A NOTE ON KOTTWITZ'S INVARIANT $e(G)$

WEE TECK GAN

Abstract. Kottwitz has defined an invariant $e(G)$ for any reductive group G. In this note, we give an interpretation of $e(G)$ in terms of the Killing forms of G and its quasi-split inner form.

1. The Invariants $d(G)$ and $e(G)$

Let G be a reductive group over a field k with $char(k) \neq 2$, and let G_s be its split form. Hence, G defines an element a_G in $H^1(k, Aut_{k^s}(G_s))$. Let G_s^{ad} be the quotient of G_s by its center. Also, let Φ be the based root datum of G_s. Then we have the following exact sequence of Galois modules:

$$1 \to G_s^{ad} \to Aut_{k^s}(G_s) \to Aut(\Phi) \to 1$$

We also have a determinant map $det : Aut(\Phi) \to \langle \pm 1 \rangle$. From the composite:

$$d : H^1(k, Aut_{k^s}(G_s)) \to H^1(k, Aut_{k^s}(\Phi)) \to H^1(k, \langle \pm 1 \rangle)$$

we get an elementary cohomological invariant of G, which is given by:

$$d(G) = d(a_G) \in H^1(k, \langle \pm 1 \rangle) = H^1(k, \mu_2)$$

since $char(k) \neq 2$. Here, the groups or pointed sets $H^*(k, -)$ refer to those of Galois cohomology, or equivalently those with respect to the étale topology on $Spec(k)$.

On the other hand, Kottwitz [K] has defined an invariant $e(G) \in H^2(k, \mu_2)$ as follows. Let G' be the quasi-split inner form of G. So G gives a well-defined $Aut_k(G'_{ad})$-orbit in $H^1(k, G'_{ad})$. Here again, G'_{ad} is the quotient of G' by its center. From the exact sequence:

$$1 \to Z \to G'_{sc} \to G'_{ad} \to 1$$

of sheaves in the flat topology on $Spec(k)$, we get a map

$$\delta : H^1(k, G'_{ad}) \to H^1_f(k, G'_{ad}) \to H^2_f(k, Z)$$

where $H^*_f(k, -)$ refers to flat cohomology.

Let (T, B) be a pair consisting of a maximal torus and a Borel subgroup of G'_{sc}. Let ρ be the character of T given by half the sum of the positive roots (positive with respect to B), and let λ be the restriction of ρ to Z. Then Kottwitz showed that λ is independent of the choice of (T, B), is defined over k, and is stable under the action of $Aut_k(G'_{sc})$. Moreover, since ρ^2 is the identity on Z, we have $\lambda : Z \to \mu_2$. Hence the composite:

$$\lambda \circ \delta : H^1(k, G'_{ad}) \to H^2_f(k, Z) \to H^2_f(k, \mu_2) \cong H^2(k, \mu_2)$$

gives an invariant $e(G) = \lambda \circ \delta(c_G)$, where c_G is any element of $H^1(k, G'_{ad})$ in the orbit determined by G. As discussed in [K], the need to use flat cohomology arises because $char(k)$ may divide the order of Z. Over a local field, the invariant $e(G)$ plays a role in the study of the trace formula and orbital integrals of G.

Now, let $W(k)$ be the Witt ring of non-degenerate quadratic forms over k, and let I be the ideal of even rank forms [S]. Then $I/I^2 \cong H^1(k, \mu_2)$ and $I^2/I^3 \cong H^2(k, \mu_2)$. Hence the invariants $d(G)$ and $e(G)$ can be regarded as elements in I/I^2 and I^2/I^3 respectively. It is thus natural to ask whether they can be interpreted in terms of quadratic forms.

2. Killing form

Now assume that G is semi-simple and that the Killing form B_G of G is non-degenerate on the Lie algebra $\mathfrak{g}$ of G. This latter assumption holds, for example, when $char(k) = 0$, or $char(k)$ is greater than the Coxeter numbers of each of the simple factors of G. Let O denote the orthogonal group of B_{G_s} and SO its connected component (of index 2). We have a natural morphism $Aut_{k^s}(G_s) \to O$ such that its restriction to G_s^{ad} is simply the adjoint representation $Ad : G_s^{ad} \to SO$. We thus have the following commutative diagram of Galois modules with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_s^{ad} & \longrightarrow & Aut_{k^s}G_s & \longrightarrow & Aut\Phi & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & SO & \longrightarrow & O & \longrightarrow & \langle \pm 1 \rangle & \longrightarrow & 1 \end{array}$$

From this, we get the following commutative diagram:

$$\begin{array}{ccc} H^1(k, Aut_{k_s}G_s) & \longrightarrow & H^1(k, Aut\Phi) \\ {\scriptstyle Ad}\downarrow & & \downarrow{\scriptstyle det} \\ H^1(k, O) & \xrightarrow{det} & H^1(k, \langle \pm 1 \rangle) \end{array}$$

This implies that $d(G) = det \circ Ad(a_G)$. One checks that $Ad(a_G) \in H^1(k, O)$ is simply the class corresponding to the Killing form B_G of G, and that:

$$det \circ Ad(a_G) = det(B_G)/det(B_{G_s})$$

Hence $d(G)$ corresponds to the element $(\mathfrak{g} \oplus \mathfrak{g}_s, B_G \oplus B_{G_s}) \in I$.

