elementary Weil representations.** We come now to the genuine representation theory of $\mathrm{Mp}(W)$ and first introduce the elementary Weil representations of $\mathrm{Mp}(W)$, which is a family of degenerate representations. For each nontrivial additive character ψ of k, one has a representation ω_ψ of $\mathrm{Mp}(W)$ which is realized on the space $S(k)$ of Schwarz-Bruhat functions on k and given by the following formula:

$$\begin{cases} \omega_\psi(t(a))\phi(y) = |a|^{1/2}\chi_\psi(a) \cdot \phi(ay) \\ \omega_\psi(n(x))\phi(y) = \psi(xy^2) \cdot \phi(y) \\ \omega_\psi(w)\phi(y) = \gamma(\psi) \cdot \int_k \psi(-2zy) \cdot \phi(z)\, dz \end{cases}$$

where dz is the measure on k which is self-dual with respect to the Fourier transform using the additive character $\psi_2(x) = \psi(2x)$.

We note:

- the representation ω_ψ is reducible, and decomposes as

$$\omega_\psi = \omega_\psi^+ \oplus \omega_\psi^-,$$

  where ω_ψ^+ is realized on the subspace of even functions (i.e. $f(-x) = f(x)$), and ω_ψ^- is realized on the space of odd functions (i.e. $f(-x) = -f(x)$).
- We have:

$$\omega_{\psi_a} \cong \omega_{\psi_b} \Longleftrightarrow a/b \in k^{\times 2}.$$

  Thus, for each $a \in k^\times/k^{\times 2}$, or equivalently, for every quadratic character χ of $k^\times$, one has a representation

$$\omega_{\psi_a} = \omega_{\psi,\chi} = \omega_{\psi,\chi}^+ \oplus \omega_{\psi,\chi}^-.$$

- The representations $\omega_\psi^\pm$ has nonzero Whittaker functionals with respect to ψ_a for $a \in k^{\times 2}$, but not with respect to any other ψ_b. Indeed, they are characterized by this property.

2.8. **Clarification.** Let us clarify a possible misconception about the representation ω_ψ. As is well-known, the representation ω_ψ of $\mathrm{Mp}(W)$ is obtained by a standard construction from an irreducible representation of the Heisenberg group $H(W) = W \times k$ associated to W. However, this irreducible representation of $H(W)$ is not the one with central character ψ; rather, it is the one with central character ψ_2.

In fact, it is better to regard the representation ω_ψ as a representation of $\mathrm{Mp}(W) \times \mathrm{O}(L_1)$ where L_1 is the rank 1 quadratic space with quadratic form $q_1 : x \mapsto x^2$ so that

$$\mathrm{O}(L_1) \cong \langle \pm 1 \rangle.$$

The representation ω_ψ is then obtained from the usual construction via the irreducible representation with central character ψ of the Heisenberg group $H(W \otimes V_1)$ associated to the symplectic space $W \otimes V_1$, where V_1 is equipped with the symmetric bilinear form

$$b_1(x, y) = q_1(x + y) - q_1(x) - q_1(y) = 2xy.$$

The action of the nontrivial element s of $\mathrm{O}(L_1)$ on $S(k)$ is given by:

$$\omega_\psi(s)\phi(x) = \phi(-x).$$

Thus the decomposition of ω_ψ above is that of it as an $\mathrm{O}(L_1) \times \mathrm{Mp}(W)$-module.

2.9. **Principal series representation of $\mathrm{Mp}(W)$.** We can now define principal series representations of $\mathrm{Mp}(W)$. Given a character μ of T, we set

$$\pi_\psi(\mu) = \mathrm{Ind}_{\tilde{B}}^{\mathrm{Mp}(W)} \mu \cdot \chi_\psi,$$

where we have used normalized induction here. The following proposition summarizes the reducibility properties of these principal series representations.

Proposition 2.1. *(i) $\pi_\psi(\mu)$ is irreducible if and only if $\mu^2 \neq |-|^{\pm 1}$, in which case $\pi_\psi(\mu) \cong \pi_\psi(\mu^{-1})$.*

(ii) If $\mu = \chi \cdot |-|^{1/2}$ where χ is a quadratic character, then we have a short exact sequence:

$$0 \longrightarrow st_\psi(\chi) \longrightarrow \pi_\psi(\mu) \longrightarrow \omega^+_{\psi,\chi} \longrightarrow 0.$$

We call $st_\psi(\chi)$ the Steinberg representation associated to (ψ, χ). When $\chi = 1$, we shall simply write st_ψ.

(iii) If $\mu = \chi \cdot |-|^{-1/2}$, then we have a short exact sequence,

$$0 \longrightarrow \omega^+_{\psi,\chi} \longrightarrow \pi_\psi(\mu) \longrightarrow sp_\psi(\chi) \longrightarrow 0.$$

The proposition gives all the non-supercuspidal genuine representations of $\mathrm{Mp}(W)$. The other irreducible representations of $\mathrm{Mp}(W)$ are all supercuspidal. For example, the odd Weil representations $\omega^-_{\psi,\chi}$ are supercuspidal.

2.10. **Representations of** $\mathrm{SO}(V)$**.** Recall that every quaternion algebra gives a rank 3 quadratic space $V_B = (B_0, -N_B)$ of discriminant 1, with associated orthogonal group $\mathrm{O}(V_B) = \mathrm{SO}(V_B) \times \langle \pm 1 \rangle$. Thus, any irreducible representation π of $\mathrm{SO}(V_B)$ has two possible extension to $\mathrm{O}(V_B)$, denoted by π^+ or π^- according to whether the action of $-1 \in \mathrm{O}(V_B)$ is trivial or not.

Since $\mathrm{SO}(V^-) = \mathrm{PD}^\times$ is compact, all its irreducible representations are finite-dimensional. One has the Jacquet-Langlands correspondence, which gives an injection:

$$JL : \mathrm{Irr}(\mathrm{SO}(V^-)) \hookrightarrow \mathrm{Irr}(\mathrm{SO}(V^+)).$$

The image of JL is the subset of discrete series representations.

On the other hand, for $\mathrm{SO}(V^+) = \mathrm{PGL}_2(k)$, one has the principal series representations

$$\pi(\mu, \mu^{-1}) = \mathrm{Ind}_B^{\mathrm{PGL}_2} \mu \times \mu^{-1}$$

where B is the Borel subgroup of upper triangular matrices in PGL_2 and $\mu \times \mu^{-1}$ is a character of the diagonal torus. The reducibility points of the family of principal series representations are well-known:

Proposition 2.2. *(i) The representation* $\pi(\mu, \mu^{-1})$ *is irreducible if and only if* $\mu^2 \neq |-|^{\pm 1}$*, in which case* $\pi(\mu, \mu^{-1}) \cong \pi(\mu^{-1}, \mu)$*.*

(ii) If $\mu = \chi \cdot |-|^{-1/2}$*, where* χ *is a quadratic character, then we have a short exact sequence*

$$0 \longrightarrow st_\chi \longrightarrow \pi(\mu, \mu^{-1}) \longrightarrow \chi \circ \det \longrightarrow 0$$

where st_χ *is the twisted Steinberg representation. If* $\chi = 1$*, we shall simply write* st *(the Steinberg representation).*

(iii) If $\mu = \chi \cdot |-|^{-1/2}$*, then we have a short exact sequence*

$$0 \longrightarrow \chi \circ \det \longrightarrow \pi(\mu, \mu^{-1}) \longrightarrow st_\chi \longrightarrow 0$$

The proposition gives all the non-supercuspidal representations of $\mathrm{SO}(V^+)$; all other irreducible representations of $\mathrm{SO}(V^+)$ are supercuspidal.

2.11. **Naive Bijection.** By comparing Proposition 2.2 with Proposition 2.1, we see at once that there is a natural bijection between the non-supercuspidal representations of $\mathrm{Mp}(W)$ and $\mathrm{SO}(V^+)$, given by:

$$\begin{cases} \pi_\psi(\mu) \leftrightarrow \pi(\mu, \mu^{-1}) \\ st_{\psi,\chi} \leftrightarrow st_\chi \\ \omega_\psi^+ \leftrightarrow \chi \circ \det \end{cases}$$

Given this naive bijection, it is reasonable to ask if the supercuspidal representations of $\mathrm{Mp}(W)$ and $\mathrm{SO}(V^\pm)$ are also related, thus giving the bijection Sh_ψ. The mechanism for doing this is the theory of theta correspondence.

2.12. **Standard local factors.** To an irreducible representation $\pi \boxtimes \mu$ of $\mathrm{SO}(V) \times \mathrm{GL}_1$ and the additive character ψ, one can attach a family of rational functions

$$\gamma(s, \pi \times \mu, \psi), \quad L(s, \pi \times \mu) \quad \epsilon(s, \pi \times \mu, \psi).$$

This is done using the doubling zeta integral, and the definitive reference is the paper of Lapid-Rallis. These standard local factors are preserved by the local Langlands correspondence and Jacquet-Langlands correspondence for PGL_2. Moreover, they satisfy a number of expected properties; we shall only mention one here:

$$\gamma(s, \pi \times \mu, \psi_a) = \mu(a)^2 \cdot |a|^{2s-1} \cdot \gamma(s, \pi \times \mu, \psi).$$

Similarly, to an irreducible representation $\sigma \boxtimes \mu$ of $\mathrm{Mp}(W) \times \mathrm{GL}_1$, one can associate

$$\gamma(s, \sigma \times \mu, \psi), \quad L(s, \sigma \times \mu, \psi) \quad \epsilon(s, \sigma \times \mu, \psi)$$

using the doubling zeta integral. The main difference here is that the L-function depends on ψ as well. Moreover,

$$\gamma(s, \pi \times \mu, \psi_a) = \mu(a)^2 \cdot |a|^{2s-1} \cdot \gamma(s, \pi \times \mu\chi_a, \psi).$$

2.13. **Weil representation of** $\mathrm{Mp}(W) \times \mathrm{O}(V)$**.** Now we come to the Weil representation $\omega_{\psi,W,V}$ of $\mathrm{Mp}(W) \times \mathrm{O}(V)$ associated to a nontrivial additive character ψ of k. This representation is realized on the space of $S(f \otimes V) = S(V)$ of Schwarz-Bruhat functions on V, with action given by:

$$\begin{cases} \omega_{\psi,W,V}(h)\phi(v) = \phi(h^{-1}v), \text{ for } h \in \mathrm{O}(V); \\ \omega_\psi(t(a))\phi(v) = |a|^{3/2}\chi_\psi(a) \cdot \phi(av) \\ \omega_\psi(n(x))\phi(y) = \psi(xq(v)) \cdot \phi(v) \\ \omega_\psi(w)\phi(y) = \gamma(\psi \circ q) \cdot \int_V \psi(-b(v,z)) \cdot \phi(z)\, dz \end{cases}$$

where dz is the measure on k which is self-dual with respect to the Fourier transform using the additive character $\psi(b(-,-))$ on $V \times V$.

2.14. **Local Theta correspondence.** We can now consider the local theta correspondence given by the representation $\omega_{\pi,W,V}$. Namely, for an irreducible representation π^ϵ of $\mathrm{O}(V)$, the maximal π^ϵ-isotypic component of $\omega_{\psi,W,V}$ has the form

$$\pi \boxtimes \Theta_{\psi,W,V}(\pi)$$

for some smooth representation $\Theta_{\psi,W,V}(\pi)$, which we will call the big theta lift of π. Similarly, for an irreducible genuine representation σ of $\mathrm{Mp}(W)$, the maximal σ-isotypic quotient of $\omega_{\psi,W,V}$ has the form

$$\sigma \boxtimes \Theta_{\psi,W,V}(\sigma)$$

for some smooth representation $\Theta_{\psi,W,V}(\sigma)$ of $\mathrm{O}(V)$, which we will call the big theta lift of σ.

2.15. **The Main Local Theorem.** The following summarizes the key results about the local theta correspondence, and will likely be shown in Ichino's lectures:

Theorem 2.1. *(i) Given an irreducible representation* π *of* $\mathrm{SO}(V)$*, exactly one extension* π^{ϵ} *of* π *to* $\mathrm{O}(V) = \mathrm{SO}(V) \times \{\pm 1\}$ *satisfies* $\Theta_{\psi,W,V}(\pi^{\epsilon}) \neq 0$*. For this extension,* $\Theta_{\psi,W,V}(\pi^{\epsilon})$ *has a unique irreducible quotient* $\theta_{\psi,W,V}(\pi)$*; indeed,* $\Theta_{\psi,W,V}(\pi^{\epsilon})$ *is irreducible unless* π *is 1-dimensional. Moreover, this unique extension* π^{ϵ} *is given by*

$$\epsilon = \epsilon(V) \cdot \epsilon(1/2, \pi, \psi),$$

where $\epsilon(s,\pi,\psi)$ *is the local standard epsilon factor of* π*, defined for example by the doubling method [LR].*

(ii) Given an irreducible representation σ *of* $\mathrm{Mp}(W)$*,* $\Theta_{\psi,W,V}(\sigma) \neq 0$ *for exactly one* V*. For this* V*,* $\Theta_{\psi,W,V}(\sigma)$ *has a unqiue irreducible quotient* $\theta_{\psi,W,V}(\sigma)$*; indeed* $\Theta_{\psi,W,V}(\sigma)$ *is irreducible unless* $\sigma = \omega^{+}_{\psi,\chi}$*. Moreover, this distinguished* V *is given by*

$$\epsilon(V) = z_{\psi}(\sigma) \cdot \epsilon(1/2, \sigma, \psi),$$

where $\epsilon(1/2,\sigma,\psi)$ *is the local standard* ϵ*-factor defined by the doubling method.*

(iii) The map $\sigma \mapsto \theta_{\psi,W,V}(\sigma)$ *gives a bijection*

$$Sh_{\psi} : \mathrm{Irr}(\mathrm{Mp}(W)) \longleftrightarrow \mathrm{Irr}(\mathrm{SO}(V^{+})) \sqcup \mathrm{Irr}(\mathrm{SO}(V^{-})).$$

We will call this bijection the local Shimura correspondence *with respect to* ψ*.*

(iv) The bijection respects the local γ*-factors,* L*-factors and* ϵ*-factors defined by the doubling method, i.e. for each character* μ *of* $k^{\times}$*,*

$$\begin{cases} \gamma(s, \sigma \times \mu, \psi) = \gamma(s, Sh_{\psi}(\sigma) \times \mu, \psi) \\ L(s, \sigma \times \mu, \psi) = L(s, Sh_{\psi}(\sigma) \times \mu) \\ \epsilon(s, \sigma \times \mu, \psi) = \epsilon(s, Sh_{\psi}(\sigma) \times \mu, \psi). \end{cases}$$

2.16. **Local Waldspurger packets.** By combining with the Jacquet-Langlands transfer, the local Shimura correspondence gives a surjective map

$$Wd_{\psi} : \mathrm{Irr}(\mathrm{Mp}(W)) \longrightarrow \mathrm{Irr}(\mathrm{SO}(V^{+})) = \mathrm{Irr}(\mathrm{PGL}_2(k))$$

whose fibers are of size 1 or 2. For an irreducible representation π of $\mathrm{PGL}_2(k)$, the fiber $A_{\psi}(\pi)$ of Wd_{ψ} over π has size 2 precisely when π is a discrete series representation. In any case, we write

$$A_{\psi}(\pi) = \{\sigma^{+} = \theta_{\psi,W,V^{+}}(\pi),\ \sigma^{-} = \theta_{\psi,W,V^{-}}(\pi_D)\}$$

where π_D is the (possibly zero) Jacquet-Langlands lift of π to $\mathrm{PD}^{\times}$. The set $A_{\psi}(\pi)$ is called a Waldspurger packet of $\mathrm{Mp}(W)$, so that we have a partition

$$\mathrm{Irr}(\mathrm{Mp}(W)) = \bigsqcup_{\pi \in \mathrm{Irr}(\mathrm{PGL}_2)} A_{\psi}(\pi).$$

2.17. **Explicit local Shimura correspondence.** We can now try to describe the set $A_\psi(\pi)$ as explicitly as possible; this amounts to an explciit description of the local theta correspondence. The results are summarized in the following table.

$\pi \in \mathrm{Irr}(\mathrm{PGL}_2)$	$\pi(\mu, \mu^{-1})$	st_χ, $\chi \neq 1$	st	$\chi \circ \det$	supercuspidal
$\sigma = \theta_{\psi,W,V^+}(\pi) \in \mathrm{Irr}(\mathrm{Mp}(W))$	$\pi_\psi(\mu)$	$st_{\psi,\chi}$	ω_ψ^-	$\omega_{\psi,\chi}^+$	supercuspidal

$\pi \in \mathrm{Irr}(\mathrm{PD}^\times)$	$\chi \circ N_D$, $\chi \neq 1$	$\mathbf{1}$	$\dim > 1$
$\sigma = \theta_{\psi,W,V^-}(\pi) \in \mathrm{Irr}(\mathrm{Mp}(W))$	$\omega_{\psi,\chi}^-$	st_ψ	supercuspidal

Observe that the local Shimura correspondence, when restricted to the non-supercuspidal representations of $\mathrm{Mp}(W)$, does not completely agree with the naive bijection suggested by Propositions 2.1 and 2.2.

2.18. **Central signs and Whittaker functionals.** The elements $\sigma^\pm$ in a Waldspurger packet $A_\psi(\pi)$ can be readily distinguished from each other in the following ways:

Proposition 2.3. *(i) (central character) The central signs of $\sigma^\pm$ are different and are explicitly given by:*

$$z_\psi(\sigma^\epsilon) = \epsilon \cdot \epsilon(1/2, \pi, \psi).$$

(ii) (Whittaker functionals) The representation $\sigma^\epsilon \neq \omega_{\psi,\chi}^+$ has ψ-Whittaker functional iff $\epsilon = +$.

Proof. (i) The actions of the central elements $-1_V \in \mathrm{O}(V)$ and $(-1_W, 1) \in \mathrm{Mp}(W)$ on the Weil representation $\omega_{\psi,W,V}$ differ by $\chi_\psi(-1)$. The result then follows from part (i) of the main local theorem.

(ii) This is proved in Corollary 5.7 below. □

2.19. **Variation of ψ.** Finally, Waldspurger describes how Sh_ψ varies with the choice of ψ.

Theorem 2.4. *(i) For $a \in k^\times$, let ψ_a denote the additive character given by $\psi_a(x) = \psi(ax)$ and let χ_a be the quadratic character associated to the class of $a \in k^\times/k^{\times 2}$. Then*

$$A_\psi(\pi) = A_{\psi_a}(\pi \otimes \chi_a),$$

so that the partition of $\mathrm{Irr}(\mathrm{Mp}(W))$ into the disjoint union of finite subsets is canonical, except that the labelling of these sets by elements of $\mathrm{Irr}(\mathrm{PGL}_2)$ depends on ψ. Moreover, the labelling of the representations by $\pm$ in these two (identical) sets may differ.

(ii) Fix a representation $\sigma \in A_\psi(\pi) = A_{\psi_a}(\pi \otimes \chi_a)$. Then we have

$$z_{\psi_a}(\sigma) = z_\psi(\sigma) \cdot \chi_a(-1),$$

and the labellings $\epsilon = \pm$ and ϵ' of σ as an element in $A_\psi(\pi)$ and $A_{\psi_a}(\pi \otimes \chi_a)$ are related by:

$$\epsilon'(\sigma) \cdot \epsilon(\sigma) = \epsilon(1/2, \pi \otimes \chi_a) \cdot \epsilon(1/2, \pi) \cdot \chi_a(-1).$$

Further, σ has ψ_a-Whittaker functional iff

$$\epsilon(\sigma) = \epsilon(1/2, \pi \otimes \chi_a) \cdot \epsilon(1/2, \pi) \cdot \chi_a(-1).$$

Proof. (i) Given $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$, we need to show that $\Theta_{\psi,W,V}(\sigma)$ and $\Theta_{\psi_a,W,V'}(\sigma) \otimes \chi_a$ are Jacquet-Langlands transfer of each other. If σ is not supercuspidal, or if $\sigma = \omega^-_{\psi,\chi}$, this is easily checked using the tables describing the explicit local Shimura correspondence; I'll leave this as an exercise. When $\sigma \neq \omega^-_{\psi,\chi}$ is supercuspidal, so are the two representations $\Theta_{\psi,W,V}(\sigma)$ and $\Theta_{\psi_a,W,V'}(\sigma) \otimes \chi_a$. In this case, it suffices to show that

$$\gamma(s, \Theta_{\psi,W,V}(\sigma) \times \mu, \psi) = \gamma(s, \Theta_{\psi_a,W,V'}(\sigma) \times \chi_a\mu, \psi)$$

for any character μ of $k^\times$.

To see this, we consider

$$\gamma(s, \Theta_{\psi_a,W,V'}(\sigma) \times \chi_a\mu, \psi_a).$$

On one hand, by the property of γ-factors, it is equal to

$$\mu(a)^2 \cdot |a|^{2s-1} \cdot \gamma(s, \Theta_{\psi_a,W,V'}(\sigma) \times \chi_a\mu, \psi).$$

On the other, using property (iv) of the main local theorem, it is equal to

$$\mu(a)^2 \cdot |a|^{2s-1} \cdot \gamma(s, \sigma \times \mu, \psi).$$

Thus, we deduce that

$$\gamma(s, \Theta_{\psi_a,W,V'}(\sigma) \times \chi_a\mu, \psi) = \gamma(s, \sigma \times \mu, \psi) = \gamma(s, \Theta_{\psi,W,V}(\sigma) \times \mu, \psi)$$

where the second equality follows by property (iv) of the main local theorem again. This proves (i).

(ii)The relation of central characters follows immediately from the fact that $\chi_{\psi_a} = \chi_\psi \cdot \chi_a$ as characters of $\tilde{Z}$. For the relation of $\epsilon(\sigma)$ and $\epsilon'(\sigma)$, we express the central character of σ in two ways. On one hand,

$$\omega_\sigma(-1) = \chi_\psi(-1) \cdot \epsilon(\sigma) \cdot \epsilon(1/2, \pi),$$

and on the other,

$$\omega_\sigma(-1) = \chi_{\psi_a}(-1) \cdot \epsilon'(\sigma) \cdot \epsilon(1/2, \pi \otimes \chi_a).$$

Equating gives the desired result. Finally, we saw that $\sigma \in A_\psi(\pi)$ has ψ_a-Whittaker coefficient iff its labelling $\epsilon'(\sigma)$ as an element of $A_{\psi_a}(\pi \otimes \chi_a)$ is $+1$. Thus the desired result follows from the relation of $\epsilon(\sigma)$ and $\epsilon'(\sigma)$ shown above. □

3. **Global Shimura Correspondence**

We come now to the global setting. Let F be a number field with ring of adeles $\mathbb{A}$ and let W be a 2-dimensional symplectic vector space over F with Witt basis $\{e, f\}$.

3.1. **Global metaplectic group.** We will be interested in the global metaplectic group attached to W which is defined as follows. Consider the restricted direct product $\prod'_v \mathrm{Mp}(W_v)$ with respect to the family of open compact subgroup K_v. This is a central extension of $\mathrm{Sp}(W_{\mathbb{A}})$ by the compact group $\oplus_v \mu_{2,v}$. Let

$$Z_0 = \{(\epsilon_v) \in \oplus_v \mu_{2,v} : \prod_v \epsilon_v = 1\}$$

and set

$$\mathrm{Mp}(W_{\mathbb{A}}) = \left(\prod_v{}' \mathrm{Mp}(W_v)\right) / Z_0.$$

This is the global metaplectic group and we have

$$0 \longrightarrow \mu_2 \longrightarrow \mathrm{Mp}(W_{\mathbb{A}}) \longrightarrow \mathrm{Sp}(W_{\mathbb{A}}) \longrightarrow 0.$$

A key fact is that this double cover of $\mathrm{Sp}(W_{\mathbb{A}})$ is split over the subgroup $\mathrm{Sp}(W_F)$. Moreover, the splitting is unique since $\mathrm{Sp}(W_F)$ is perfect.

3.2. **Automorphic representations.** As such, we may talk about the space $\mathcal{A}(\mathrm{Mp}(W))$ of genuine automorphic forms on $\mathrm{Sp}(W_F)\backslash\mathrm{Mp}(W_{\mathbb{A}})$. Observe that an irreducible genuine representation of $\mathrm{Mp}(W_{\mathbb{A}})$ is of the form

$$\sigma = \otimes_v \sigma_v$$

where σ_v is an irreducible genuine representation of $\mathrm{Mp}(W_v)$. In particular, every irreducible automorphic representation is of this form. We shall also look at the unitary representation

$$L^2(\mathrm{Mp}(W)) := L^2(\mathrm{Sp}(W_F)\backslash\mathrm{Mp}(W_{\mathbb{A}})),$$

and in particular its discrete spectrum $L^2_{disc}(\mathrm{Mp}(W))$.

Goal of the global Shimura correspondence: Describe the spectral decomposition of $L^2_{disc}(\mathrm{Mp}(W))$ into irreducible representations of $\mathrm{Mp}(W_{\mathbb{A}})$.

As suggested by the local Shimura correspondence, we expect this decomposition to be described in terms of the spectral decomposition of $L^2_{disc}(\mathrm{PGL}_2)$.

3.3. **Near equivalence.** As a first step, we may define an equivalence relation on the set of irreducible representations of $\mathrm{Mp}(W_{\mathbb{A}})$. We say that two such representations $\sigma = \otimes_v \sigma_v$ and $\sigma' = \otimes_v \sigma'_v$ are nearly equivalent if $\sigma_v \cong \sigma'_v$ for almost all v. The spectral decomposition of $L^2_{disc}(\mathrm{Mp}(W))$ can be separated into two parts:

- describe the near equivalence classes in $L^2_{disc}(\mathrm{Mp}(W))$;
- describe the decomposition of each near equivalence class.

3.4. **Elementary Theta functions (ETF)..** The first examples of automorphic representations of $\mathrm{Mp}(W_{\mathbb{A}})$ are the global analog of the Weil representations introduced in §. Let $\psi = \prod_v \psi_v$ be a non-trivial additive character of $F\backslash\mathbb{A}$. One may consider the restricted tensor product of the local Weil representations ω_{ψ_v}:

$$\omega_\psi = \otimes'_v \omega_{\psi_v}.$$

As an abstract representation, this is realized on the space $S(\mathbb{A})$ of Schwarz-Bruhat functions. Moreover, it is a highly reducible rpresentation:

$$\omega_\psi = \bigoplus_S \omega_{\psi,S}$$

where S ranges over all finite subsets of places of F and

$$\omega_{\psi,S} = \left(\bigotimes_{v\in S} \omega^-_{\psi_v}\right) \otimes \left(\bigotimes_{v\notin S} \omega^+_{\psi_v}\right).$$

Consider the map

$$\theta_\psi : \omega_\psi \longrightarrow \{\text{Functions on } \mathrm{Mp}(W_\mathbb{A})\}$$

given by

$$\theta_\psi(\phi)(g) = \sum_{x\in F} (\omega_\psi(g)\phi)(x).$$

The basic facts about the map θ_ψ is summarized in the following proposition:

Proposition 3.1. *(i) For each $\phi \in S(\mathbb{A})$, the function $\theta_\psi(\phi)$ on $\mathrm{Mp}(W_\mathbb{A})$ is left-invariant under $\mathrm{Sp}(W_F)$, so that θ_ψ gives a $\mathrm{Mp}(W_\mathbb{A})$-equivariant map*

$$\theta_\psi : \omega_\psi \longrightarrow \mathcal{A}(\mathrm{Mp}(W_\mathbb{A})).$$

(ii) We have:

$$\mathrm{Ker}(\theta_\psi) = \bigoplus_{\#S \text{ odd}} \omega_{\psi,S} \quad \text{and} \quad \mathrm{Im}(\theta_\psi) \cong \bigoplus_{\#S \text{ even}} \omega_{\psi,S}.$$

(iii) The image of θ_ψ is contained in $L^2_{disc}(\mathrm{Mp}(W))$ and is a full near equivalence class in $L^2_{disc}(\mathrm{Mp}(W))$. Moreover, $\omega_{\psi,S}$ is cuspidal if and only if S is nonempty.

The automorphic forms in $\mathrm{Im}(\theta_\psi)$ are sometimes called elementary theta functions (ETF). They are very degenerate automorphic forms in the sense that they have only one orbit of non-constant Fourier coefficients.

If we fix the additive character ψ, then all others are of the form $\psi_a(x) = \psi(ax)$ for $a \in F^\times$. As in the local case, one has

$$\omega_{\psi_a} \cong \omega_{\psi_b} \iff a/b \in F^{\times 2}.$$

For each $a \in F^\times/F^{\times 2}$ with associated quadratic (Hecke) character χ of $F^\times\backslash\mathbb{A}^\times$, we set

$$\omega_{\psi,a} = \omega_{\psi,\chi} := \omega_{\psi_a}.$$

We have thus constructed a subspace of $L^2(\mathrm{Mp}(W))$ isomorphic to

$$\bigoplus_\chi \left(\bigoplus_{\#S \text{ even}} \omega_{\psi,\chi,S}\right).$$

We shall be interested in understanding the orthogonal complement of this submodule of elementary theta functions. The global Shimura correspondence relates this orthogonal complement, which is contained in the space of cusp forms, to the space of cusp forms on PGL_2.

Henceforth, to simplify terminology, by a cuspidal representation of $\mathrm{Mp}(W_{\mathbb{A}})$, we shall mean a cuspidal representation which is orthogonal to the space of ETF's.

3.5. **The orthogonal groups.** Let V be a rank 3 quadratic space over F of discriminant 1. Then $V = V_B = (B_0, N_B)$ for some quaternion F-algebra B. One then has the associated special orthogonal group $\mathrm{SO}(V) \cong PB^{\times}$ and one may consider the space of automorphic forms $\mathcal{A}(\mathrm{SO}(V))$ and the L^2-space $L^2_{disc}(\mathrm{SO}(V))$. By the strong multipicity one theorem and the Jacquet-Langlands correspondence, the near equivalence classes in $L^2_{disc}(\mathrm{SO}(V))$ are singletons, and $L^2_{disc}(\mathrm{SO}(V))$ is multiplicity-free.

3.6. **Global Waldspurger lift Wd_{ψ}.** Let σ be an irreducible representation of $\mathrm{Mp}(W_{\mathbb{A}})$ (automorphic or not). For each v, we have the representation $Wd_{\psi_v}(\sigma_v) \in \mathrm{Irr}(\mathrm{PGL}_2(F_v))$ which is unramified for almost all v. So we may set

$$Wd_{\psi}(\sigma) = \otimes'_v Wd_{\psi_v}(\sigma_v).$$

We call this the global Waldspurger lift of σ with respect to ψ. A basic question is:

If σ is a cuspidal representation of $\mathrm{Mp}(W_{\mathbb{A}})$, is its Waldspurger lift $Wd_{\psi}(\sigma)$ a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$?

We shall see that this is the case later on.

3.7. **Global Waldspurger packet.** We now define the global Waldspurger packet associated to a representation π of $\mathrm{PGL}_2(\mathbb{A})$ to be the set $A_{\psi}(\pi) := Wd_{\psi}^{-1}(\pi)$. If $\pi = \otimes_v \pi_v$ is a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$, then for each v, we have the local Waldspurger packet

$$A_{\psi_v}(\pi_v) = \{\sigma_v^+, \sigma_v^-\}$$

where σ_v^- is interpreted as 0 if π_v is not discrete series. Then

$$A_{\psi}(\pi) = \{\sigma^{\epsilon} := \bigotimes_v \sigma_v^{\epsilon_v} : \epsilon_v = + \text{ for almost all } v\}.$$

This is the global Waldspurger packet associated to π, which is a finite set of cardinality $2^{\#S_{\pi}}$, where S_{π} is the set of places v of F such that π_v is discrete series.

If π is a cuspidal representation, we shall see that roughly half the elements in $A_{\psi}(\pi)$ occur in $L^2_{disc}(\mathrm{Mp}(W))$.

3.8. **A necessary condition for automorphy.** Before doing so, we shall first take note of a negative result, namely that roughly half of the elements in $A_{\psi}(\pi)$ cannot be automorphic for essentially trivial reasons. Recall that the central character of $\sigma_v^{\epsilon_v}$ is given by

$$\omega_{\sigma_v^{\epsilon_v}} / \chi_{\psi_v} = \epsilon_v \cdot \epsilon(1/2, \pi_v),$$

where both sides are interpreted as characters of $Z(F_v) = \langle \pm 1 \rangle$. Now the character $\chi_{\psi} = \otimes_v \chi_{\psi_v}$ is a global automorphic character of $\tilde{T}_{\mathbb{A}}$ and hence $\tilde{Z}_{\mathbb{A}}$, and so is the central character $\omega_{\sigma^{\epsilon}} = \otimes_v \omega_{\sigma_v^{\epsilon_v}}$ if σ^{ϵ} is automorphic. Thus, the character $\omega_{\sigma}/\chi_{\psi}$ is trivial when restricted to $Z(F) = \langle \pm 1 \rangle$ and we have:

Proposition 3.2. *If σ^ϵ is automorphic, then we must have:*

$$\prod_v \epsilon_v = \epsilon(1/2, \pi).$$

What is not immediately clear is the converse (when π is cuspidal).

3.9. **The Main Global Theorem.** The following is the main global theorem of this course:

Theorem 3.3. *We have:*

$$L^2_{disc}(\mathrm{Mp}(W)) = (\mathrm{ETF}) \bigoplus \left(\bigoplus_\pi L^2_\pi \right)$$

as π ranges over all cuspidal representations of $\mathrm{PGL}_2(\mathbb{A})$. Moreover, each L^2_π is a full near equivalence class and is described by

$$L^2_\pi = \bigoplus_{\sigma^\epsilon \in A_\psi(\pi)} m_\psi(\epsilon) \cdot \sigma^\epsilon \subset L^2_{disc}(\mathrm{Mp}(W)),$$

where

$$m_\psi(\sigma^\epsilon) = \begin{cases} 1, & \text{if } \prod_v \epsilon_v = \epsilon(1/2,\pi); \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $L^2_{disc}(\mathrm{Mp}(W))$ is multiplicity-free. Thus the summands L^2_π are canonical, but their labeling by cuspidal representations of $\mathrm{PGL}_2(\mathbb{A})$ depends on ψ.

The rest of the course is devoted to the proof of this theorem.

4. **Global Theta Correspondence**

Not surprisingly, the main technique for the proof of the global theorem is global theta correspondence. In this section, we introduce some general results about the global theta correspondence.

4.1. **Global Weil representation.** For the fixed additive character ψ of $F\backslash\mathbb{A}$, one has the global Weil representation of $\mathrm{Mp}(W_\mathbb{A}) \times \mathrm{O}(V_\mathbb{A})$:

$$\omega_{\psi,W,V} = \otimes'_v \omega_{\psi_v,W_v,V_v}.$$

As an abstract representation, it is realized on the space $S(f \otimes V_\mathbb{A})$ of Schwarz-Bruhat functions on $f \otimes V_\mathbb{A}$ and given by analogous formulas as in §2.13. There is a natural $\mathrm{Mp}(W_\mathbb{A}) \times \mathrm{SO}(V_\mathbb{A})$-equivariant map

$$\theta_{\psi,W,V} : \omega_{\psi,W,V} \longrightarrow \{\text{Functions on } \mathrm{Sp}(W_F)\backslash\mathrm{Mp}(W_\mathbb{A}) \times \mathrm{SO}(V_F)\backslash\mathrm{SO}(V_\mathbb{A})\}$$

defined by:

$$\theta_{\psi,W,V}(\phi)(g,h) = \sum_{x \in V_F} (\omega_{\psi,W,V}(g,h)\phi)(x).$$

4.2. **Global Theta correspondence.** We may use the functions in $\mathrm{Im}\theta_{\psi,W,V}$ as kernel functions to lift automorphic forms on $\mathrm{O}(V_{\mathbb{A}})$ to those on $\mathrm{Mp}(W_{\mathbb{A}})$ and vice versa. More precisely, given an automorphic form f on $\mathrm{O}(V_{\mathbb{A}})$ and $\phi \in S(V_{\mathbb{A}})$, we define an automorphic form on $\mathrm{Mp}(W_{\mathbb{A}})$ by

$$\theta_{\psi,W,V}(\phi, f)(g) = \int_{\mathrm{O}(V_F)\backslash \mathrm{O}(V_{\mathbb{A}})} \theta(\phi)(g,h) \cdot \overline{f(h)}\, dh,$$

where dh is the Tamagawa measure on $\mathrm{O}(V_{\mathbb{A}})$. To ensure the convergence of this integral, we shall assume that f is cuspidal (though this condition may not be necessary).

If π is a cuspidal representation of $\mathrm{O}(V_{\mathbb{A}})$, we let

$$\Theta_{\psi,W,V}(\pi) = \langle \theta_{\psi,W,V}(\phi, f) : \phi \in S(V_{\mathbb{A}}),\ f \in \pi \rangle.$$

Then $\Theta_{\psi,W,V}(\pi) \subset \mathcal{A}(\mathrm{Mp}(W))$ is a submodule, and is called the *global theta lift* of π (with respect to $\omega_{\psi,V,W}$).

4.3. **Compatibility with local theta correspondence.** We thus have a map

$$\theta_{\psi,W,V} : \omega_{\psi,W,V} \otimes \overline{\pi} \longrightarrow \Theta_{\psi,W,V}(\pi) \subset \mathcal{A}(\mathrm{Mp}(W))$$

which is $\mathrm{Mp}(W_{\mathbb{A}}) \times \mathrm{O}(V_{\mathbb{A}})$-equivariant, where the action of $\mathrm{SO}(V_{\mathbb{A}})$ on the domain is diagonal, and its action on the image is trivial. Since π is cuspidal and hence unitary, $\overline{\pi} \cong \pi^{\vee}$ and the above map induces

$$\theta_{\psi,W,V} : \omega_{\psi,W,V} \longrightarrow \pi \boxtimes \Theta_{\psi,W,V}(\pi).$$

This map has to factor through the abstract projection

$$\omega_{\psi,W,V} \cong \bigotimes_v \omega_{\psi_v,W_v,V_v} \longrightarrow \bigotimes_v \pi_v \boxtimes \Theta_{\psi_v,W_v,V_v}(\pi_v),$$

so that one has a surjection

$$\bigotimes_v \Theta_{\psi_v,W_v,V_v}(\pi_v) \twoheadrightarrow \Theta_{\psi,W,V}(\pi).$$

Thus, we deduce:

Lemma 4.1. *The automorphic representation $\Theta_{\psi,W,V}(\pi)$, if nonzero, has a unique irreducible quotient isomorphic to $\otimes_v \theta_{\psi_v,W_v,V_v}(\pi_v)$. In particular, if $\Theta_{\psi,W,V}(\pi)$ is contained in the space of cusp forms, so that it is semisimple, then*

$$\Theta_{\psi,W,V}(\pi) \cong \otimes_v \theta_{\psi_v,W_v,V_v}(\pi_v).$$

4.4. **Back and forth.** By symmetry, if σ is a genuine cuspidal representation of $\mathrm{Mp}(W_{\mathbb{A}})$, we can analogously define its global theta lift $\tau := \Theta_{\psi,W,V}(\sigma) \subset \mathcal{A}(\mathrm{SO}(V))$. If this global theta lift τ is contained in the space of cusp forms, then it is isomorphic to $\otimes_v \theta_{\psi_v,W_v,V_v}(\sigma_v)$.

Now suppose we consider the global theta lift of τ back to $\mathrm{Mp}(W_{\mathbb{A}})$. Then it follows from definition that $\Theta_{\psi,W,V}(\tau)$ is not orthogonal to σ. In particular, it is not zero and isomorphic as an abstract representation to σ . However, at this point, we cannot say that $\Theta_{\psi,W,V}$ is equal to σ as a space of automorphic forms (though we shall show this later on).

Suppose now we had started with a cuspidal representation π of $\mathrm{SO}(V_{\mathbb{A}})$, consider its theta lift $\sigma := \Theta_{\psi,W,V}(\pi)$ on $\mathrm{Mp}(W_{\mathbb{A}})$ (say σ is cuspidal), and then the theta lift $\Theta_{\psi,V,W}(\sigma)$ of σ

back to $\mathrm{SO}(V)$. As above, it follows from definition that $\Theta_{\psi,W,V}(\sigma)$ is not orthogonal to π. However, because the multiplicity one theorem holds for $\mathrm{SO}(V)$, we conclude that in fact

$$\pi = \Theta_{\psi,W,V}(\sigma) \quad \text{as a space of automorphic forms.}$$

4.5. **Basic questions.** The basic questions we need to address concerning the global theta correspondence are: given a cuspidal representation π of $\mathrm{O}(V_{\mathbb{A}})$,

- Is $\Theta_{\psi,W,V}(\pi)$ contained in the space of cusp forms?
- Is $\Theta_{\psi,W,V}(\pi)$ nonzero?

If the answers to both questions are affirmative, then the above lemma would tell us exactly what $\Theta_{\psi,W,V}(\pi)$ is, or rather reduce its determination to local problems.