Remarks: The assumption that G is semi-simple above is for convenience. If G is reductive, say $G = Z \cdot G_0$, with Z central and G_0 the derived group of G, then we have:

$$d(G)/d(Z) = det(B_{G_0})/det(B_{(G_0)_s}).$$

Now, for $e(G)$, we shall use the following commutative diagram with exact rows of sheaves in the flat topology:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & G'_{sc} & \longrightarrow & G'_{ad} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \lambda'} & & \downarrow & & \downarrow{\scriptstyle Ad} & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin & \longrightarrow & SO & \longrightarrow & 1 \end{array}$$

Here, SO denotes the special orthogonal group for the Killing form $B_{G'_{ad}}$. This gives:

$$\begin{array}{ccc} H^1_f(k, G'_{ad}) & \xrightarrow{\delta} & H^2_f(k, Z) \\ \downarrow{\scriptstyle Ad} & & \downarrow{\scriptstyle \lambda'} \\ H^1_f(k, SO) & \xrightarrow{\delta} & H^2_f(k, \mu_2) \end{array}$$

We define:

$$e'(G) = \delta \circ Ad(c_G) \in H^2_f(k, \mu_2) \cong H^2(k, \mu_2)$$

where c_G is any class in $H^1(k, G'_{ad})$ representing G, or rather its image in $H^1_f(k, G'_{ad})$. Note that the definition of e' uses only the Spin exact sequence. Hence, since $char(k) \neq 2$ by assumption, e' can be defined purely in terms of Galois cohomology. Moreover, the above commutative diagram shows that $e'(G) = \lambda' \circ \delta(c_G)$.

Of course we have to show that $e'(G)$ is well-defined, i.e. independent of the choice of c_G. To see this, note that if b_G is another class in $H^1(k, G'_{ad})$ representing G, then $Ad(b_G)$ and $Ad(c_G)$ are two classes in $H^1(k, SO)$, which have the same image in $H^1(k, O)$, since they both represent the Killing form B_G. From the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_2 & \longrightarrow & Spin & \longrightarrow & SO & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mu_2 & \longrightarrow & Pin & \longrightarrow & O & \longrightarrow & 1 \end{array}$$

we have:

$$\begin{array}{ccc} H^1(k, SO) & \longrightarrow & H^2(k, \mu_2) \\ \downarrow & & \downarrow{\scriptstyle id} \\ H^1(k, O) & \longrightarrow & H^2(k, \mu_2) \end{array}$$

Hence the images of $Ad(b_G)$ and $Ad(c_G)$ in $H^2(k, \mu_2)$ are the same, i.e. $e'(G)$ is well-defined. Now for any reductive group G with centre C, we set $e'(G) := e'(G/C)$.

The following properties of $e'(G)$ are easy to check:

Proposition 1. *(i)* $e'(G) = w(B_G)/w(B_{G'})$*, where* $w(Q)$ *is the Hasse-Witt invariant of the quadratic form* Q *(see [S] for the definition of* w*, and [Se] for this equality)*

(ii) $e'(G_1 \times G_2) = e'(G_1)e'(G_2)$*, using multiplicative notation in* $H^2(k, \mu_2)$*.*

(iii) $e'(G) = e'(G/Z)$*, where* Z *is central in* G*.*

(iv) $e'(G \otimes_k E) = Res(e'(G))$*, where* E *is a field extension of* k*, and* $Res : H^2(k, \mu_2) \to H^2(E, \mu_2)$ *is the usual restriction map.*

(v) $e'(Res_{E/k} G_E) = Cor(e'(G_E))$*, where* E *is a finite extension of* k *and* $Cor : H^2(E, \mu_2) \to H^2(k, \mu_2)$ *is the usual corestriction map.*

We want to show that $e(G) = e'(G)$. It suffices to show that $\lambda = \lambda'$. By above proposition, we only have to do this for G absolutely quasi-simple.