4.6. **Passing between** O **and** SO**.** Because $\mathrm{O}(V) \cong \mathrm{SO}(V) \times \mu_2$, it is easy to extend automorphic representations of $\mathrm{SO}(V)$ to $\mathrm{O}(V)$. More precisely, given a finite set S of places of F of *even* cardinality, we let sgn_S denote the automorphic character of $\mu_2(F)\backslash\mu_2(\mathbb{A})$ defined by

$$\mathrm{sgn}_S = \left(\bigotimes_{v\in S} \mathrm{sgn}_v\right) \otimes \left(\bigotimes_{v\notin S} 1_v\right).$$

Then given an automorphic representation π of $\mathrm{SO}(V_{\mathbb{A}})$, the representation $\pi \boxtimes \mathrm{sgn}_S$ is an automorphic representation of $\mathrm{O}(V_{\mathbb{A}})$, and all such automorphic extensions of π are constructed in this way. Consequently, for f on $\mathrm{SO}(V_{\mathbb{A}})$ and $\phi \in S(V_{\mathbb{A}})$, if we set

$$\theta_{\psi,W,V}(\phi, f)(g) = \int_{\mathrm{O}(V_F)\backslash \mathrm{O}(V_{\mathbb{A}})} \theta(\phi)(g,h) \cdot \overline{f(h)}\, dh,$$

and define $\Theta_{\psi,W,V}(\pi)$ analogously as above, but with $\mathrm{O}(V)$ replaced by $\mathrm{SO}(V)$, then we see that

$$\Theta_{\psi,W,V}(\pi) = \sum_{S:\#S \text{ even}} \Theta_{\psi,W,V}(\pi \boxtimes \mathrm{sgn}_S).$$

4.7. **Vanishing criterion.** Let us see how the main local theorem gives us a sufficient condition for the vanishing of the global theta lift $\Theta_{\psi,W,V}(\pi)$. As a consequence of the main local theorem and the above lemma, we deduce:

Proposition 4.2. *If π is a cuspidal representation of $\mathrm{SO}(V_{\mathbb{A}})$, then there is at most one S such that $\Theta_{\psi,W,V}(\pi \boxtimes \mathrm{sgn}_S)$ can be nonzero. Moreover, this S is characterized by*

$$v \in S \iff \epsilon(1/2, \pi_v, \psi_v) = -1,$$

so that

$$\#S \text{ is even} \iff \epsilon(1/2, \pi) = 1.$$

In particular, if $\epsilon(1/2,\pi) = -1$, then $\Theta_{\psi,W,V}(\pi) = 0$.

Besides the vanishing criterion, this result implies that there is no loss of generality in studying global theta correspondences from $\mathrm{SO}(V)$ to $\mathrm{Mp}(W)$, in place of $\mathrm{O}(V)$ to $\mathrm{Mp}(W)$.

Henceforth, we shall consider cuspidal representations of $\mathrm{SO}(V_{\mathbb{A}})$ as opposed to $\mathrm{O}(V_{\mathbb{A}})$.

5. From $\mathrm{SO}(V)$ to $\mathrm{Mp}(W)$

Given $\phi \in \omega_{\psi,W,V}$ and $f \in \pi$, we would like to see if the automorphic form $\theta_{\psi,W,V}(\phi, f)$ is a nonzero cusp form on $\mathrm{Mp}(W_{\mathbb{A}})$.

5.1. **Periods.** One way of showing that an automorphic form on a group G is nonzero is to show that it has a nonzero period integral.To be more precise, suppose that H is a subgroup of G and χ is a character of $H(F)\backslash H(\mathbb{A})$ (a so-called automorphic character of H). Then the (H, χ) period of an automorphic form f on G is the integral

$$f_{H,\chi}(g) = \int_{H(F)\backslash H(\mathbb{A})} f(hg) \cdot \overline{\chi(h)}\, dh,$$

and f is said to have nonzero (H, χ) period if $f_{H,\chi}$ is nonzero as a function on $G(\mathbb{A})$. Equivalently, we consider the linear functional

$$\mathcal{P}_{H,\chi} : f \mapsto f_{H,\chi}(1)$$

on $\mathcal{A}(G)$, and an irreducible submodule $\pi \subset \mathcal{A}(G)$ is (H, χ)-distinguished if $\mathcal{P}_{H,\chi}$ is nonzero on π. If χ is trivial, we shall simply write f_H and P_H.

5.2. **Fourier coefficients.** The simplest example of periods is Fourier coefficients. In the context of an automorphic form φ on $\mathrm{Mp}(W)$, one may consider the Fourier expansion of φ along the subgroup N. More precisely, for any character χ of $N(F)\backslash N(\mathbb{A})$, the χ-th Fourier expansion of φ along N is by definition $f_{N,\chi}$.

Since we have identified N with $\mathbb{G}_a$, and we have fixed an additive character ψ of $F\backslash\mathbb{A}$, we may regard ψ as a character of $N(F)\backslash N(\mathbb{A})$. Then all such χ's as above are of the form ψ_a for some $a \in F$. When $a = 0$, the ψ_a-th Fourier coefficient is called the *constant term* of φ, and is denoted by φ_N. By definition, φ is cuspidal iff $\varphi_N = 0$.

5.3. **Computation of constant term.** To detect if $\theta_{\psi,W,V}(\phi, f)$ is cuspidal, we shall compute its constant term $\mathcal{P}_N(\theta_{\psi,W,V}(\phi, f))$.

$$\begin{aligned}
&\mathcal{P}_N(\theta_{\psi,W,V}(\phi,f))\\
&=\int_{N(F)\backslash N(\mathbb{A})}\theta_{\psi,W,V}(\phi,f)(n)\,dn\\
&=\int_{N(F)\backslash N(\mathbb{A})}\int_{\mathrm{SO}(V_F)\backslash\mathrm{SO}(V_{\mathbb{A}})}\theta(\phi)(n,h)\cdot\overline{f(h)}\,dh\,dn\\
&=\int_{N(F)\backslash N(\mathbb{A})}\int_{\mathrm{SO}(V_F)\backslash\mathrm{SO}(V_{\mathbb{A}})}\left(\sum_{x\in V_F}(\omega_{\psi,W,V}(n,h)\phi)(x)\right)\cdot\overline{f(h)}dh\;dn\\
&=\int_{N(F)\backslash N(\mathbb{A})}\int_{\mathrm{SO}(V_F)\backslash\mathrm{SO}(V_{\mathbb{A}})}\overline{f(h)}\cdot\left(\sum_{x\in V_F}\psi(nq(x))\cdot\phi)(h^{-1}x)\right)dh\;dn\\
&=\int_{\mathrm{SO}(V_F)\backslash\mathrm{SO}(V_{\mathbb{A}})}\overline{f(h)}\cdot\sum_{x\in V_F}\phi)(h^{-1}x)\cdot\left(\int_{N(F)\backslash N(\mathbb{A})}\psi(nq(x))\,dn\right)\,dh\\
&=\int_{\mathrm{SO}(V_F)\backslash\mathrm{SO}(V_{\mathbb{A}})}\overline{f(h)}\cdot\left(\sum_{x\in V_F:q(x)=0}\phi(h^{-1}x)\right)\,dh.
\end{aligned}$$

With $V=V_B$, we see that if B is a division algebra, then the only $x\in V_F$ such that $q(x)=0$ is $x=0$, in which case we see that

$$\theta_{\psi,W,V}(\phi,f)_N(g)=\left(\int_{\mathrm{SO}(V_F)\backslash\mathrm{SO}(V_{\mathbb{A}})}\overline{f(h)}\,dh\right)\cdot\phi(0).$$

This is zero if and only if f is not the constant function. Thus, $\Theta_{\psi,W,V}(\pi)$ is contained in the space of cusp forms if and only if π is not the trivial representation of $\mathrm{SO}(V_{\mathbb{A}})$.

On the other hand, if B is split, then besides $x=0$, there will be nonzero isotropic x's in V_F. However, all such nonzero isotropic x's are in one orbit under the action of $\mathrm{SO}(V_F)$. If we fix such an x_0, then its stabilizer in $\mathrm{SO}(V_F)$ is the unipotent radical U of a Borel subgroup. Thus, we have

$$\{x\in V_F:q(x)=0\}=\{0\}\cup\mathrm{SO}(V_F)/U(F)\cdot x_0.$$

Then the constant term $\theta_{\psi,W,V}(\phi,f)_N(g)$ is the sum of

$$\left(\int_{\mathrm{SO}(V_F)\backslash\mathrm{SO}(V_{\mathbb{A}})}\overline{f(h)}\,dh\right)\cdot\phi(0),$$

and

$$\begin{aligned}
&\int_{\mathrm{SO}(V_F)\backslash \mathrm{SO}(V_{\mathbb{A}})} \overline{f(h)} \cdot \left(\sum_{\gamma \in U(F)\backslash SO(V_F)} \phi(h^{-1}\gamma^{-1}x_0) \right) dh. \\
&= \int_{U(F)\backslash \mathrm{SO}(V_{\mathbb{A}})} \overline{f(h)} \cdot \phi(h^{-1}x_0)\, dh \\
&= \int_{U(\mathbb{A})\backslash \mathrm{SO}(V_{\mathbb{A}})} \int_{U(F)\backslash U(\mathbb{A})} \overline{f(uh)} \cdot h^{-1}u^{-1}x_0)\, du\, dh \\
&= \int_{U(\mathbb{A})\backslash \mathrm{SO}(V_{\mathbb{A}})} \phi(h^{-1}x_0) \cdot \left(\int_{U(F)\backslash U(\mathbb{A})} \overline{f(uh)}\, du \right) dh
\end{aligned}$$

As before the first term above is zero since f is cuspidal, and thus is orthogonal to the constant functions. Similarly, the cuspidality of f implies that the inner integral in the second term vanishes.

To summarize, we have shown

Proposition 5.1. *The constant term map $\mathcal{P}_N$ is zero on the global theta lift $\Theta_{\psi,W,V}(\pi)$, so that $\Theta_{\psi,W,V}(\pi)$ is cuspidal, unless V is anisotropic and π is the trivial representation of* $\mathrm{SO}(V_{\mathbb{A}})$.

5.4. **Computation of ψ_a-th Fourier coefficients.** Now, with $a \in F^{\times}$, we can compute the ψ_a-th Fourier coefficient of $\theta_{\psi,W,V}(\phi, f)$ by a similar computation. We obtain:

$$\mathcal{P}_{N,\psi_a}(\theta_{\psi,W,V}(\phi, f)) = \int_{\mathrm{SO}(V_F)\backslash \mathrm{SO}(V_{\mathbb{A}})} \overline{f(h)} \cdot \left(\sum_{x \in V_F : q(x)=a} \phi(h^{-1}x) \right) dh.$$

We now need to examine the set

$$X_a(F) = \{x \in V_F : q(x) = a\}.$$

If $V = V_B = (B_0, -N_B)$, then $X_a(F)$ is equal to

$$\{x \in B_0 : N_B(x) = -a\} = \{x \in B_0 : x \text{ generates a quadratic subalgebra } F(x)/(x^2 - a)\}$$

By the Hasse principle, $X_a(F)$ is nonempty if and only if $X_a(F_v)$ is nonempty for all v. This is equivalent to: $a \notin F_v^{\times 2}$ for all v where B is ramified.

Assuming that $X_a(F)$ is nonempty, it follows by Witt's theorem that $\mathrm{SO}(V_F)$ acts transitively on $X_a(F)$. If we fix an $x_a \in X_a$, and let $V_a = x_a^{\perp}$ so that

$$V = F \cdot x_a \oplus V_a$$

then the stabilizer of x_a in $\mathrm{SO}(V_F)$ is the subgroup $\mathrm{SO}(V^a)$ (acting trivially on x_a). Thus,

$$\begin{aligned}
&\mathcal{P}_{N,\psi_a}(\theta_{\psi,W,V}(\phi,f)) \\
&= \int_{\mathrm{SO}(V_F)\backslash\mathrm{SO}(V_{\mathbb{A}})} \overline{f(h)} \cdot \left(\sum_{\gamma \in SO(V_F^a)\backslash \mathrm{SO}(V_F)} \phi(h^{-1}\gamma^{-1}x_a) \right) dh \\
&= \int_{\mathrm{SO}(V_F^a)\backslash\mathrm{SO}(V_{\mathbb{A}})} \overline{f(h)} \cdot \phi(h^{-1}x_a)\, dh \\
&= \int_{\mathrm{SO}(V_{\mathbb{A}}^a)\backslash\mathrm{SO}(V_{\mathbb{A}})} \phi(h^{-1}x_a) \cdot \left(\int_{\mathrm{SO}(V_F^a)\backslash\mathrm{SO}(V_{\mathbb{A}}^a)} \overline{f(th)}\, dt \right) dh
\end{aligned}$$

Hence, we see that the nonvanishing of the ψ_a-th Fourier coefficient of $\theta_{\psi,W,V}(\phi,f)$ is related to the nonvanishing of the period integral

$$P_{\mathrm{SO}(V^a)}(f) := \int_{\mathrm{SO}(V_F^a)\backslash\mathrm{SO}(V_{\mathbb{A}}^a)} \overline{f(t)}\, dt$$

on π.

The above computation illustrates the following principle in the theory of theta correspondences:

Principle: A period $\mathcal{P}_1$ of the theta lift $\Theta(\pi)$ of a cuspidal representation π is related to a corresponding period $\mathcal{P}_2$ on π.

To summarize, we have more or less shown:

Proposition 5.2. *The ψ_a-th Fourier coefficient of $\Theta_{\psi,W,V}(\pi)$ is nonzero if and only if the period integral P_{V^a} is nonzero on π. In particular, if $\mathcal{P}_{V^a}$ is nonzero on π for some a, then the global theta lift $\Theta_{\psi,W,V}(\pi)$ is nonzero.*

Proof. The only point which is not clear is that if $\mathcal{P}_{\mathrm{SO}(V^a)}$ is nonzero on π, then $\mathcal{P}_{N,\psi_a}$ is nonzero on $\Theta_{\psi,W,V}(\pi)$. We may choose $f \in \pi$ such that the function $h \mapsto \mathcal{P}_{\mathrm{SO}(V^a)}(h \cdot f)$ is a nonzero function on $\mathrm{SO}(V_{\mathbb{A}})$. Since

$$\mathcal{P}_{N,\psi_a}(\theta_{\psi,W,V}(\phi,f)) = \int_{\mathrm{SO}(V_{\mathbb{A}}^a)\backslash\mathrm{SO}(V_{\mathbb{A}})} \phi(h^{-1} \cdot x_a) \cdot \mathcal{P}_{\mathrm{SO}(V^a)}(h \cdot f) dh,$$

we need to show that one can find ϕ so that the above integral is nonzero. But note that $X_a \subset V$ is a Zariski-closed subset, and

$$\mathrm{SO}(V_{\mathbb{A}}^a)\backslash\mathrm{SO}(V_{\mathbb{A}}) \cong X_a(\mathbb{A})$$

via $h \mapsto h^{-1} \cdot x_a$. So it suffices to see that the restriction map

$$S(V_{\mathbb{A}}) \longrightarrow S(X_a(\mathbb{A}))$$

is surjective.

For each local field F_v, the surjectivity of $S(V \otimes F_v) \to S(X_a(F_v))$ is clear. But the adelic statement has an additional subtlety which is often not pointed out explicitly. Namely, the spaces $S(V_{\mathbb{A}})$ and $S(V_a(\mathbb{A}))$ are restricted tensor products $\otimes_v' S(V \otimes F_v)$, and $\otimes_v' S(X_a(\mathbb{A}))$ with respect to a family of distinguished vectors $\{\phi_v^0\}$ and $\{\varphi_v^0\}$ for almost all v. We need to take note that the restriction map takes ϕ_v^0 to φ_v^0 for almost all v. For the case at hand, we can choose a basis $\mathcal{B}$ of V which endows V and X_a with an integral structure. For almost all v, we may take $\phi_v^0$and φ_v^0 to be the characteristic functions of $V(\mathcal{O}_{F_v})$ and $X_a(\mathcal{O}_{F_v})$ respectively. The result then follows from the fact that

$$X_a(\mathcal{O}_v) = X_a(F_v) \cap V(\mathcal{O}_{F_v}).$$

□

Note that the proposition simply transfers the question of nonvanishing of the global theta lift on $\mathrm{Mp}(W)$ to another period problem on $\mathrm{O}(V)$, but it is not clear that this second period problem is any easier! In general, a nonvanishing criterion given in terms of periods is not very useful, because the nonvanishing of periods a priori depends on the automorphic realization of the representation in question (though this is not an issue for $\mathrm{SO}(V)$, because of the multiplicity one theorem). It will be more useful to have a nonvanishing criterion in terms of a representation theoretic invariant, i.e. something which depends only on the isomorphism class of the representation in question. We shall see an instance of this presently.

5.5. **Split torus periods.** We first examine the case when $a \in F^{\times 2}$, in which case T_a is a split torus; this is only relevant when B is split. Thus, if π is a cuspidal representation of $PGL_2(\mathbb{A})$ and $f \in \pi$, we are interested in studying

$$\mathcal{P}(f) = \int_{T(F)\backslash T(\mathbb{A})} f(t)\, dt = \int_{F^\times\backslash\mathbb{A}^\times} f\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} da$$

We shall see that the nonvanishing of $\mathcal{P}$ on π is controlled by an L-value.

The key is the following result of Hecke-Jacquet-Langlands:

Proposition 5.3. *If we set*

$$Z(s,f) = \int_{F^\times\backslash\mathbb{A}^\times} f\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot |a|^{s-\frac{1}{2}}\, da,$$

then

$$Z(s,f) = \prod_{v\in S} Z_v^{\#}(s,f_v) \cdot L(s,\pi),$$

where S is a large finite set of places outside of which π is unramified, and $Z_v^{\#}(s,f_v)$ is an entire function of s. Moreover, given any s_0, there is a choice of f_v such that $Z_v^{\#}(s_0,f_v) \neq 0$.

Corollary 5.4. *The split torus period $\mathcal{P}_T$ is nonzero on π if and only if $L(1/2,\pi) \neq 0$. Thus, for π a cuspidal representation of* $\mathrm{PGL}_2(\mathbb{A})$,

$$L(1/2,\pi) \neq 0 \Longrightarrow \Theta_{\psi,V,W}(\pi) \neq 0.$$

Note that the condition $L(1/2,\pi) \neq 0$ is independent of ψ. We shall see the converse at the next section.

5.6. **A local analog.** One could do the analogous local calculation to compute $\Theta_{\psi,W,V}(\pi_v)_{N,\psi_{a,v}}$ for a representation π_v of $\mathrm{O}(V_v)$. To simplify notation, we will revert to the notation of §2, where the local field is denoted by k.

Proposition 5.5. *For $a \in k^\times$,*

$$(\omega_{\psi,W,V})_{N,\psi_a} \cong \mathrm{ind}_{\mathrm{O}(V^a)}^{\mathrm{O}(V)} 1$$

if X_a is nonempty; otherwise it is 0. In particular, if π is an irreducible representation of $\mathrm{O}(V)$, then

$$\Theta_{\psi,W,V}(\pi)_{N,\psi_a} \cong \mathrm{Hom}_{\mathrm{O}(V^a)}(\pi,\mathbb{C})$$

if X_a is nonempty, and is otherwise 0.

Proof. Recall that $\omega_{\psi,W,V}$ is realized on the space $S(V\otimes f)$ of Schwarz-Bruhat functions on $V\otimes f$, and the action of $n(x)\in N$ is given by

$$\omega_{\psi,W,V}(n(x))\phi(v) = \psi(x\cdot q(v))\cdot\phi(v).$$

From this, we deduce that the natural $\mathrm{O}(V)$-equivariant projection map

$$\omega_{\psi,W,V} \longrightarrow (\omega_{\psi,V,W})_{N,\psi_a}$$

is given by the restriction-of-functions map

$$S(V\otimes f) \longrightarrow S(X_a).$$

Moreover, the $\mathrm{O}(V)$-action on $S(X_a)$ is given by

$$(h\cdot\phi)(v) = \phi(h^{-1}\cdot v).$$

Since $\mathrm{O}(V)$ acts transitively on X_a, and the stabilizer of $x_a\in X_a$ is $\mathrm{O}(V^a)$ (as described above), we see that

$$(\omega_{\psi,W,V})_{N,\psi} \cong S(X_a) \cong \mathrm{ind}_{\mathrm{O}(V^a)}^{\mathrm{O}(V)} 1.$$

This proves the first statement. For the second, consider the maximal π-isotypic quotient on both sides, which gives

$$\Theta_{\psi,W,V}(\pi)_{N,\psi}^* \cong \mathrm{Hom}_{\mathrm{O}(V)}(\mathrm{ind}_{\mathrm{O}(V^a)}^{\mathrm{O}(V)} 1,\pi) \cong \mathrm{Hom}_{\mathrm{O}(V^a)}(\pi^\vee,\mathbb{C}).$$

The second statement then follows from the fact that $\pi^\vee\cong\pi$ and $\Theta_{\psi,V,W}(\pi)$ is finite length, so that $\Theta_{\psi,W,V}(\pi)_{N,\psi}^*$ is finite-dimensional. □

Corollary 5.6. *Suppose that π is an irreducible representation of $\mathrm{SO}(V)$. Then there exists $a\in k^\times$ such that $\mathrm{Hom}_{\mathrm{SO}(V^a)}(\pi,\mathbb{C})\neq 0$. Moreover, $\dim \mathrm{Hom}_{\mathrm{SO}(V^a)}(\pi,\mathbb{C})\le 1$.*

Proof. Note that

$$\mathrm{Hom}_{\mathrm{SO}(V^a)}(\pi,\mathbb{C}) \cong \mathrm{Hom}_{\mathrm{O}(V^a)}(\pi^+,\mathbb{C}) \oplus \mathrm{Hom}_{\mathrm{O}(V^a)}(\pi^-,\mathbb{C}).$$

Let π^ϵ be the unique extension of π to $\mathrm{O}(V)$ so that $\Theta_{\psi,W,V}(\pi^\epsilon)\neq 0$. Then the proposition implies that

$$\mathrm{Hom}_{\mathrm{O}(V^a)}(\pi^{-\epsilon},\mathbb{C}) = 0 \quad \text{and} \quad \dim\mathrm{Hom}_{\mathrm{O}(V^a)}(\pi^\epsilon,\mathbb{C}) = \dim\Theta_{\psi,W,V}(\pi^\epsilon)_{N,\psi_a} \le 1.$$

This proves the second statement. Now we can find $a \in k^\times$ such that $\Theta_{\psi,W,V}(\pi^\epsilon)_{N,\psi_a} \neq 0$. By the proposition again, this implies

$$\dim \mathrm{Hom}_{\mathrm{SO}(V^a)}(\pi, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}(V^a)}(\pi^\epsilon, \mathbb{C}) \neq 0.$$

□

The following is a restatement of Proposition 2.3(ii).

Corollary 5.7. *Given $\sigma^\epsilon \in A_\psi(\pi)$, σ has ψ-Whittaker functional if and only if $\epsilon = +$.*

Proof. We take $a = 1$ in the proposition. The proposition implies immediately that $\sigma^-_{N,\psi} = 0$, since then X_1 is empty. On the other hand, for $\sigma^+ = \Theta_{\psi,V^+,W}(\pi)$, we have $\mathrm{SO}(V^1) = T$ and so we need to show that $\mathrm{Hom}_T(\pi, \mathbb{C}) \neq 0$ when π is infinite-dimensional. This follows from the fact that one has a short exact sequence of T-modules:

$$0 \longrightarrow S(T) \longrightarrow \pi \longrightarrow \pi_N \longrightarrow 0.$$

□

Observe that the computation in this subsection is entirely parallel to the global computation in the earlier subsections. The reader might try his/her hand in computing the local analog of the constant term of $\omega_{\psi,W,V}$ along N, i.e. taking $a = 0$ in the above proposition. The general case of such a computation is covered in Ichino's lecture.

6. From $\mathrm{Mp}(W)$ to $\mathrm{SO}(V_0)$.

We can now turn the tables around and consider the lifting of cusp forms from $\mathrm{Mp}(W)$ to $\mathrm{SO}(V)$. If $V = V_0$ is the split quadratic space so that $\mathrm{SO}(V_0) = \mathrm{PGL}_2$, then we can detect the nonvanishing of this lift by computing Fourier coefficients of the theta lifts. We describe this computation here, as it requires us to use a different model of the Weil representation ω_{ψ,W,V_0}: the so-called *mixed model*.

In our usual model of the Weil representation, the group $\mathrm{O}(V_0)$ acts geometrically, whereas the Borel subgroup $\tilde{B}$ of $\mathrm{Mp}(W)$ acts in a way similar to that in the Kirillov model. We would like to change model, so that $\mathrm{Mp}(W)$ acts geometrically (at least partially) and the Borel subgroup $B_0 = T_0 \cdot U_0$ of $\mathrm{SO}(V_0)$ acts analogously as in the Kirillov model (at least partially). The reason for the adjective "mixed" is because we can only achieve this partially.

6.1. Mixed model.

Since $V = V_0$ is split, we can write

$$V = Fv \oplus L_1 \oplus Fv^*$$

with $b(v, v^*) = 1$ and where we recall that L_1 is the quadratic space with form $x \mapsto x^2$. The stabilizer $B_0 = T_0U_0$ of v is then a Borel subgroup of $\mathrm{SO}(V)$, with

$$T_0 = \mathrm{GL}(F \cdot v) \quad \text{and} \quad U_0 \subset \mathrm{Hom}(v^*, L_1).$$

We can consider the maximal isotropic subspace

$$X = (W \otimes v^*) \oplus (f \otimes L_1) \subset W \otimes V,$$

and write an element of X as

$$(w, l) = (a, b, l) \in W \oplus L_1 = Fe \oplus Ff \oplus L_1.$$

Then the partial Fourier transform defines an isomorphism of topological vector spaces

$$\iota : S(f \otimes V) \longrightarrow S(X).$$

This is given by

$$\iota(\phi)(a,b,l) = \int_{F f \otimes v} \phi(y,l,b) \cdot \psi(ay)\, dy.$$

We can transport the action of $\mathrm{Mp}(W) \times \mathrm{O}(V)$ to $S(X)$, denoting this action by $\omega'_{\psi,W,V}$. We may also regard elements φ of $S(X)$ as functions on $W \otimes v^*$ taking values in $S(f \otimes L_1)$, in which case we shall write $\varphi(w)(l)$ in place of $\varphi(w,l)$. Recall that there is an elementary Weil representation $\omega_\psi = \omega_{\psi,W,L_1}$ on $S(f \otimes L_1)$.

Now one has the following formula for the action of $\mathrm{Mp}(W) \times B_0$:

$$\begin{cases} \omega'(g)\varphi(w \otimes v^*) = \omega_{\psi,W,L_1}(g)\varphi(g^{-1} \cdot w \otimes v^*) \text{ as elements of } S(f \otimes L_1) \\ \omega'(t_0(a))\varphi(w \otimes v^*) = |a|^{1/2} \cdot \varphi(a \cdot w \otimes v^*) \\ \omega'(u_0(x))\varphi(w \otimes v^*) = \omega_\psi(h(x \cdot w \otimes L_1))\phi(w \otimes v^*) \end{cases}$$

where in the last formula,

$$h(xw \otimes L_1) \in H(W \otimes L_1) = (W \otimes L_1) \times \mathbb{G}_a$$

is an element of the Heisenberg group $H(W \otimes L_1)$, and its action on $\phi(w \otimes v^*) \in S(f \otimes L_1)$ is by the Heisenberg representation with central character ψ.

6.2. **Automorphic realization.** The formula

$$\theta'(\varphi)(g,h) = \sum_{x \in X_F} \omega'(g,h)\varphi)(x)$$

defines a function on $\mathrm{Mp}(W_{\mathbb{A}}) \times \mathrm{O}(V_{\mathbb{A}})$. A well-known result of Weil says that for $\phi \in S(f \otimes V)$,

$$\theta(\phi) = \theta'(\iota(\phi)).$$

6.3. **Fourier coefficients.** For a cusp form $f \in \sigma$ and $\phi \in S(f \otimes V_{\mathbb{A}})$, we can now compute the ψ-th Fourier coefficient of the global theta lift $\theta(\phi, f)$. Let us set $\varphi = \iota(\phi)$. Then we have:

$$\begin{aligned}
&\mathcal{P}_{U_0,\psi}(\theta(\phi,f))\\
&=\int_{U_0(F)\backslash U_0(\mathbb{A})}\overline{\psi(u)}\cdot\int_{\mathrm{Sp}(W_F)\backslash\mathrm{Mp}(W_{\mathbb{A}})}\overline{f(g)}\cdot\theta'(\iota(\phi))(g,u)\cdot\overline{f(g)}\,dg\,du\\
&=\int_{U_0(F)\backslash U_0(\mathbb{A})}\overline{\psi(u)}\cdot\int_{\mathrm{Sp}(W_F)\backslash\mathrm{Mp}(W_{\mathbb{A}})}\overline{f(g)}\cdot\sum_{w\in v^*\otimes W_F}\sum_{y\in f\otimes L_1}\omega'(u,g)\varphi(w,y)\,dg\,du\\
&=\int_{U_0(F)\backslash U_0(\mathbb{A})}\overline{\psi(u)}\cdot\int_{\mathrm{Sp}(W_F)\backslash\mathrm{Mp}(W_{\mathbb{A}})}\overline{f(g)}\cdot\sum_{\gamma\in N(F)\backslash\mathrm{Sp}(W_F)}\sum_{y\in f\otimes L_1}\omega'(u,g)\varphi(\gamma^{-1}e,y)\,dg\,du\\
&=\int_{U_0(F)\backslash U_0(\mathbb{A})}\overline{\psi(u)}\cdot\int_{\mathrm{Sp}(W_F)\backslash\mathrm{Mp}(W_{\mathbb{A}})}\overline{f(g)}\cdot\sum_{\gamma\in N(F)\backslash\mathrm{Sp}(W_F)}\sum_{y\in f\otimes L_1}\omega_{W,L_1}(\gamma)\omega'(u,g)\varphi(\gamma^{-1}e,y)\,dg\,du\\
&=\int_{U_0(F)\backslash U_0(\mathbb{A})}\overline{\psi(u)}\cdot\int_{\mathrm{Sp}(W_F)\backslash\mathrm{Mp}(W_{\mathbb{A}})}\overline{f(g)}\cdot\sum_{\gamma\in N(F)\backslash\mathrm{Sp}(W_F)}\sum_{y\in f\otimes L_1}\omega'(u,\gamma g)\varphi(e,y)\,dg\,du\\
&=\int_{U_0(F)\backslash U_0(\mathbb{A})}\overline{\psi(u)}\cdot\int_{N(F)\backslash\mathrm{Mp}(W_{\mathbb{A}})}\overline{f(g)}\cdot\sum_{y\in f\otimes L_1}\omega'(u,g)\varphi(e,y)\,dg\,du\\
&=\int_{U_0(F)\backslash U_0(\mathbb{A})}\overline{\psi(u)}\cdot\int_{N(F)\backslash\mathrm{Mp}(W_{\mathbb{A}})}\overline{f(g)}\cdot\sum_{y\in f\otimes L_1}\psi(2uy)\cdot\omega'(g)\varphi(e,y)\,dg\,du\\
&=\int_{N(F)\backslash\mathrm{Mp}(W_{\mathbb{A}})}\overline{f(g)}\omega'(g)\varphi(e,\frac{1}{2})\,dg\\
&=\int_{N(\mathbb{A})\backslash\mathrm{Mp}(W_{\mathbb{A}})}\int_{N(F)\backslash N(\mathbb{A})}\omega'(g)\varphi(e,\frac{1}{2})\cdot\psi(\frac{1}{4}x)\cdot\overline{f(n(x)g)}\,dx\,dg\\
&=\int_{N(\mathbb{A})\backslash\mathrm{Mp}(W_{\mathbb{A}})}\omega'(g)\varphi(e,\frac{1}{2})\cdot\overline{\mathcal{P}_{N,\psi_{1/4}}(g\cdot f)}\,dg
\end{aligned}$$

One can do a similar computation of the constant term $\mathcal{P}_{U_0}(\theta(\phi,f))$, by omitting the term $\psi(u)$ in the above computation. We have thus shown the following proposition:

Proposition 6.1. *If σ is a cuspidal representation of $\mathrm{Mp}(W)$ orthogonal to (ETF), then Θ_{ψ,W,V_0} is cuspidal. Further,*

$$\Theta_{\psi,W,V_0}(\sigma)\neq 0 \Longleftrightarrow \mathcal{P}_{N,\psi} \text{ is nonzero on } \sigma,$$

i.e. iff σ has nonzero ψ-th Fourier coefficient.

As before, this proposition merely transfers the nonvanishing of the global theta lift to a question about periods on $\mathrm{Mp}(W)$. Since we do not at this moment know that $\mathrm{Mp}(W)$ has the multiplicity one theorem, it is possible that the nonvanishing of the ψ-th Fourier coefficient depends on the automorphic realization of σ and not only on the isomorphism class of σ. However, we can deduce the following corollary:

Corollary 6.2. *If σ is a cuspidal representation of $\mathrm{Mp}(W_{\mathbb{A}})$, then its global Waldspurger lift $Wd_\psi(\sigma)$ is a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$.*

Proof. There is an $a \in F^\times$ such that σ has nonzero ψ_a-th Fourier coefficient. Then the proposition implies that $\Theta_{\psi_a,W,V_0}(\sigma)$ is a nonzero cuspidal representation of PGL_2. Moreover,

$$Wd_\psi(\sigma) = \Theta_{\psi_a,W,V_0}(\sigma) \otimes \chi_a.$$

Thus, $Wd_\psi(\sigma)$ is a cuspidal representation as well. □

6.4. **Local analog.** Performing the analogous local computation, we obtain:

Proposition 6.3. *With the local notation of §2, one has an isomorphism of* $\mathrm{Mp}(W)$*-modules:*

$$(\omega_{\psi,W,V_0})_{U_0,\psi} \cong ind_N^{\mathrm{Mp}(W)}\psi.$$

Thus, if $\sigma \in \mathrm{Irr}(\mathrm{Mp}(W))$*, then* $\Theta_{\psi,W,V_0}(\sigma)$ *is nonzero if and only if* σ *has nonzero* ψ*-Whittaker functional.*

6.5. **Nonvanishing of global theta lifts.** Combining the results of this and the previous section, we can now obtain a useful nonvanishing criterion for global theta lifts from $\mathrm{SO}(V_0) \cong \mathrm{PGL}_2$ in terms of representation theoretic invariants.

Theorem 6.4. *Let* π *be a cuspidal representation of* $\mathrm{SO}(V_0) \cong \mathrm{PGL}_2$*. The following are equiavalent:*

- (i) $L(1/2,\pi) \neq 0$;
- (ii) π *has nonzero split torus period;*
- (iii) π *has nonzero period over some torus* T_a*;*
- (iv) $\Theta_{\psi,W,V_0}(\pi) \neq 0$.

Proof. The equivalence of (i) and (ii) is Corollary 5.4. That (ii) implies (iii) is clear. That (iii) implies (iv) is Proposiiton 5.2. Finally, suppose that $\sigma := \Theta_{\psi,W,V_0}(\pi) \neq 0$. Then $\Theta_{\psi,W,V_0}(\sigma)$ is nonzero, and is not orthogonal to π. By the multiplicity one theorem for PGL_2, we must have $\pi = \Theta_{\psi,W,V_0}(\sigma)$. Now Proposition 6.1 implies that σ has nonzero ψ-th Whittaker coefficient. Hence, Proposition 5.2 implies that π has nonzero split torus period, and Corollary 5.4 implies that $L(1/2,\pi) \neq 0$. □

This theorem gives a complete understanding of the nonvanishing of global theta lifts from $\mathrm{SO}(V_0)$ to $\mathrm{Mp}(W)$. However, at this point, we still do not have good understanding of the global theta liftings from a general $\mathrm{SO}(V)$ to $\mathrm{Mp}(W)$. This will be rectified in the following section.

7. **Rallis Inner product Formula**

In this section, we shall study the nonvanishing of theta lifts by a method which is more along the lines of the analogous nonvanishing results in the local setting. This alternative argument is due to Rallis.

The point is that: to see if the cusp form $\theta(\phi,f)$ is nonzero, we can compute its Petersson inner product:

$$\langle \theta(\phi,f), \theta(\phi,f)\rangle = \int_{\mathrm{Sp}(W_F)\backslash \mathrm{Sp}(W_{\mathbb{A}})} \theta(\phi,f)(g) \cdot \overline{\theta(\phi,f)(g)}\, dg.$$

The question is: how does one compute this more explicitly?

7.1. **The doubling see-saw.** Consider the doubled quadratic space

$$\mathbb{V} = V + V^-$$

where V^- is the quadratic space $(V, -q)$. Then observe that $\mathbb{V}$ is the split quadratic space, even if V is not. Indeed, the subspaces

$$V^\Delta = \{(v, v) : v \in V\} \quad \text{and} \quad V^\nabla = \{(v, -v) : v \in V\},$$

are maximal isotropic subspaces such that

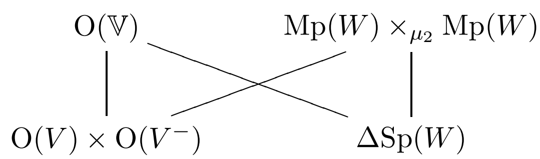
$$\mathbb{V} = V^\Delta \oplus V^\nabla.$$

Thus, $\mathrm{O}(\mathbb{V})$ is a split orthogonal group containing $\mathrm{O}(V) \times \mathrm{O}(V^-)$ as a subgroup. We shall identify $\mathrm{O}(V^-)$ with $\mathrm{O}(V)$ by the "identity map".

Now note that in the symplectic group $\mathrm{Sp}(W \otimes \mathbb{V})$, we have the following two dual pairs:

$$\mathrm{Sp}(W) \times \mathrm{O}(\mathbb{V}) \quad \text{and} \quad (\mathrm{Sp}(W) \times \mathrm{Sp}(W)) \times (\mathrm{O}(V) \times \mathrm{O}(V^-)).$$

The key mechanism for computing the above inner product is the following see-saw diagram of dual pairs, which is known as the doubling see-saw:



In addition, as representations of $\mathrm{O}(V) \times \mathrm{O}(V) \times \Delta\mathrm{Sp}(W)$ realized on

$$S(f \otimes \mathbb{V}) = S(f \otimes V) \otimes S(f \otimes V^-),$$

we have:

$$\omega_{\psi,W,\mathbb{V}} = \omega_{\psi,W,V} \otimes \omega_{\psi,W,V^-} \cong \omega_{\psi,W,V} \otimes \overline{\omega_{\psi,W,V}}.$$

7.2. **Formal computation.** Starting with cusp forms $f_1 \in \pi$ and $\overline{f_2} \in \overline{\pi}$ on $\mathrm{O}(V)$, we consider their theta lifts $\theta(\phi_1, f_1)$ and $\overline{\theta(\phi_2, \phi_2)}$ on $\mathrm{Mp}(W)$ and integrate their product over $\Delta\mathrm{Sp}(W)$. This is a computation happening on the RHS of the see-saw diagram. By an exchange of order of integration, we shall transfer this to a computation on the LHS of the see-saw. More precisely, we have:

$$\begin{aligned}
&\langle \theta(\phi_1, f_1), \theta(\phi_2, f_2) \rangle \\
&= \int_{\mathrm{Sp}(W_F)\backslash \mathrm{Sp}(W_\mathbb{A})} \theta(\phi_1, f_1)(g) \cdot \overline{\theta(\phi_2, f_2)(g)}\, dg \\
&= \int_{\mathrm{Sp}(W_F)\backslash \mathrm{Sp}(W_\mathbb{A})} \int_{\mathrm{O}(V_F)^2\backslash \mathrm{O}(V_\mathbb{A})^2} \theta(\phi_1)(g, h_1) \cdot \overline{f_1(h_1)} \cdot \overline{\theta(\phi_2)(g, h_2)} \cdot f_2(h_2)\, dh_1\, dh_2, dg \\
&= \int_{\mathrm{O}(V_F)^2\backslash \mathrm{O}(V_\mathbb{A})^2} \overline{f_1(h_1)} \cdot f_2(h_2) \cdot \left(\int_{\mathrm{Sp}(W_F)\backslash \mathrm{Sp}(W_\mathbb{A})} \theta_{\psi,\mathbb{V},W}(\phi_1 \otimes \overline{\phi_2})(g, (h_1, h_2))\, dg \right) dh_1\, dh_2
\end{aligned}$$

Here, for the last equality, we have exchanged the order of integration, and the inner integral is the global theta lift of the constant function 1 on $\Delta\mathrm{Sp}(W)$ to the group $\mathrm{O}(\mathbb{V})$. However, it turns out that this inner integral does not converge, so that this exchange is not justified.

At the moment, let us proceed formally and pretend that it is convergent; the next task is then to identify the inner integral as an automorphic form on $O(\mathbb{V})$, or rather to give an alternative construction of this automorphic form.