Proposition 2. $\lambda = \lambda'$

Proof. This is just a question about coweight and coroot lattices. Indeed, the adjoint representation $ad : \mathfrak{g}' \to \mathfrak{so}$ induces a map from the coroot lattice $\wedge_r$ of $\mathfrak{g}'$ to the coroot lattice Λ_r of $\mathfrak{so}$. It also maps the coweight lattice $\wedge_w$ of $\mathfrak{g}'$ into a lattice Λ which contains Λ_r as a sublattice of index 2. Λ is simply the lattice corresponding to the group SO. Note that if $dim(\mathfrak{g}')$ is odd, then Λ is simply the coweight lattice. Otherwise, it is a lattice of index 2 in the coweight lattice. Hence, we get an induced map:

$$Z(k^s) \cong \wedge_w/\wedge_r \longrightarrow \Lambda/\Lambda_r \cong \mu_2(k^s)$$

This induced map is simply λ'.

To compute this map of lattices, it suffices to assume that $\mathfrak{g}'$ is defined over $\mathbb{C}$, so that we can regard the lattices above as sitting in $\mathfrak{g}'$ and $\mathfrak{so}$. So let $\{H_\alpha, X_\alpha, Y_\alpha\}$ be a Chevalley basis of $\mathfrak{g}'$. Using the isomorphism $\mathfrak{so} \cong \wedge^2 \mathfrak{g}'$ described in [F-H] Pg. 303, we see that for any $t \in \mathfrak{t}$, the Lie algebra of the torus of G', we have:

$$ad : t \mapsto \sum_{\alpha \in \Phi^+} \alpha(t) \frac{1}{2} X_\alpha \wedge Y_\alpha \in \mathfrak{t}(\mathfrak{so})$$

Let $t_\alpha = \frac{1}{2} X_\alpha \wedge Y_\alpha$. Then the t_α's lie in Λ and can be completed to a basis of Λ. Also, the intersection of the $\mathbb{Z}$-span of the t_α's with Λ_r is generated by $\{t_\alpha \pm t_\beta : \alpha, \beta \in \Phi^+\}$. Notice that if $t \in \wedge_w$, then $\alpha(t) \in \mathbb{Z}$ for all α, and so $ad(t) \in \Lambda$. Similarly, if $t \in \wedge_r$, then $\alpha(t) \in 2\mathbb{Z}$ for all α, and so $ad(t) \in \Lambda_r$.

We can check if an element of $\wedge_w$ lands in Λ_r as follows. There is an augmentation map:

$$\begin{array}{cccc} \varepsilon : & \mathbb{Z}[t_\alpha] & \longrightarrow & \frac{1}{2}\mathbb{Z} \\ & c = \sum c_\alpha t_\alpha & \mapsto & \frac{1}{2} \sum c_\alpha \end{array}$$

Then c lies in Λ_r if and only if $\varepsilon(c) \in \mathbb{Z}$, ie:

$$\varepsilon : \mathbb{Z}[t_\alpha]/\Lambda_r \cap \mathbb{Z}[t_\alpha] \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$$

Hence $ad(t)$ lies in Λ_r if and only if $\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha(t) \in \mathbb{Z}$, ie if and only if $\rho(t) \in \mathbb{Z}$. Hence $\varepsilon \circ \lambda' = \rho|_Z$, as required. □

Corollary 3. *$e(G) = e'(G)$.*

Example: Suppose $k = \mathbb{R}$. Then we have the Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{g}'$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$
$$\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$$

The Killing form is positive definite (respectively negative definite) on $\mathfrak{p}$ and $\mathfrak{p}'$ (respectively $\mathfrak{k}$ and $\mathfrak{k}'$). Let $n = dim(\mathfrak{k})$ and $m = dim(\mathfrak{k}')$. Then we have:

$$w(B_G) = (-1)^{\frac{1}{2}n(n-1)}$$
$$w(B_{G'}) = (-1)^{\frac{1}{2}m(m-1)}$$

Since G and G' are inner twists of each other, n and m must have the same parity. Hence,

$$w(B_G)/w(B_{G'}) = (-1)^{\frac{1}{2}(n-m)} = (-1)^{\frac{1}{2}(dimG/K - dimG'/K')}$$

where K (respectively K') is the maximal compact subgroup of G (respectively G').

Remarks: (1) This computation of $e(G)$ for $k = \mathbb{R}$ seems to be simpler than that given in [K].
(2) It would be nice to show using this definition that if k is p-adic, then $e'(G)$ is as computed in [K]. However I am unable to prove this.

References

[F-H] W. Fulton and J. Harris, *Representation Theory*, Graduate Text in Math. 129, Springer-Verlag.

[K] R. Kottwitz, *Sign Changes in Harmonic Analysis on Reductive Groups*, Trans. of AMS 278(1983), Pg. 289-297.

[S] W. Scharlau, *Quadratic and Hermitian Forms*, Springer Grund. der Math. Wissen. 270.

[Se] J.P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Math. No. 5, Springer-Verlag.

Mathematics Department, Harvard University, 1 Oxford Street, Cambridge MA 02138, U.S.A.