7.3. **Degenerate principal series.** Let us denote the inner integral by

$$I(\phi_1 \otimes \overline{\phi_2})(h) = \int_{\mathrm{Sp}(W_F)\backslash \mathrm{Sp}(W_{\mathbb{A}})} \theta_{\psi,\mathbb{V},W}(\phi_1 \otimes \overline{\phi_2})(g,(h_1,h_2))\, dg.$$

Then I defines a map

$$S(f \otimes \mathbb{V}) \longrightarrow \mathcal{A}(\mathrm{O}(\mathbb{V})),$$

which is $\Delta\mathrm{Sp}(W_{\mathbb{A}})$-invariant and $\mathrm{O}(\mathbb{V})$-equivariant. Thus, it must factor through the projection

$$\omega_{\psi,W,\mathbb{V}} \longrightarrow (\omega_{\psi,W,\mathbb{V}})_{\mathrm{Sp}(W)}$$

This last projection can be described concretely as follows.

By taking Fourier transform, we have an isomorphism

$$\iota : S(\mathbb{V}_{\mathbb{A}}) \longrightarrow S(W_{\mathbb{A}} \otimes V_{\mathbb{A}}^{\nabla}).$$

For $\varphi \in S(f \otimes \mathbb{V})$, we consider the function on $\mathrm{O}(\mathbb{V}_{\mathbb{A}})$ defined by

$$\mathcal{F}_{\phi}(h) = (\omega_{\psi,W,\mathbb{V}}(h)\iota(\phi))(0).$$

The function $\mathcal{F}_{\phi}$ lies in a degenerate principal series representation of $\mathrm{O}(\mathbb{V}_{\mathbb{A}})$. Namely, the stabilizer $P(V^{\Delta}) = M(V^{\Delta}) \cdot N(V^{\Delta})$ of the maximal isotropic subspace V^{Δ} is a Siegel parabolic subgroup with Levi factor

$$M(V^{\Delta}) \cong \mathrm{GL}(V^{\Delta}).$$

For a character χ of $\mathrm{GL}(V_{\mathbb{A}}^{\Delta})$ and $s \in \mathbb{C}$, we may consider the degenerate principal series representation of $\mathrm{O}(\mathbb{V}_{\mathbb{A}})$:

$$I_{P(V^{\Delta})}(s,\chi) = \mathrm{Ind}_{P(V^{\Delta})}^{\mathrm{O}(\mathbb{V}}\chi \cdot |\det_{V^{\Delta}}|^{s}.$$

If χ is trivial, we shall simply write $I(s)$. Then, from the definition of the Weil representation, it is easy to see that

$$\mathcal{F}_{\phi} \in I(0).$$

Thus, we have a map

$$\mathcal{F} : \omega_{\psi,W,\mathbb{V}} \longrightarrow I(0)$$

which is $\mathrm{Sp}(W)$-invariant and $\mathrm{O}(\mathbb{V}_{\mathbb{A}})$-equivariant.

A result of Rallis says that the image of $\mathcal{F}$ is isomorphic to $(\omega_{\psi,W,\mathbb{V}})_{\mathrm{Sp}(W)}$. We can describe the image of $\mathcal{F}$ very concretely, once we know the structure of the representation $I(0)$. This is given by:

Lemma 7.1. *For each place v, $I_v(0)$ is the direct sum of two irreducible representations, each of which is the twist of the other by the determinant character* det *of $\mathrm{O}(\mathbb{V}_v)$. The image of $\mathcal{F}_v$ is the unique irreducible submodule which contains nonzero spherical vectors for a hyperspecial maximal copact subgroup of $\mathrm{O}(\mathbb{V})_v$.*

7.4. **Eisenstein series.** Now there is a standard way of defining an intertwining map from $I(s)$ to $\mathcal{A}(\mathrm{O}(\mathbb{V}))$: the theory of Eisenstein series. For the case at hand, the theory of Eisenstein series gives a map

$$E(s,-) : I(s) \longrightarrow \mathcal{A}(\mathrm{O}(\mathbb{V})),$$

and we shall write E for $E(0,-)$. Composing with the map $\mathcal{F}$, we obtain the map

$$\mathcal{E} : \omega_{\psi,W,\mathbb{V}} \longrightarrow \mathcal{A}(\mathrm{O}(\mathbb{V}))$$
$$\phi \mapsto E(\mathcal{F}_\phi),$$

which is $\mathrm{Sp}(W)$-invariant and $\mathrm{O}(\mathbb{V})$-equivariant.

7.5. **Siegel-Weil formula.** The Siegel-Weil formula says that for any $\phi \in S(f \otimes \mathbb{V}_{\mathbb{A}})$, one has an identity

$$I(\phi) = \mathcal{E}(\phi).$$

7.6. **Doubling zeta integral.** Continuing with our computation of the inner product of global theta lifts, and using the Siegel-Weil formula to replace the inner integral $I(\phi_1 \otimes \overline{\phi_2})$ by $\mathcal{E}(\phi_1 \otimes \overline{\phi_2})$, we see that

$$\langle \theta(\phi_1, f_1), \theta(\phi_2, f_2) \rangle = \int_{\mathrm{O}(V_F)^2 \backslash \mathrm{O}(V_{\mathbb{A}})^2} \overline{f_1(h_1)} \cdot f_2(h_2) \cdot \mathcal{E}(\phi_1 \otimes \overline{\phi_2})(h_1, h_2)\, dh_1\, dh_2.$$

We are thus led to considering the so-called doubling zeta integral:

$$Z(s, \mathcal{F}, f_1, f_2) := \int_{\mathrm{O}(V_F)^2 \backslash \mathrm{O}(V_{\mathbb{A}})^2} \overline{f_1(h_1)} \cdot f_2(h_2) \cdot E(s, \mathcal{F}, (h_1, h_2))\, dh_1\, dh_2,$$

for all standard sections $\mathcal{F}_s \in I(s)$ and $f_1, f_2 \in \pi$. This is a meromorphic function in s.

This doubling zeta integral was studied by Piatetski-Shapiro and Rallis, who showed:

Theorem 7.2. *For a sufficiently large set S of places of F, one has*

$$Z(s, \mathcal{F}, f_1, f_2) = \prod_{v \in S} Z_v(s, \mathcal{F}_v, f_{1,v}, f_{2,v}) \cdot L^S(s + \frac{1}{2}, \pi),$$

where the local zeta integral Z_v is given for $Re(s)$ sufficiently large by

$$Z_v(s, \mathcal{F}_v, f_{1,v}, f_{2,v}) = \int_{\mathrm{O}(V_v)} \mathcal{F}_v(h, 1) \cdot \overline{\langle f_{1,v}, f_{2,v} \rangle_v}\, dh$$

with $\langle -, - \rangle_v$ an invariant inner product on π_v.

We note:

Lemma 7.3. *The global zeta integral $Z(s)$ is holomorphic at $s = 0$. If π_v is the local component of a cuspidal representation π (so that π_v is close to being tempered), the local zeta integral Z_v is convergent at $s = 0$.*

In particular, the local zeta integral $Z_v(0)$ defines a *nonzero* linear functional

$$Z_v(0) : I_v(0) \otimes \overline{\pi} \otimes \pi \longrightarrow \mathbb{C},$$

which is $\mathrm{O}(V_v) \times \mathrm{O}(V_v^-)$-invariant.

7.7. **End of computation.** Back to our computation of the inner product, we now have

Rallis inner product formula

$$\langle \theta(\phi_1, f_1), \theta(\phi_2, f_2)\rangle = Z(0, \mathcal{F}_{\phi_1 \otimes \overline{\phi_2}}, f_1, f_2) = \left(\prod_{v \in S} Z_v(0, \mathcal{F}_{\phi_{1,v} \otimes \overline{\phi_{2,v}}}, f_{1,v}, f_{2,v})\right) \cdot L^S(1/2, \pi).$$

Thus we conclude:

Proposition 7.4. *The global theta lift* $\Theta_{\psi,W,V}(\pi)$ *is nonzero if and only if*

- $L^S(1/2, \pi) \neq 0$*;*
- *for all* $v \in S$*,* Z_v *is nonzero on the submodule*

$$\mathrm{Im}\mathcal{F} \otimes \overline{\pi} \otimes \pi \subset I_v(0) \otimes \overline{\pi} \otimes \pi.$$

7.8. **The two conditions.** Let us study these two conditions in the proposition more closely. Firstly, because we know that π is close to tempered, there is no difference between the analytic properties of $L^S(s, \pi)$ and $L(s, \pi)$ at $s = 1/2$. Thus the first condition is equivalent to $L(1/2, \pi) \neq 0$. For the second condition, we note that

Lemma 7.5. *The local zeta integral* $Z_v(0)$ *does not vanish on* $\mathrm{Im}\mathcal{F} \otimes \overline{\pi_v} \otimes \pi_v$ *if and only if* $\Theta_{\psi_v, W_v, V_v}(\pi_v) \neq 0$*.*

Proof. The local see-saw identity gives:

$$\mathrm{Hom}_{\Delta \mathrm{Sp}(W_v)}(\Theta_{\psi_v,W_v,V_v}(\pi_v) \otimes \overline{\Theta_{\psi_v,W_v,V_v}(\pi_v)}, \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{O}(V_v) \times \mathrm{O}(V_v^-)}(\mathrm{Im}\mathcal{F}, \pi \otimes \overline{\pi}).$$

Thus, it is clear that if $Z_v(0)$ is nonzero on $\mathrm{Im}\mathcal{F} \otimes \overline{\pi_v} \otimes \pi_v$, then $\Theta_{\psi_v,W_v,V_v}(\pi_v) \neq 0$. Conversely, if $Z_v(0)$ vanishes on the given submodule, then $Z_v(0)$ is nonzero on $(\mathrm{Im}\mathcal{F}) \otimes \det$, which in turn implies that $\Theta_{\psi_v,W_v,V_v}(\pi_v \otimes \det) \neq 0$. By the main local theorem, this implies that $\Theta_{\psi_v,W_v,V_v}(\pi_v) = 0$. □

7.9. **Nonvanishing of global theta lifts.** To conclude, we have shown:

Theorem 7.6. *Let* π *be a non-trivial cuspidal representation of* $\mathrm{SO}(V)$*. Then* $\Theta_{\psi,W,V}(\pi) \neq 0$ *if and only if* $L(1/2, \pi) \neq 0$*.*

Proof. In Proposition 4.2, we have seen that if $\epsilon(1/2, \pi) = -1$, then $\Theta_{\psi,W,V}(\pi) = 0$, and of course $L(1/2, \pi) = 0$ also. Thus, we assume $\epsilon(1/2, \pi) = 1$. In this case, one chooses the unique automorphic extension π_{S_0} of π to $\mathrm{O}(V_{\mathbb{A}})$ such that each $\pi_{S_0,v}$ has nonzero theta lift to $\mathrm{Mp}(W_v)$. Then the assertion follows from the previous Proposition and Lemma. □

7.10. **Regularization.** Now the reader will no doubt complain that our proof of the theorem relies on an invalid exchange of order of integration in the computation of the inner product. We will close this section by explaining how one overcomes this analytic difficulty by regularizing the inner integral $I(\phi)$ so that the ensuing computation makes sense.

Let's consider the integral

$$I(\phi)(h) = \int_{\mathrm{Sp}(W_F) \backslash \mathrm{Sp}(W_{\mathbb{A}})} \theta(\phi)(g, h)\, dg$$

for $\phi \in S(f \otimes \mathbb{V})$. It turns out that there is a nonzero $\mathrm{Sp}(W) \times \mathrm{O}(\mathbb{V})$- submodule

$$\Sigma \subset S(f \otimes \mathbb{V})$$

such that $I(\phi$ converges if $\phi \in \Sigma$. Indeed, we have:

Lemma 7.7. *Choose a place v_0 of F.*

(i) Let Σ_{v_0} denote the subspace of those $\phi_{v_0} \in S(f \otimes \mathbb{V}_{v_0})$ such that $\omega_{\psi_{v_0}, W_{v_0}, \mathbb{V}_{v_0}}(g)\phi$ vanishes at 0 for all $g \in \mathrm{Sp}(W_{v_0})$. Then Σ_{v_0} is a nonzero $\mathrm{Sp}(W_{v_0}) \times \mathrm{O}(\mathbb{V}_{v_0})$- submodule.

(ii) Set

$$\Sigma = \Sigma_{v_0} \otimes (\otimes_{v \neq v_0} S(f \otimes \mathbb{V}_v))$$

Then for any $\phi \in \Sigma$, ϕ is a rapidly decreasing function on $\mathrm{Sp}(W_F)\backslash\mathrm{Sp}(W_{\mathbb{A}})$ so that $I(\phi)(h)$ converges absolutely.

Now we shall see that one can extend I from Σ to the whole of $S(f \otimes \mathbb{V})$. Let's reformulate the definition of Σ slightly. Consider the map

$$J : S(f \otimes \mathbb{V}_{v_0}) \longrightarrow \{\text{Functions on } \mathrm{Sp}(W_{v_0})\}$$

defined by

$$J(\phi)(g) = \omega_{\psi, W, \mathbb{V}}(g)\phi(0),$$

so that $\Sigma_{v_0} = \mathrm{Ker} J$ by definition. It is easy to check that the image of J is contained in the irreducible principal series representation

$$I(2) = \mathrm{Ind}_B^{\mathrm{SL}_2} |-|^2 \quad \text{of } \mathrm{Sp}(W_{v_0}) \text{ (normalized induction).}$$

Since this irreducible unramified representation is distinct from the trivial representation of $\mathrm{Sp}(W_{v_0})$, one can find an element z_0 of the Bernstein center of $\mathrm{Sp}(W_{v_0})$ such that $z_0 = 1$ on the trivial representation, but $z_0 = 0$ on $I(2)$. The element z_0 gives a map

$$z_0 : S(f \otimes \mathbb{V}) \longrightarrow \Sigma$$

and we set

$$I^{reg}(\phi) = I(z_0 \cdot \phi).$$

It was shown by Ichino that that I^{reg} is the unique equivariant extension of I on Σ, so that the choice of z_0 is really not serious.

The map I^{reg} is the *regularized* theta integral, and the *regularized Siegel-Weil formula* is the identity

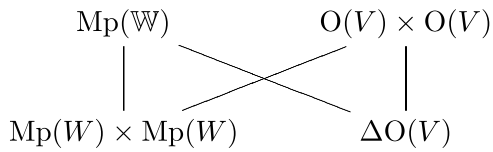
$$I^{reg}(\phi) = \mathcal{E}(\phi).$$

Given this, the way to make sense of the computation of the inner product above is to start with the expression:

$$\int_{\mathrm{O}(V_F)^2\backslash \mathrm{O}(V_{\mathbb{A}})^2} \overline{f_1(h_1)} \cdot f_2(h_2) \cdot \left(\int_{\mathrm{Sp}(W_F)\backslash\mathrm{Sp}(W_{\mathbb{A}})} \theta_{\psi, \mathbb{V}, W}(z_0(\phi_1 \otimes \overline{\phi_2}))(g, (h_1, h_2))\, dg \right) dh_1\, dh_2$$

which is absolutely convergent, and then derive two different expressions by performing the two integrals in different orders. We omit the details here.

7.11. **From** $\mathrm{Mp}(W)$ **to** $\mathrm{SO}(V)$**.** We can consider the theta lifting from $\mathrm{Mp}(W)$ to $\mathrm{SO}(V)$ using a similar idea based on the see-saw:



Then one obtains:

Theorem 7.8. *Given a cuspidal represetation* σ *of* $\mathrm{Mp}(W)$*,* $\Theta_{\psi,W,V}(\sigma) \neq 0$ *if and only if the following conditions hold:*

- $L(1/2, \sigma, \psi) \neq 0$*;*
- *for all* v*,* $\theta_{\psi_v, W_v, V_v}(\sigma_v) \neq 0$*.*

Here $L(s, \sigma, \psi)$ *is the* L*-function associated to* (σ, ψ) *by the doubling method, and one has*

$$L(s, \sigma, \psi) = L(s, Wd_{\psi}(\sigma)).$$

If we consider the theta correspondence for $\mathrm{Mp}(W) \times \mathrm{O}(V_0)$ and combine this theorem with Proposition 6.1 (which gives another nonvanishing criterion for global theta lifting) and its local analog Proposition 6.3, we obtain the following important theorem:

Theorem 7.9. *Let* σ *be a cuspidal representation of* $\mathrm{Mp}(W)$*. Then* $\mathcal{P}_{U_0,\psi}$ *is nonzero on* σ *if and only if*

- $L(1/2, \sigma, \psi) \neq 0$*;*
- *for all* v*,* σ_v *has nonzero* ψ_v*-Whittaker functional.*

The importance of this theorem lies in the fact that it characterizes the nonvanishing of a period, which a priori depends on the automorphic realization of σ in terms of a representation theoretic invariant, which depends only on the isomorphism class of σ. We shall apply this theorem immediately at the beginning of the next section.

8. Global Shimura Correspondence

We can finally assemble the results of the previous sections to give a rather precise description of the discrete spectrum of $\mathrm{Mp}(W_{\mathbb{A}})$.

8.1. **Multiplicity One Theorem.** We first show that $L^2_{disc}(\mathrm{Mp}(W))$ is multiplicity-free, since it is a consequence of the last theorem of the previous section. This is clear for the submodule (ETF), so we focus on the orthogonal complement.

It is instructive to recall the proof of the multiplicty one theorem for $L^2_{cusp}(\mathrm{PGL}_2)$. For an abstractly generic representation π of $\mathrm{PGL}_2(\mathbb{A})$, composition with the Fourier coefficient map gives:

$$\mathrm{Hom}_{\mathrm{PGL}_2(\mathbb{A})}(\pi, L^2_{cusp}(\mathrm{PGL}_2)) \longrightarrow \mathrm{Hom}_{U_0(\mathbb{A})}(\pi, \mathbb{C}_{\psi}).$$

Now we note:

- the target space is 1-dimensional by the local uniqeness of Whittaker functionals;
- the map is injective, since every cuspidal representation has nonzero ψ-th Foureir coefficient.

To adapt this argument to $\mathrm{Mp}(W)$, suppose that $\mathrm{Hom}_{\mathrm{Mp}(W)}(\sigma, L^2_{cusp}(\mathrm{Mp}(W))) \neq 0$. Choose $a \in F^\times$ such that some automorphic realization of σ has nonzero ψ_a-th Fourier coefficient. Composition with the Fourier coefficient map $\mathcal{P}_{U_0,\psi_a}$ gives a nonzero map

$$\mathrm{Hom}_{\mathrm{Mp}(W)}(\sigma, \mathcal{A}(\mathrm{Mp}(W))) \longrightarrow \mathrm{Hom}_{N(\mathbb{A})}(\sigma, \psi_a).$$

The latter space is 1-dimensional by the local uniqueness of Whittaker functionals. Thus it remains to show that this map is injective, or equivalently, that any automorphic realization of σ in $L^2_{cusp}(\mathrm{Mp}(W))$ has nonzero ψ_a-th Fourier coefficient. This follows from Theorem 7.9, which says that the nonvanishing of the ψ_a-th Fourier coefficient depends only on the isomorphism class of σ.

Since it is easy to see that (ETF) is spectrally disjoint from its orthogonal complement, we have shown:

Theorem 8.1. *The representation* $L^2_{disc}(\mathrm{Mp}(W))$ *is multiplicity-free.*

8.2. **Global Waldspurger packet.** If $\pi = \otimes_v \pi_v$ is a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$, then recall that for each v, we have already defined the local Waldspurger packet

$$A_{\psi_v}(\pi_v) = \{\sigma_v^+, \sigma_v^-\}$$

where σ_v^- is interpreted as 0 if π_v is not discrete series. We have defined the global Waldspurger packet to be

$$A_\psi(\pi) = \{\sigma^\epsilon := \bigotimes_v \sigma_v^{\epsilon_v} : \epsilon_v = + \text{ for almost all } v\}.$$

It is a finite set of cardinality $2^{\# S_\pi}$, where S_π is the set of places v of F such that π_v is discrete series. We have also shown:

Proposition 8.2. *If* σ^ϵ *is automorphic, then we must have:*

$$\prod_v \epsilon_v = \epsilon(1/2, \pi).$$

What is not immediately clear is the converse. Note also that for any $a \in F^\times$, we have

$$A_{\psi_a}(\pi \otimes \chi_a) = A_\psi(\pi).$$

8.3. **Near equivalence class.** Our next goal is to show that if the condition of the above proposition holds, then σ^ϵ is automorphic.

For a cuspidal representation π of $\mathrm{PGL}_2(\mathbb{A})$, we consider the submodule of $L^2_{disc}(\mathrm{Mp}(W))$

$$L^2_\pi := \sum_{a \in F^\times / F^{\times 2}} \Theta_{\psi_a}(\pi \otimes \chi_a).$$

We have the following key theorem:

Theorem 8.3. *(i) Any irreducible cuspidal representation of* $\mathrm{Mp}(W)$ *is contained in some* L^2_π*, so that*

$$L^2_{disc}(\mathrm{Mp}(W)) \cong (ETF) \oplus \bigoplus_\pi L^2_\pi,$$

where the sum ranges over all cuspidal representation π *of* $\mathrm{PGL}_2(\mathbb{A})$ *such that*

$$L(1/2, \pi \otimes \chi_a) \neq 0 \quad \textit{for some } a \in F^\times.$$

(ii) The submodule L^2_π *is a full near equivalence class and each irreducible summand in* L^2_π *is isomorphic to an element of* $A_\psi(\pi)$*. Indeed,*

$$L^2_\pi \cong \bigoplus_{\epsilon : \prod_v \epsilon_v = \epsilon(1/2,\pi)} \sigma^\epsilon.$$

Proof. (i) Given a cuspidal σ, there is an $a \in F^\times$ such that σ has nonzero ψ_a-th Fourier coefficient. Then $\Theta_{\psi_a,W,V_0}(\sigma)$ is nonzero cuspidal. If we set $\pi := \Theta_{\psi_a}(\sigma) \otimes \chi_a$, we see that σ is not orthogonal to $\Theta_{\psi_a,W,V_0}(\pi \otimes \chi_a)$, and hence is isomorphic to it. By the multiplicity one theorem for $\mathrm{Mp}(W)$, σ must be equal to $\Theta_{\psi_a,W,V_0}(\pi \otimes \chi_a)$ as a space of automorphic forms, so that $\sigma \subset L^2_\pi$.

(ii) If σ is nearly equivalent to the elements of L^2_π, then the argument in (i) shows that for some $a \in F^\times$, $\Theta_{\psi_a,W,V_0}(\sigma) \otimes \chi_a$ is a nonzero cuspidal representation, which is nearly equivalent to π. The strong multiplicity one theorem for PGL_2 implies that $\pi = \Theta_{\psi_a,W,V_0}(\sigma) \otimes \chi_a$. Then it follows as in (i) that $\sigma \subset L^2_\pi$.

It is clear that the irreducible summands in L^2_π are in $A_\psi(\pi)$. Moreover, for the last displayed equality, we have noted that the LHS is contained in the RHS. Suppose that $L(1/2, \pi \otimes \chi_a) \neq 0$. We shall consider the set

$$\mathcal{B}_{\pi \otimes \chi_a} = \{\text{quaternion } F\text{-algebra } B\colon\ S_B \subset S_{\pi \otimes \chi_a}\}.$$

Since $\#S_B$ is necessarily even, we see that

$$\#\mathcal{B}_{\pi \otimes \chi_a} = \#\mathcal{B}_\pi = \begin{cases} 2^{\#S_\pi - 1}, & \text{if } \#S_\pi \neq 0; \\ 1, & \text{if } \#S_\pi = 0. \end{cases}$$

For each $B \in \mathcal{B}_\pi$, we may consider the Jacquet-Langlands transfer π_B of π to $PB^\times$. Then we may consider the global theta lift $\Theta_{\psi_a,W,V_B}(\pi_B \otimes \chi_a)$ of π_B to $\mathrm{Mp}(W_\mathbb{A})$. By Theorem, we see that

$$\Theta_{\psi,W,V_B}(\pi_B) \neq 0$$

and is cuspidal. We have thus produced $\#A_\psi(\pi)$ elements of L^2_π. This shows that the RHS is contained in the LHS. □

8.4. **A key analytic input.** At this point, we have almost proved the main global theorem (Theorem 3.3), excep that in Theorem 3.3, the sum runs over all cuspidal representations π of PGL_2 whereas in the above theorem, it runs over π such that $L(1/2, \pi \otimes \chi) \neq 0$ for some quadratic character χ. This extra condition on π is not a very good condition, because it is not something which is easy to check. Thus the last remaining issue is to show that this extra condition on π is not necessary.

Observe that a necessary condition for the existence of χ such that $L(1/2, \pi \otimes \chi) \neq 0$ is the existence of χ such that $\epsilon(1/2, \pi \otimes \chi) = 1$, which is a much more reasonable condition. It turns out that they are in fact equivalent! This is not easy at all, and is an input from analytic number theory, namely the following theorem of Friedberg-Hoffstein.

Theorem 8.4. *Let π be a cuspidal representation of* PGL_2*, and suppose that there is a quadratic Hecke character χ_0 such that*

$$\epsilon(1/2, \pi \otimes \chi_0) = 1.$$

Then there exists infinitely many quadratic Hecke characters χ such that

$$L(1/2, \pi \times \chi) \neq 0.$$

Moreover, one can require χ to be fixed at a given finite set of places.

Corollary 8.5. *The following are equivalent:*

- *there is a quadratic χ such that $L(1/2, \pi \otimes \chi) \neq 0$;*
- *there is a quadratic χ such that $\epsilon(1/2, \pi \otimes \chi) = 1$.*

In particular, in Theorem 8.3(ii), the sum over π can be taken over those π satisfying the second condition.

We consider now the π's when

$$\epsilon(1/2, \pi \otimes \chi) = -1 \quad \text{for all quadratic } \chi.$$

For this, we note:

Lemma 8.6. *If π is such that $\epsilon(1/2, \pi \otimes \chi) = -1$ for all quadratic χ, then S_π is empty, so that $\# A_\psi(\pi) = 1$. Moreover, for the unique $\sigma^+ \in A_\psi(\pi)$, $\prod_v \epsilon_v = 1 \neq \epsilon(1/2, \pi)$.*

What the lemma says is that there is no harm including these π in the sum in Theorem 8.3(ii) since $m(\sigma^+) = 0$ for the unique $\sigma^+ \in A_\psi(\pi)$. Thus, we have completed the proof of Theorem 3.3.

9. Local Torus Periods and Root Numbers

We have seen that nonvanishing of global theta lifts from $\mathrm{SO}(V)$ to $\mathrm{Mp}(W)$ is governed by two different sets of conditions: the nonvanishing of $L(1/2, \pi)$ and the nonvanishing of the periods over $\mathrm{SO}(V^a)$ for some $a \in F^\times$. This certainly suggests that the periods over $\mathrm{SO}(V^a)$ is related to central L-value, and we have seen this in the case when $a \in F^{\times 2}$ so that $\mathrm{SO}(V^a)$ is the split torus. In the final two sections, we investigate this connection for general $\mathrm{SO}(V^a) \subset \mathrm{SO}(V)$. It turns out that this is a very subtle and rich question, which is finer than the question of nonvanishing of theta lifts. I did not have time to cover these topics in the course, but I include then here as bonus material. In the investigation of these topics, it turns out that we shall need to consider an extension of the theta correspondence for *similitude groups.*

9.1. Torus periods.

If we take $V = V_B = (B_0, -N_B)$, so that $\mathrm{SO}(V) = PB^\times$, and take $x_a \in B_0$ with $x^2 = a$, then we obtain an algebra embedding

$$F(a) \hookrightarrow B,$$

and the subgroup $\mathrm{SO}(V^a)$ is equal to the 1-dimensional torus

$$T_a = \text{image of } F(a)^\times \text{ in } PB^\times.$$

Thus, if we regard π as a cuspidal representation of $PB_{\mathbb{A}}^{\times}$, then the period $\mathbb{P}_{\mathrm{SO}(V^a)}$ is a torus period.

In the final two sections, we shall discuss this question of torus period in detail. As in any adelic question, there is an analogous local question which one should really try to understand first. In this case, the local problem is to determine the Hom space

$$\mathrm{Hom}_{T_a}(\pi_B, \mathbb{C})$$

when π_B is an irreducible representation of $PB^{\times}$.

9.2. **Local torus periods.** Since we are working locally, we shall revert to the notation of §2. Given an irreducible representation π of $\mathrm{PGL}_2(k)$, let π_D be its Jacquet-Langlands transfer to $PD^{\times}$ (interpreted as 0 if it does not exist). For any $a \in k^{\times}$, we have the torus embedding

$$T_a \hookrightarrow PD^{\times}$$

unless $a \in k^{\times 2}$. The following special case of a result of Saito-Tunnell answers the local question completely.

Theorem 9.1. *Let π be an infinite-dimensional representation of $\mathrm{PGL}_2(k)$. Then we have:*

$$\dim \mathrm{Hom}_{T_a}(\pi, \mathbb{C}) + \dim \mathrm{Hom}_{T_a}(\pi_D, \mathbb{C}) = 1.$$

Moreover,

$$\mathrm{Hom}_{T_a}(\pi, \mathbb{C}) \neq 0 \Longleftrightarrow \epsilon(1/2, \pi, \psi) \cdot \epsilon(1/2, \pi \otimes \chi_a, \psi) \cdot \chi_a(-1) = 1,$$

and

$$\mathrm{Hom}_{T_a}(\pi_D, \mathbb{C}) \neq 0 \Longleftrightarrow \epsilon(1/2, \pi, \psi) \cdot \epsilon(1/2, \pi \otimes \chi_a, \psi) \cdot \chi_a(-1) = -1,$$

The point is: in the global case, we are trying to use torus periods to help us understand the Whittaker-Fourier coefficients of the theta lift, but in the local case, we have already understood the existence of Whittaker functionals (see Proposition 2.3(ii) and Corollary 5.7, and so we can turn the table around and use this knowledge to understand local torus period.

Proof. The fact that the spaces of local torus periods are of dimension ≤ 1 has been shown in Corollary 5.6. We have also shown a special case of the theorem, when $T_a = T$ is the split torus, in the proof of Corollary 5.7. Thus, it remains to consider the case when T_a is nonsplit. By Proposition 5.5 and Theorem 2.4(ii), we see that

$$\mathrm{Hom}_{T_a}(\pi, \mathbb{C}) \neq 0 \Longleftrightarrow \Theta_{\psi, V^+, W}(\pi)_{N, \psi_a} \neq 0 \Longleftrightarrow \epsilon(1/2, \pi, \psi) \cdot \epsilon(1/2, \pi \otimes \chi_a, \psi) \cdot \chi_a(-1) = 1,$$

and

$$\mathrm{Hom}_{T_a}(\pi_D, \mathbb{C}) \neq 0 \Longleftrightarrow \Theta_{\psi, V^-, W}(\pi_D)_{N, \psi_a} \neq 0 \Longleftrightarrow \epsilon(1/2, \pi, \psi) \cdot \epsilon(1/2, \pi \otimes \chi_a, \psi) \cdot \chi_a(-1) = -1.$$

This proves the theorem. □

9.3. **Theorem of Tunnell-Saito.** The above theorem is actually a special case of the theorem of Tunnell-Saito, which treats the existence of local (T_a, χ)-periods for any character χ of T_a. More precisely, we have:

Theorem 9.2. *Let π be an infinite-dimensional representation of $\mathrm{GL}_2(k)$ and χ a character of the torus $T_a \cong k(a)^\times$ such that $\chi|_{k^\times} = \omega_\pi$. Then we have:*

$$\dim \mathrm{Hom}_{T_a}(\pi, \chi) + \dim \mathrm{Hom}_{T_a}(\pi_D, \chi) = 1.$$

Moreover, if $\pi(\chi)$ is the dihedral representation of $\mathrm{GL}_2(k)$ associated to χ, then

$$\mathrm{Hom}_{T_a}(\pi, \chi) \neq 0 \Longleftrightarrow \epsilon(1/2, \pi \times \pi(\chi)^\vee, \psi) \cdot \chi_a(-1) = 1,$$

and

$$\mathrm{Hom}_{T_a}(\pi_D, \chi) \neq 0 \Longleftrightarrow \epsilon(1/2, \pi \otimes \pi(\chi)^\vee, \psi) \cdot \chi_a(-1) = -1,$$

This theorem is the simplest case of the local Gross-Prasad conjecture, (which has recently been shown by Waldspurger, modulo the work of Arthur). It was first proved by Tunnell by a direct computation, and then given a cleaner proof by H. Saito using the base change character identities. Yet another proof was recently given by D. Prasad, using a global-to-local argument. In the following, we shall give a fourth proof based on theta correspondence and the properties of the local Rankin-Selberg integrals for $\mathrm{GL}(2) \times \mathrm{GL}(2)$. The proof we give below is natural in the sense that it works both locally and globally, thus giving parallel proofs of the local Theorem 9.2 and the global Theorems 10.1 and 10.2 below.

9.4. **Dual pairs.** We come now to the proofs of the above theorems. Consider the 4-dimensional quadratic space (B, N_B). Then the identity component of the associated similitude group

$$\mathrm{GSO}(B) \cong (B^\times \times B^\times)/\Delta\mathbb{G}_m$$

with (b_1, b_2) on the RHS acting on B by

$$(b_1, b_2) : b \mapsto b_1 \cdot b \cdot b_2^{-1}.$$

The similitude group $\mathrm{GO}(B)$ is generated by $\mathrm{GSO}(B)$ together with the element $b \mapsto \overline{b}$ which has determinant -1. Over a local field k, we set $\epsilon(B) = +1$ or -1 depending on whether B splits or not.

With W denoting the 2-dimensional symplectic space as before, one has a dual pair $\mathrm{Sp}(W) \times \mathrm{O}(B)$ in $\mathrm{Sp}(W \otimes B)$ and a similitude version $\mathrm{GSp}(W) \times \mathrm{GO}(B)$.

Given a quadratic k-algebra E, with associated quadratic character χ_E of $k^\times$, fix an embedding $E \hookrightarrow B$ if it exists, and write

$$B = E \oplus E \cdot b,$$

where the conjugation action of b on E is the Galois action. Then we note that

$$\epsilon(B) = \chi_E(-\mathbb{N}b).$$

The two dimensional quadratic subspaces E and $E \cdot b$ have discriminant algebra equal to E. For such a quadratic space V_E, we set $\epsilon(V_E) = +1$ or -1 according to whether V_E is isomorphic to (E, N_E) or not.

In any case, $\mathrm{GSO}(E) \cong \mathrm{GSO}(E \cdot b) \cong E^\times$, with $E^\times$ acting by left multiplication on E and $E \cdot b$, and

$$\iota : (\mathrm{GSO}(E) \times \mathrm{GSO}(E \cdot b))^0 \hookrightarrow \mathrm{GSO}(B) \cong (B^\times \times B^\times)/\Delta k^\times,$$

where the LHS consists of elements $(h_1, h_2) \in E^\times \times E^\times$ such that $N(h_1) = N(h_2)$. The image of this embedding is the subgroup

$$(E^\times \times E^\times)/\Delta k^\times \subset (B^\times \times B^\times)/\Delta k^\times.$$

More precisely ,if

$$(h_1, h_2) \mapsto (b_1, b_2) \in (E^\times \times E^\times)/\Delta k^\times,$$

then

$$h_1 = b_1/b_2 \quad \text{and} \quad h_2 = b_1/b_2^\sigma$$

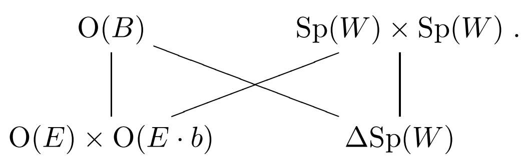
so that

$$h_1/h_2 = b_2^\sigma/b_2 \quad \text{and} \quad h_2^\sigma/h_1 = b_1^\sigma/b_1.$$

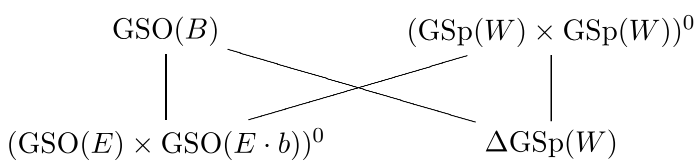
In particular, observe that for a given character χ of $E^\times$

$$(\chi \boxtimes \chi^{-1}) \cdot \iota = \chi \boxtimes 1.$$

9.5. **A see-saw diagram.** In view of the above discussion, one has the following see-saw diagram:



In fact, we shall consider its similitude version:



Since we will be using similitude theta correspondence, we shall review it briefly.

9.6. **Similitude theta correspondence.** For the dual pair $\mathrm{Sp}(W) \times \mathrm{O}(B)$, the Weil representation $\omega_{\psi,W,B}$ can be naturally extended to a slightly bigger group:

$$R_{W,B} = \{(g,h) \in \mathrm{GSp}(W) \times \mathrm{GO}(B) : \det(g) = \mathbb{N}(h)\} \subset \mathrm{GSp}(W) \times \mathrm{GO}(B).$$

Note here that the maps det and $\mathbb{N}$ are both surjective onto $F^\times$. Then one may form the compactly-induced representation

$$\Omega_{W,B} = \mathrm{ind}_{R_{W,B}}^{\mathrm{GSp}(W) \times \mathrm{GO}(B)} \omega_{\psi,W,B}$$

of $\mathrm{GSp}(W) \times \mathrm{GO}(B)$, which is independent of ψ. We may thus consider the theta correspondence for this similitude dual pair:

Proposition 9.3. *If π is an infinite-dimensional representation of* $\mathrm{GSp}(W)$*, then*

$$\Theta_{W,B}(\pi) = \pi_B \boxtimes \pi_B^\vee$$

as a representation of $(B^\times \times B^\times)/\Delta k^\times$*.*

Similarly, for the pair $\mathrm{GO}(E)\times\mathrm{GSp}(W)$ (or $\mathrm{GO}(E\cdot b)\times\mathrm{GSp}(W)$), the Weil representation extends to

$$R_{W,E} = \{(g,h) \in \mathrm{GSp}(W) \times \mathrm{GO}(E) : \det(g) = \mathbb{N}h\}.$$

Note however that the norm map on $E^\times$ is not surjective onto $k^\times$ if E is a field, so that $R_{W,E}$ is actually a subgroup of $\mathrm{GSp}(W)^+ \times \mathrm{GO}(E)$ where

$$\mathrm{GSp}(W)^+ = \{g \in \mathrm{GSp}(W) : \det(g) \in \mathbb{N}E^\times\}.$$

In this case, we consider the representation of $\mathrm{GSp}(W)^+ \times \mathrm{GO}(E)$ defined by:

$$\Omega^+_{W,E} = \mathrm{ind}_{R_{W,E}}^{\mathrm{GSp}(W)^+\times\mathrm{GO}(E)} \omega_{\psi,W,E},$$

and similarly for $\Omega^+_{W,E\cdot b}$. Using these, one may consider theta correspondences for these similitude dual pairs.

It is important to note that we do not induce the Weil representation all the way up to $\mathrm{GSp}(W) \times \mathrm{GO}(E)$ when E is a field, but to a subgroup of index 2. This may seem like a minor technicality, but this annoying minor detail is the *rasion d'être* for the dichotomy of torus period in the local theorem. In any case, the representation $\Omega^+_{W,E}$ depends on the $\mathbb{N}E^\times$-orbit of ψ. If ψ' is an additive character which is in the other $\mathbb{N}E^\times$-orbit, then we set $\Omega^-_{W,E}$ for the induction of $\omega_{\psi',W,E}$. Note that

$$\Omega^+_{W,Eb} \cong \Omega^{\chi_E(\mathbb{N}b)}_{W,E} \quad \text{and} \quad \Omega^-_{W,Eb} \cong \Omega^{-\chi_E(\mathbb{N}b)}_{W,E}.$$

Proposition 9.4. *Consider the theta correspondence for* $\mathrm{GSp}(W)^+ \times \mathrm{GSO}(E)$ *under* $\Omega^\pm_{W,E}$*. If χ is a character of* $E^\times \cong \mathrm{GSO}(E)$*, then*

$$\pi(\chi)|_{\mathrm{GSp}(W)^+} \cong \Theta^+_{W,E}(\chi) \oplus \Theta^-_{W,E}(\chi).$$

Moreover, the constituent $\Theta^+_{W,E}(\chi)$ is characterised as the unique ψ-generic summand.

We shall set

$$\pi(\chi)^+ = \Theta^+_{W,E}(\chi) \quad \text{and} \quad \pi(\chi)^- = \Theta^-_{W,E}(\chi).$$

On the other hand, one has:

Proposition 9.5. *Consider the principal series representation $I(1,\chi_E)$ of* $\mathrm{GSp}(W) \cong \mathrm{GL}_2(k)$*, and write*

$$I(1,\chi_E) = I(1,\chi_E)^+ \oplus I(1,\chi_E)^-$$

with $I(1,\chi_E)^+$ the unique ψ-generic summand. Then under the theta correspondence for $\mathrm{GSp}(W)^+ \times \mathrm{GSO}(Eb)$*, one has*

$$\Theta^+_{W,Eb}(1) = I(1,\chi_E)^{\chi_E(\mathbb{N}b)} \textit{ and } \quad \Theta^-_{W,Eb}(1) = I(1,\chi_E)^{-\chi_E(\mathbb{N}b)}.$$

9.7. **Local see-saw identity.** We can now begin the proof of Theorem 9.2. Consider the space

$$\Sigma_B := \mathrm{Hom}_{\mathrm{GSp}(W)\times H}(\Omega_B, \pi \boxtimes (\chi \boxtimes 1)),$$

where

$$H = (\mathrm{GSO}(E) \times \mathrm{GSO}(Eb))^0.$$

On one hand, it is equal to

$$\mathrm{Hom}_H(\Theta_{W,B}(\pi), \chi \boxtimes 1) \cong \mathrm{Hom}_{E^\times}(\pi_B, \chi) \otimes \mathrm{Hom}_{E^\times}(\pi_B^\vee, \chi^\vee),$$

which is precisely the space we want to understand. On the other hand, by induction in stages,

$$\Omega_{W,B} = \mathrm{ind}_{\mathrm{GSp}(W)^+\times H}^{\mathrm{GSp}(W)\times H} \mathrm{ind}_{(\mathrm{GSp}(W)\times H)^0}^{\mathrm{GSp}(W)^+\times H} \omega_{\psi,W,B}.$$

Thus, by Frobenius reciprocity,

$$\Sigma_B = \mathrm{Hom}_{\mathrm{GL}_2^+\times H}(\mathrm{ind}_{(\mathrm{GL}_2\times H)^0}^{\mathrm{GL}_2^+\times H} \omega_{\psi,W,E} \otimes \omega_{\psi,W,Eb}, \pi \boxtimes \chi \boxtimes 1)$$

which is in turn equal to

$$\mathrm{Hom}_{\mathrm{GL}_2^+}(\Theta^+_{W,E}(\chi) \boxtimes \Theta^+_{W,Eb}(1), \pi).$$

By Frobenius reciprocity again, we obtain

$$\Sigma_B = \mathrm{Hom}_{\mathrm{GSp}(W)}(\mathrm{Ind}_{\mathrm{GSp}(W)^+}^{\mathrm{GSp}(W)}(\Theta^+_{W,E}(\chi) \boxtimes \Theta^+_{W,Eb}(1), \pi).$$

Now we note:

Lemma 9.6. *As a representation of* $\mathrm{GSp}(W)$,

$$\Pi_B := \mathrm{Ind}_{\mathrm{GSp}(W)^+}^{\mathrm{GSp}(W)}(\Theta^+_{W,E}(\chi) \otimes \Theta^+_{W,Eb}(1)) = (\Theta^+_{W,E}(\chi) \otimes \Theta^+_{W,Eb}(1)) + (\Theta^-_{W,E}(\chi) \boxtimes \Theta^-_{W,Eb}(1)).$$

Thus, if we sum over the two possible B's, we obtain

$$\begin{aligned} &\bigoplus_B \mathrm{Hom}_{E^\times}(\pi_B, \chi) \otimes \mathrm{Hom}_{E^\times}(\pi_B^\vee, \chi^{-1}) \\ \cong &\bigoplus_B \mathrm{Hom}_{\mathrm{GSp}(W)}(\Pi_B, \pi) \\ = &\mathrm{Hom}_{\mathrm{GSp}(W)}(\pi(\chi) \otimes I(1, \chi_E) \otimes \pi^\vee, \mathbb{C}). \end{aligned}$$

This latter space of trilinear forms can be rather easily shown to be 1-dimensional (by Mackey theory). Hence we obtain

$$\sum_B \dim \mathrm{Hom}_{E^\times}(\pi_B, \chi) = 1.$$

This proves the first part of Theorem 9.2. Moreover, we have seen that

$$\mathrm{Hom}_{E^\times}(\pi_B, \chi) \neq 0 \Longleftrightarrow \mathrm{Hom}_{\mathrm{GSp}(W)}(\Pi_B, \pi) \neq 0.$$

9.8. **Local Rankin-Selberg integral.** To prove the epsilon dichotomy part of Theorem 9.2, we need to introduce the local Rankin-Selberg integrals for the local factors of $\mathrm{GL}(2)\times\mathrm{GL}(2)$, which was studied by Jacquet.

Start with irreducible generic representations π_1 and π_2 of $\mathrm{GSp}(W)\cong\mathrm{GL}_2(k)$ and let

$$l_1\in\mathrm{Hom}_N(\pi_1,\psi)\quad\text{and}\quad l_2\in\mathrm{Hom}_N(\pi_2,\psi^{-1})$$

be nonzero ψ-Whittaker functionals. Consider also the family of principal series representations

$$I(s,\chi_1,\chi_2):=\pi(\chi_1|-)s,\chi_2|-|^{-s})$$

and assume that

$$\chi_1\cdot\chi_2\cdot\omega_{\pi_1}\cdot\omega_{\pi_2}=1.$$

For $v_i\in\pi_i$ and $\Phi(s)\in I(s,\chi,\chi_2)$, the associated local Rankin-Selberg integral is given by:

$$Z(s,f_1,f_2,\Phi)=\int_{N(k)\backslash\mathrm{PGL}_2(k)}l_1(gv_1)\cdot l_2(gv_2)\cdot\Phi(s,g)\,dg.$$

This converges for $Re(s)$ sufficiently large and has meromorphic continuation to all of $\mathbb{C}$. The local Rankin-Selberg L-factor $L(s,\pi_1\times\pi_2\times\chi_1)$ is defined to be the GCD of this family of zeta integrals, as $\Phi(s)$ ranges over "good" sections of $I(s,\chi_1,\chi_2)$. Thus the normalized zeta integral

$$Z^*(s,f_1,f_2,\Phi)=\frac{Z(s,f_1,f_2,\Phi)}{L(s,\pi_1\times\pi_2\times\chi_1)}$$

is thus an entire function, which is nonzero at any given $s_0\in\mathbb{C}$ for some choice of data. In particular, for each $s\in\mathbb{C}$, one obtains an $\mathrm{GL}_2(k)$-invariant map

$$Z^*(s,-,-,-):\pi_1\otimes\pi_2\otimes I(s,\chi_1,\chi_2)\longrightarrow\mathbb{C}.$$

The local zeta integrals satisfies a local functional equation which we shall describe presently. Let

$$M(s,\chi_1,\chi_2):I(s,\chi_1,\chi_2)\longrightarrow I(-s,\chi_2,\chi_1)$$

be the standard intertwining operator. Then for an appropriate normalized version $M^*_\psi(s,\chi_1,\chi_2)$, one has the identity

$$Z^*(-s,v_1,v_2,M^*(s,\chi_1,\chi_2)\Phi)=\epsilon(s+\frac{1}{2},\pi_1\times\pi_2,\psi)Z^*(s,v_1,v_2,\Phi).$$

The constant of proportionality $\epsilon(s,\pi_1\times\pi_2\times\chi_1,\psi)$ is defined to be the Rankin-Selberg epsilon factor attached to $\pi_1\times\pi_2\times\chi_1$.

9.9. **The case at hand.** We shall now specialize the above description to the case at hand, namely, set

$$\pi_1=\pi(\chi),\quad\pi_2=\pi^\vee\quad I(s,\chi_1,\chi_2)=I(s,1,\chi_E).$$

The local zeta integral $Z^*(0)$ is a nonzero element of the 1-dimensional vector space

$$\mathrm{Hom}_{\mathrm{GSp}(W)}(\pi(\chi)\otimes I(1,\chi_E)\otimes\pi^\vee,\mathbb{C}).$$

Recalling that as a $\mathrm{GSp}(W)$-module,

$$(\pi(\chi)\otimes I(1,\chi_E)=\bigoplus_B\Pi_B,$$

we deduce that

$$\mathrm{Hom}_{E^\times}(\pi_B, \chi) \neq 0 \Longleftrightarrow Z^*(0) \text{ is nonzero when restricted to } \Pi_B \otimes \pi^\vee.$$

To study the effect of $Z^*(0)$ on the submodule Π_B, we take note of the following two lemmas.

Lemma 9.7. *(i) Via the restriction of functions, we have a $\mathrm{GL}_2(k)^+$-equivariant isomorphism*

$$I(s, 1, \chi_E) \longrightarrow I(s) := \mathrm{Ind}_{B^+}^{\mathrm{GL}_2^+} |-|^s \times |-|^{-s}.$$

In particular, at $s = 0$, one has a $\mathrm{GL}_2(k)^+$-equivariant isomorphism

$$\pi(1, \chi_E) \longrightarrow I(0) = \mathrm{Ind}_{B^+}^{\mathrm{GL}_2^+} 1.$$

Similarly, the restriction of functions give an identification

$$\pi(\chi_E, 1) \longrightarrow I(0).$$

Let $I(0)^\pm$ be the image of $I(1, \chi_E)^+$ under the restriction map, so that $I(0)^+$ is ψ-generic.

(ii) The normalized intertwining operator $M^*(0, 1, \chi)$ induces a $\mathrm{GL}_2(k)^+$-equivariant non-scalar map

$$M_\psi^*(0) : I(0) \longrightarrow I(0).$$

satisfying $M_\psi^*(0)^2 = 1$. Further:

$$M_\psi^*(0) = \begin{cases} +1 & \text{on } I(0)^+; \\ -1 & \text{on } I(0)^-. \end{cases}$$

Lemma 9.8. *Consider the modified zeta integrals*

$$Z_+(s, v_1, v_2, \Phi) = \int_{N(k)\backslash \mathrm{PGL}_2(k)^+} l_1(gv_1) \cdot l_2(gv_2) \cdot \Phi_s(g)\, dg.$$

Then the family

$$\{Z_+(s, v_1, v_2, \Phi) : v_1 \in \pi(\chi)^+, v_2 \in \pi^\vee \text{ and } \Phi \in I(s) \text{ arbitrary}\}$$

is identical to the family of Rankin-Selberg zeta integrals.

Proof. Let $c \in \mathrm{GL}_2(k) \smallsetminus \mathrm{GL}_2(k)^+$. Then we can write

$$Z(s, v_1, v_2, \Phi) =$$

$$\int_{N(k)\backslash \mathrm{PGL}_2(k)^+} l_1(gv_1) \cdot l_2(gv_2) \cdot \Phi_s(g)\, dg + \int_{N(k)\backslash \mathrm{PGL}_2(k)^+} l_1(gcv_1) \cdot l_2(gcv_2) \cdot \Phi_s(g)\, dg.$$

Now observe that for $v_1 \in \pi(\chi)^+$,

$$l_1(gcv_1) = 0 \quad \text{for } v_1 \in \pi(\chi)^+ \text{ and } g \in \mathrm{GL}_2(k)^+,$$

so that the second term in the above identity vanishes and

$$Z(s, v_1, v_2, \Phi) = Z_+(s, v_1, v_2, \Phi).$$

Similarly, if $v_1 \in \pi(\chi)^-$, then the first term vanishes, so that

$$Z(s, v_1, v_2, \Phi) = Z_+(s, c \cdot v_1, v_2, \Phi).$$

This proves the lemma. □

9.10. **Proof of Theorem 9.2.** We can now complete the proof of Theorem 9.2. We have:

$$\begin{aligned}
&\mathrm{Hom}_{E^\times}(\pi_B, \chi) \neq 0 \\
\Longleftrightarrow &\mathrm{Hom}_{\mathrm{GL}_2}(\pi^\vee \otimes \Pi_B, \mathbb{C}) \neq 0 \\
\Longleftrightarrow &Z^*(0) \text{ is nonzero on } \pi^\vee \otimes \Pi_B \\
\Longleftrightarrow &Z_+^*(0) \text{ is nonzero on } \pi^\vee \otimes \pi(\chi)^+ \otimes I(0)^{\epsilon(B)\cdot\chi_E(-1)}.
\end{aligned}$$

Using the local functional equation

$$Z_+^*(0) \circ (1_{\pi(\chi)^+} \otimes 1_{\pi^\vee} \otimes M_\psi^*(0)) = \epsilon(1/2, \pi(\chi) \times \pi^\vee, \psi) \cdot Z_+^*(0)$$

and taking note of Lemma 9.7, we see that

$$Z_+^*(0) \text{ is nonzero on } \pi^\vee \otimes \pi(\chi)^+ \otimes I(0)^{\epsilon(B)\cdot\chi_E(-1)}$$

if and only if

$$\epsilon(1/2, \pi(\chi) \times \pi^\vee, \psi) = \epsilon(B) \cdot \chi_E(-1).$$

This proves Theorem 9.2.

10. Global Torus Periods and Central L-values

In this final section, we shall analyze the global (twisted) torus periods on cuspidal representations of $PB^\times$. The treatment will be completely parallel to the local proof of the previous section.

10.1. **Global torus periods.** The analysis of the global (T_a, χ) periods was done by Waldpsurger, who proved the following theorem in [Wa3].

Theorem 10.1. *Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, and let π_B be its Jacquet-Langlands transfer to $B^\times$. Let χ be an automorphic character of $T_a(\mathbb{A})$ such that $\chi|_{\mathbb{A}^\times} = \omega_\pi$.*

There is at most one quaternion F-algebra B such that the torus period $\mathcal{P}_{T_a,\chi^\vee}$ may be nonzero on π_B. This B is characterized by the requirement that

$$B \text{ ramifies at } v \Longleftrightarrow \epsilon(1/2, \pi_{B,v} \times \pi(\chi_v)^\vee)\chi_{a,v}(-1) = -1.$$

In particular, this distinguished B does not exist if

$$\epsilon(1/2, \pi \times \pi(\chi)^\vee) = -1.$$

If $\epsilon(1/2, \pi \times \pi(\chi)^\vee) = 1$, then $\mathcal{P}_{T_a,\chi^\vee}$ is nonzero on π_B if and only if

$$L(1/2, \pi_B \times \pi(\chi)^\vee) \neq 0.$$

10.2. **An explicit formula.** In fact, in [Wa3], Waldspurger proves an exact formula relating the period $\mathcal{P}_{T_a,\chi^\vee}$ and the central L-value $L(1/2, \pi \times \pi(\chi)^\vee)$.

Theorem 10.2. *Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, and let π_B be its Jacquet-Langlands transfer to $B^\times$. Let χ be an automorphic character of $T_a(\mathbb{A})$ such that $\chi|_{\mathbb{A}^\times} = \omega_\pi$. For any $f = \otimes_v f_v \in \pi_B$, one has:*

$$|\mathcal{P}_{T_a,\chi}(f)|^2 = \frac{\zeta_F(2) \cdot L(1/2, \pi \times \pi(\chi)^\vee)}{2 \cdot L(1, \pi, Ad) \cdot L(1, \chi_E)} \cdot \prod_v \mathcal{P}_{T_a,\chi_v}(f_v).$$

Here,

- *$L(1, \pi, Ad)$ is the adjoint L-function of π evaluated at $s = 1$;*
- *the local factors $\mathcal{P}_v(f_v, \chi_v)$ are defined by the normalized integral of matrix coefficients:*

$$\mathcal{P}_v(f_v, \chi_v) = \frac{L(1, \pi_v, Ad) \cdot L(1, \chi_{E,v})}{\zeta_{F_v}(2) \cdot L(1/2, \pi_v \times \pi(\chi_v)^\vee)} \cdot \int_{F_v^\times \backslash T_{a,v}} B_{\pi_v}(t_v \cdot f_v, f_v) \cdot \chi_v(t_v)^{-1} \, dt_v$$

where B_{π_v} are local inner products on π_v such that $\prod_v B_{\pi_v}$ is equal to the Petersson inner product on π, and dt_v are local Haar measures such that $\prod_v dt_v$ is equal to the Tamagawa measure of the torus $F^\times \backslash T_a$.

Note that the local factor $\mathcal{P}_{T_a,\chi_v}(f_v)$ is equal to 1 for almost all v, so that the Euler produt is actually a finite one. Moreover, the local factor $\mathcal{P}_{T_a,\chi_v}(f_v)$ is nonzero for some $f_v \in \pi_v$ if and only if $\mathrm{Hom}_{T_a}(\pi_v, \chi_v) \neq 0$.

This is the simplest case of the *Ichino-Ikeda conjecture*, also known as the *refined Gross-Prasad conjecture.*

10.3. **Global similitude theta lifting.** We come now to the global setting, so that F is a number field with adele ring $\mathbb{A}$. Let E be a quadratic field extension of F and suppose that there is an embedding $E \hookrightarrow B$. We may then write

$$B = E \oplus E \cdot b$$

as in the local case. The setup of the see-saw works in the same way as in the local case, and the reader will no doubt notice that the proof of Theorem 9.3 to be given below is exactly the global analog of the local proof of Theorem 9.2 given above.

We begin by defining the notion of global theta lifting in the similitude setting. If π is a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, we define its global theta lift $\Theta_B(\pi)$ to $\mathrm{GSO}(B_\mathbb{A}) \cong (B_\mathbb{A}^\times \times B_\mathbb{A}^\times)/\Delta\mathbb{A}^\times$ to be the span of the functions

$$\theta_{W,B}(\phi, f)(h) = \int_{\mathrm{SL}_2(F) \backslash \mathrm{SL}_2(\mathbb{A})} \theta(\phi)(h, \lambda(h)g) \cdot \overline{f(g)} \, dg$$

where $\lambda(h)$ is any element in $\mathrm{GL}_2(\mathbb{A})$ whose determinant is the similitude of h. Here and elsewhere, we will always use the Tamagawa measure on adelic groups. It is well-known that

$$\Theta_B(\pi) = \pi_B \boxtimes \pi_B^\vee.$$

On the other hand, let χ be a character of $\mathrm{GSO}(E_\mathbb{A}) \cong E^\times \backslash \mathbb{A}_E^\times$. Then we define its theta lift $\Theta_{W,E}(\chi)$ to be the $\mathrm{GL}_2(\mathbb{A})^+$-module spanned by the functions on $\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A})^+$ given by

$$\theta_{W,E,\psi}(\phi,\chi)(g) = \int_{\mathrm{SO}(E)\backslash \mathrm{SO}(E_{\mathbb{A}})} \theta(\phi)(g,\lambda(g)t)\cdot\chi(t)^{-1}\,dt, \qquad g\in \mathrm{GL}_2(\mathbb{A})^+.$$

This gives a space $\Theta^+_{W,E,\psi}(\chi)$ of automorphic forms on $\mathrm{GL}_2(\mathbb{A})^+$. If χ does not factor through the norm, then $\Theta^+_{W,E,\psi}(\chi)$ is contained in the space of cusp forms and

$$\Theta^+_{W,E,\psi}(\chi)\cong \otimes_v \theta_{W_v,E_v,\psi_v}(\chi_v)$$

as $\mathrm{GL}_2(\mathbb{A})^+$-modules. Note that $\mathrm{GL}_2(F)\cdot\mathrm{GL}_2(\mathbb{A})^+$ is a subgroup of index 2 in $\mathrm{GL}_2(\mathbb{A})$. If we extend the functions $\theta_{W,E,\psi}(\phi,\chi)$ on

$$\mathrm{GL}_2(F)^+\backslash\mathrm{GL}_2(\mathbb{A})^+ = \mathrm{GL}_2(F)\backslash\mathrm{GL}_2(F)\cdot\mathrm{GL}_2(\mathbb{A})^+$$

by zero to $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})$, then the resulting automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ generate a representation of $\mathrm{GL}_2(\mathbb{A})$ isomorphic to

$$\pi(\chi)=\otimes_v\pi(\chi_v)=\bigotimes_v \mathrm{Ind}^{\mathrm{GL}_2(F_v)}_{\mathrm{GL}_2(F_v)^+}\pi(\chi_v)^+.$$

10.4. **A Siegel-Weil formula.** Similarly, one has the theta lift $\Theta_{W,Eb}(1)$: it is the $\mathrm{GL}_2(\mathbb{A})^+$-module spanned by the functions on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})^+$ given by

$$I_{W,Eb,\psi}(\phi)(g)\int_{\mathrm{SO}(Eb)\backslash\mathrm{SO}(Eb_{\mathbb{A}})}\theta(\phi)(g,\lambda(g)t)\,dt, \qquad g\in\mathrm{GL}_2(\mathbb{A})^+.$$

The Siegel-Weil formula now identifies the above function as an Eisenstein series.

More precisely, there is a $R_{W,Eb}(\mathbb{A}) = (\mathrm{GSO}(Eb_{\mathbb{A}})\times\mathrm{GSp}(W_{\mathbb{A}}))^0$-equivariant map

$$\mathcal{F}: S(Eb_{\mathbb{A}})\longrightarrow I(0)\cong\pi(1\chi_E)$$

given by

$$\mathcal{F}(\phi)(g)=\omega_{W,Eb,\psi}(g)(0) \qquad \text{for } g\in R_{W,Eb}(\mathbb{A}).$$

Note that one has a short exact sequence

$$1\longrightarrow \mathrm{SO}(Eb_{\mathbb{A}})\longrightarrow R_{W,Eb}(\mathbb{A})\longrightarrow \mathrm{GSp}(W_{\mathbb{A}})^+\longrightarrow 1$$

and the function $\mathcal{F}(\phi)$ of $R_{W,Eb}(\mathbb{A})$ is invariant on the left (and right) by $\mathrm{SO}(Eb_{\mathbb{A}})$, so that it descends to a function on $\mathrm{GSp}(W_{\mathbb{A}})^+$ which one easily checks to lie in $I(0)\cong\pi(1,\chi_E)$ (by Lemma 9.9) The image of the map $\mathcal{F}$ is precisely the submodule

$$I(0)^{\epsilon(B)\cdot\chi_E(-1)} := \otimes_v I_v(0)^{\epsilon(B_v)\cdot\chi_{E_v}(-1)}.$$

One has the Eisenstein series map

$$E(0,-):\pi(1,\chi_E)\longrightarrow\mathcal{A}(\mathrm{GSp}(W)),$$

and we set

$$\mathcal{E}(\phi)=E(0,\mathcal{F}(\phi))|_{\mathrm{GSp}(W_{\mathbb{A}})^+}.$$

Now the Siegel-Weil formula says:

Proposition 10.3. *One has:*

$$I_{W,Eb,\psi}(\phi)=\mathcal{E}(\phi)$$

as functions on $\mathrm{GSp}(W_{\mathbb{A}})^+$.

Proof. We give a sketch proof of the proposition. One can verify the proposition by computing the constant terms of the two automorphic forms in question. On the LHS, one gets:

$$\mathcal{P}_N(I_{W,Eb,\psi}(\phi)) = \phi(0) \cdot \int_{\mathrm{SO}(Eb_F)\backslash \mathrm{SO}(Eb_{\mathbb{A}})} dt = 2 \cdot \phi(0)$$

since the Tamagawa number of $\mathrm{SO}(Eb)$ is 2. On the other hand, on the RHS, one gets:

$$\mathcal{P}_N(\mathcal{E}(\phi)) = \mathcal{F}(\phi)(1) + M\mathcal{F}(\phi)(1),$$

where $M : I(0) \longrightarrow I(0)$. But since $\mathcal{F}(\phi)$ lies in the submodule $I(0)^{\epsilon(B)\cdot\chi_E(-1)}$, we see that $M\mathcal{F}(\phi) = \mathcal{F}(\phi)$. Thus

$$\mathcal{P}_N(\mathcal{E}(\phi)) = 2 \cdot \mathcal{F}(\phi)(1) = 2 \cdot \phi(0).$$

□

10.5. **Computation of global torus period.** For ease of notation, let us set

$$H = (\mathrm{GSO}(E) \times \mathrm{GSO}(Eb))^0.$$

Take a Schwarz function

$$\phi = \phi_1 \otimes \phi_2 \in S(B_{\mathbb{A}}) = S(E_{\mathbb{A}}) \otimes S(Eb_{\mathbb{A}}).$$

We are interested in computing the period integral

$$\mathcal{P} := \mathcal{P}_{T_E,\chi} \boxtimes \overline{P}_{T_E,\chi^{-1}} : \Theta(\pi) = \pi_B \boxtimes \pi_B^\vee \longrightarrow \mathbb{C}.$$

We have:

$$\begin{aligned}
&\mathcal{P}(\theta(\phi, f) \\
&= \int_{\mathbb{A}^\times H(F)\backslash H(\mathbb{A})} \theta(\phi, f)(h) \cdot \overline{(\chi \otimes 1)(h)}\, dh \\
&= \int_{H(F)\backslash H(\mathbb{A})} \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} \theta(\phi)(h, \lambda(h)g) \cdot \overline{f(g)} \cdot \overline{(\chi \otimes 1)(h)}\, dg\, dh \\
&= \int_{\mathbb{A}^\times (\mathrm{GL}_2(F)^+ \times H(F))^0 \backslash (\mathrm{GL}_2(\mathbb{A})^+ \times H(\mathbb{A}))^0} \theta(\phi)(t) \cdot \overline{f(t)}\, dt \\
&= \int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})^+} \overline{f(g)} \cdot \left(\int_{\mathrm{SO}(E)\backslash SO(E_{\mathbb{A}}) \times \mathrm{SO}(Eb)\backslash SO(Eb_{\mathbb{A}})} \theta(\phi)(g, \lambda(g)h)\, dh \right) dg \\
&= \int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})^+} \overline{f(g)} \cdot \theta(\phi_1, \chi)(g) \cdot I(\phi_2)(g) \\
&= \int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})^+} \overline{f(g)} \cdot \theta(\phi_1, \chi)(g) \cdot \mathcal{E}(\phi_2)(g) \\
&= \int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})} \overline{f(g)} \cdot \theta(\phi_1, \chi)(g) \cdot \mathcal{E}(\phi_2)(g)
\end{aligned}$$

This last integral is the global Rankin-Selberg zeta integral, whose theory we now briefly recall.

10.6. **Global Rankin-Selberg integral.** Let π_1 and π_2 be cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$, Let $\pi(\chi_1|-|^s, \chi_2|-|^{-s})$ be a family of principal series representations satisfying

$$\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \chi_1\chi_2 = 1.$$

One has the associated Eisenstein series map $E(s)$. Now the global Ranin-Selberg zeta integral is defined by:

$$Z(s, f_1, f_2, \Phi) = \int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})} f_1(g) \cdot f_2(g) \cdot E(s, \Phi, g)\, dg,$$

and is a meromorphic function on $\mathbb{C}$.

Theorem 10.4. *One has:*

$$Z(s, f_1, f_2, \Phi) = \prod_v Z_v^*(s, f_{1,v}, f_{2,v}, \Phi_v) \cdot L(s + \frac{1}{2}, \pi_1 \times \pi_2 \times \chi_1),$$

where the local factors $Z_v^*(s)$ are the normalized local zeta integrals introduced in §9.8, and the Euler product is actually a finite product.

10.7. **Proof of Theorem 10.1.** Let us now specialize to the case of Theorem 10.1. Thus we apply the above theorem with

$$\pi_1 = \pi(\chi) \quad \pi_2 = \pi^\vee \quad \pi(\chi_1, \chi_2) = \pi(1, \chi_E).$$

Our computation of global torus periods and the Rankin-Selberg theory gives a formula:
(10.5)

$$\mathcal{P}(\theta(\phi, f)) = Z(0, f, \theta_{W,E,\psi}(\phi_1, \chi), \mathcal{F}(\phi_2)) = \prod_v Z^*(0, f_v, \theta_v(\phi_{1,v}, \chi_v), \mathcal{F}_v(\phi_{2,v})) \cdot L(1/2, \pi^\vee \times \pi(\chi)),$$

where the local maps

$$\theta_v : S(E_v) \otimes \chi_v^\vee \mapsto \theta_{E,W}^+(\chi_v)$$

are chosen so that

$$\theta_{W,E,\psi} = \otimes_v \theta_v$$

and

$$\mathcal{F}_v : S(E_v b) \longrightarrow \pi_v(1, \chi_{E_v})$$

is defined by the same formula as its global analog $\mathcal{F}$, so that

$$\mathcal{F} = \otimes_V \mathcal{F}_v.$$

In view of this and our local discussion in §9.9, we deduce immediately that $\mathcal{P} = \mathcal{P}_{T_E,\chi} \boxtimes \mathcal{P}_{T_E,\chi^{-1}}$ is nonzero on $\Theta(\pi) = \pi_B \boxtimes \pi_B^\vee$ if and only if the following two conditions hold:

- for all v, $\mathrm{Hom}_{E_v^\times}(\pi_v, \chi_v) \neq 0$;
- $L(1/2, \pi^\vee \times \pi(\chi)) \neq 0$.

In other words, we have verified Theorem 10.1.

10.8. **An exact formula.** The formula (10.5) already resembles the exact formula claimed in Theorem 10.2, in the sense that it captures the most important term $L(1/2, \pi^\vee \times \pi(\chi))$. However, the terms $Z_v^*(0)$ are local quantities on the RHS of the see-saw diagram in §9.5, whereas those in Theorem 10.2 are local quantities on the LHS of the see-saw. Thus, to prove Theorem 10.2, we need to interpret the normalized local zeta integral $Z_v^*(0)$ in terms of quantities on the LHS of the see-saw.

Let us examine the RHS of formula (10.5) again. Observe that the central L-value appears here because we have use the local L-factors to normalize the local zeta integrals, so that their Euler product converges. If we pretend that we could cancel these L-factors away, then (10.5) reads:

$$\mathcal{P}(\theta(\phi, f)) = \int_{N(\mathbb{A})\backslash \mathrm{PGL}_2(\mathbb{A})} \mathcal{P}_{N,\psi}(\theta_{W,E,\psi}(g \cdot \phi_1)) \cdot \overline{\mathcal{P}_{N,\psi}(g \cdot f)} \cdot \mathcal{F}(\phi_2)(g)\, dg \tag{10.6}$$

Of course, the integral on the RHS is divergent, but we know how to make sense of it!

On the other hand, let us examine the RHS of the exact formula in Theorem 10.2. We see that the various L-values appear there because we need to normalize the local integrals $\mathcal{P}_{T_v,\chi_v}$ to make their Euler product converge. If we were able to cancel the L-factors away, the formula in Theorem 10.2 would read:

$$\mathcal{P}(\theta(\phi, f)) == \frac{1}{2} \cdot \int_{\mathbb{A}^\times \backslash \mathbb{A}_E^\times} \mathcal{P}_{\Delta B^\times}(\theta(t \cdot \phi, f)) \cdot \chi(t)^{-1}\, dt. \tag{10.7}$$

Here, we have regarded the element $f_1 \otimes \overline{f_2}$ in Theorem 10.2 as an element of the theta lift $\Theta(\pi) = \pi_B \boxtimes \pi_B^\vee$. Moreover, the Petersson inner product on π_B is simply the period integral $\mathcal{P}_{\Delta B^\times}$ of $\pi_B \boxtimes \pi_B^\vee$ over the diagonal subgroup

$$\Delta B^\times / \Delta F^\times \hookrightarrow (B^\times \times B^\times)/\Delta F^\times.$$

Note that this diagonal subgroup is simply the stabilizer in $\mathrm{SO}(B)$ of the identity element 1_B of B. Again, the integral on the RHS of (10.7) diverges, but we know how to make sense of it!

Thus, our task is to identify the RHS of (10.6) with the RHS of (10.7). To explain the proof of Theorem 10.2, we shall pretend that the integrals on the RHS of (10.6) and (10.7) are both convergent. To make sense of the argument to follow is a standard procedure, and taking care of this analytic issue will only serve to obscure the transparent nature of the proof.

10.9. **Fourier coefficients of theta lifts.** We begin with the RHS of (10.6). The first thing to do is to explicate the term $\mathcal{P}_{N,\psi}(\theta_{W,E,\psi}(g \cdot \phi_1))$. This is the ψ-th Fourier coefficient of a theta lift, and we have done a similar computation in §5.4. An analogous computation gives:

Proposition 10.8. *For $\phi_1 \in S(E_{\mathbb{A}})$.*

$$\mathcal{P}_{N,\psi}(\theta_{W,E,\psi}(\phi_1, \chi)) = \int_{\mathrm{SO}(E_{\mathbb{A}})} \phi_1(t) \cdot \chi(t)^{-1}\, dt.$$

If we substitute this expression into the RHS of (10.6), we obtain

$$\begin{aligned}
\mathcal{P}(\theta(\phi,f)) &= \int_{N(\mathbb{A})\backslash \mathrm{PGL}_2(\mathbb{A})} \left(\int_{\mathrm{SO}(E_{\mathbb{A}})} \phi_1(t)\cdot \chi(t)^{-1}\, dt \right) \cdot \overline{\mathcal{P}_{N,\psi}(g\cdot f)} \cdot \mathcal{F}(\phi_2)(g)\, dg \\
&= \int_{\mathbb{A}^\times \backslash \mathbb{A}_{E^\times}} \chi(t)^{-1} \cdot \left(\int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \omega_{W,E,\psi}(g)\phi_1(t) \cdot \overline{\mathcal{P}_{N,\psi}(g\cdot f)} \cdot \mathcal{F}(\phi_2)(g)\, dg \right) dt \\
&= \int_{\mathbb{A}^\times \backslash \mathbb{A}_{E^\times}} \chi(t)^{-1} \cdot \left(\int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \omega_{W,B,\psi}(g)\phi(1_B) \cdot \overline{\mathcal{P}_{N,\psi}(g\cdot f)}\, dg \right) dt \qquad (10.9)
\end{aligned}$$

Now if we compare this with the RHS of (10.7), we see that Theorem 10.2 would follow if we can show:

$$\frac{1}{2}\cdot \mathcal{P}_{\Delta B^\times}(\theta(\phi,f)) = \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \omega_{W,B,\psi}(g)\phi(1_B)\cdot \overline{\mathcal{P}_{N,\psi}(g\cdot f)}\, dg. \qquad (10.10)$$

10.10. **The period $\mathcal{P}_{\Delta B^\times}$.** To establish equation (10.10), we shall need to explicate the period integral $\mathcal{P}_{\Delta B^\times}(\theta(\phi,f))$. For this, we shall make use of the following see-saw diagram:

$$\begin{array}{ccc}
\mathrm{SO}(B) & & \mathrm{Mp}(W)\times \mathrm{Mp}(W)\ . \\
| & \times & | \\
\mathrm{SO}(B_0)\times \mathrm{SO}(F\cdot 1_B) & & \Delta \mathrm{Sp}(W)
\end{array}$$

where B_0 is the subspace of trace zero elements in B. If we take

$$\phi = \phi_0 \boxtimes \phi_1 \in S(B_{0,\mathbb{A}}) \otimes S(\mathbb{A}\cdot 1_B) = S(B_{\mathbb{A}}),$$

and $f|_{\mathrm{SL}_2}$ on $\Delta\mathrm{Sp}(W)$, then the see-saw identity gives:

$$\mathcal{P}_{\Delta B^\times}(\theta(\phi,f)) = \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} I(\phi_0)(g)\cdot \theta_{W,\psi}(\phi_1)(g)\cdot \overline{f(g)}\, dg \qquad (10.11)$$

where

$$I(\phi_0)(g) = \int_{\mathrm{SO}(B_{0,F}\backslash \mathrm{SO}(B_{0,\mathbb{A}})} \theta_{W,B_0,\psi}(\phi_0)(gh)\, dh,$$

and $\theta_{W,\psi}(\phi_1) \in \omega_{W,F\cdot 1_B,\psi}$.

10.11. **Another Siegel-Weil formula.** Observe that $I(\phi_0)$ is an element in the theta lift of the trivial representation of $\mathrm{SO}(B_0)$ to $\mathrm{Mp}(W)$, i.e. it should be a special case of the Shimura correspondence discussed in this course! However, note that the case of the trivial representation is precisely the case which we omitted in our earlier discussion; see for example Proposition 5.1.

Nonetheless, as we have seen on a couple of occasions, the Siegel-Weil formula identifies the theta lift of the trivial representation with an Eisenstein series. As in previous occasions, one has a map

$$\mathcal{F} : S(B_{0,\mathbb{A}}) \longrightarrow \text{a principal series representation of } \mathrm{Mp}(W_{\mathbb{A}}),$$

defined by

$$\mathcal{F}(\phi_0)(g) = \omega_{W,B_0,\psi}(g)(\phi_0)(0).$$

In this case, $\mathcal{F}(\phi_0)$ lies in the principal series representation $\pi_{\psi^{-1}}(|-|^{1/2}) = \otimes_v \pi_{\psi_v^{-1}}(|-|_v^{1/2})$ (see §2.9 for this notation). Indeed, from the two tables in §2.17, one sees that:

$$\mathrm{Im}(\mathcal{F}) = \left(\otimes_{v\in S_B} st_{\psi^{-1}}\right) \otimes \left(\otimes_{v\notin S_B} \pi_{\psi^{-1}}(|-|^{1/2})\right).$$

Now one may compose $\mathcal{F}$ with the Eisenstein series map

$$E(1/2,-) : \pi_{\psi^{-1}}(|-|^{1/2}) \longrightarrow \mathcal{A}(\mathrm{Mp}(W)).$$

Note however that the Eisenstein series $E(s,-)$ has a pole of order 1 at $s = 1/2$, and the residue there produces the global even Weil representation $\omega_{\psi^{-1}}^{+}$ in $L^2_{disc}(\mathrm{Mp}(W))$. However, one can show that $E(s,\Phi)$ is holomorphic at $s = 1/2$ if $\Phi \in \mathrm{Im}(\mathcal{F})$. With this, one has:

Proposition 10.12. *For $\phi_0 \in S(B_{0,\mathbb{A}})$, one has*

$$I(\phi_0) = 2 \cdot E(1/2, \mathcal{F}(\phi_0)).$$

Proof. As we did in Proposition 10.3, let us give a sketch of the proof, especially for the appearance of the factor 2. If we compute the constant term on the LHS, we see that

$$\mathcal{P}_N(I(\phi_0)) = \phi_0(0) \cdot \int_{\mathrm{SO}(B_{0,F})\backslash \mathrm{SO}(B_{0,\mathbb{A}})} dh = 2 \cdot \phi_0(0),$$

since the Tamagawa number of $\mathrm{SO}(B_0) = PB^\times$ is 2. On the other hand, computing the constant term on the RHS, we obtain

$$\mathcal{P}_N(E(\mathcal{F}(\phi_0))) = \mathcal{F}(\phi_0)(1) + M_{1/2}\mathcal{F}(\phi_0)(1),$$

where

$$M_s : \pi_{\psi^{-1}}(|-|^s) \longrightarrow \pi_{\psi^{-1}}(|-|^{-s})$$

is the standard intertwining operator. The operator M_s has a pole at $s = 1/2$ (which is the point of interest), but if $\Phi \in \mathrm{Im}(\mathcal{F})$, $M_s\Phi$ actually vanishes at $s = 1/2$. Thus,

$$\mathcal{P}_N(E(\mathcal{F}(\phi_0))) = \mathcal{F}(\phi_0)(1) = \phi_0(0).$$

This explains the appearance of 2 on the RHS of the Siegel-Weil formula. □

10.12. **The** Sym^2 **Rankin-Selberg integral.** On applying the Siegel-Wel formula above to (10.11), we see that

$$\mathcal{P}_{\Delta B^\times}(\theta(\phi, f)) = 2 \cdot \int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} E(1/2, \mathcal{F}(\phi_0))(g) \cdot \theta_{W,\psi}(\phi_1)(g) \cdot \overline{f(g)}\, dg \tag{10.13}$$

Now the family of integrals

$$\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A})} E(s,\Phi)(g) \cdot \theta_{W,1_B,\psi}(\phi_1)(g) \cdot \overline{f(g)}\, dg,$$

with $\Phi_s \in \pi_{\psi^{-1}}(|-|^s)$, is nothing but the Rankin-Selberg integral which represents the Sym^2 L-function of π (discovered by Shimura!). Indeed, the unfolding of this Rankin-Selberg integral leads to an Eulerian integral

$$\int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \Phi_s(g)\cdot \omega_{W,\psi}(g)\phi_1(1_B)\cdot \overline{\mathcal{P}_{N,\psi}(g\cdot f)}\, dg.$$

The computation of the analogous local integral at unramified places leads to $L(s+1/2, \pi_v, Ad)$ which explains the appeareance of $L(1, \pi, Ad)$ in Theorem 10.2.

10.13. **Proof of Theorem 10.2.** Applying the above to (10.13), we see that

$$\begin{aligned}\mathcal{P}_{\Delta B^\times}(\theta(\phi,f)) =& 2\cdot \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \mathcal{F}(\phi_0)(g)\cdot \omega_{W,\psi}(g)\phi_1(1_B)\cdot \overline{\mathcal{P}_{N,\psi}(g\cdot f)}\, dg \\ =& 2\cdot \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \omega_{W,B_0,\psi}(g)\phi_0(0)\cdot \omega_{W,\psi}(\phi_1)(1_B)\cdot \overline{\mathcal{P}_{N,\psi}(g\cdot f)}\, dg \\ =& 2\cdot \int_{N(\mathbb{A})\backslash \mathrm{SL}_2(\mathbb{A})} \omega_{W,B,\psi}(g)\phi(1_B)\cdot \overline{\mathcal{P}_{N,\psi}(g\cdot f)}\, dg\end{aligned}$$

In particular, we have established (10.10) and thus Theorem 10.2 is proved, at least if we allow ourselves the liberty of working with divergent integrals!

References

[W1] J.-L. Waldspurger, *Correspondance de Shimura*, J. Math. Pures et Appl. 59 (1980), 1-133.

[W2] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*, J. Math. Pures Appl. (9) 60 (1981), no. 4, 375484.

[W3] J.-L. Waldspurger, *Quelques proprits arithmtiques de certaines formes automorphes sur GL(2)*, Compositio Math. 54 (1985), no. 2, 121-171.

[W4] J.-L. Waldspurger, *Correspondance de Shimura et quaternions*, Forum Math. 3 (1991), 219-307.

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road Singapore 119076

E-mail address: matgwt@math.nus.edu.sg