Arithmeticity for Periods of Automorphic Forms

Wee Teck Gan[1] and A. Raghuram[2]

1 Introduction

Let G be a connected reductive algebraic group over a number field F, and let (π, V_π) be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. Let H be an algebraic F-subgroup of G, and let χ be an automorphic character of $H(\mathbb{A}_F)$. We say that π has a non-vanishing (H, χ)-period if the functional

$$\phi \mapsto \ell_\chi(\phi) := \int_{[H]} \chi(h)^{-1}\phi(h)\,dh, \quad \phi \in V_\pi, \tag{1.1}$$

is nonzero, where $[H] := H(F)\backslash H(\mathbb{A}_F)$ or sometimes $[H] := Z_G(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)$. Let us now suppose that we are in an arithmetic situation, in as much as that we can talk of the automorphic representation $^\sigma\pi$ for any $\sigma \in \mathrm{Aut}(\mathbb{C})$. For example, if π is a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ then, by a result of Clozel (see Theorem 3.1 below), we know that so is $^\sigma\pi$. In this paper we study, mostly by the way of presenting a lot of examples, the dictum: *π has a non-vanishing (H, χ)-period if and only if $^\sigma\pi$ has a non-vanishing $(H, {}^\sigma\chi)$-period.* It is this dictum that we call 'arithmeticity for periods of automorphic forms.'

Let us remind the reader that automorphisms of $\mathbb{C}$, with the exceptions of the identity automorphism and complex-conjugation, are discontinuous; in particular, it is almost never the case that $\sigma(\ell_\chi(\phi)) = \ell_{\sigma\circ\chi}(\sigma(\phi))$. So one cannot naively take the σ inside the integral sign. In every example that we study, the dictum holds, and the argument is always indirect via some characterization of existence of such periods.

Let us also observe at the outset that the problem is a distinctly global problem. The corresponding local problem, at any finite place v, is trivial.

If π_v is (H_v, χ_v)-distinguished, i.e., there exists a nonzero functional $\ell :$ $\pi_v \to \mathbb{C}$ such that $\ell(\pi_v(h)v) = \chi_v(h)\ell(v)$ for all $h \in H(F_v)$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, it is easy to see that $\sigma \circ \ell$ gives a $(H_v, {}^\sigma\chi_v)$-distinguishing functional for the conjugated representation ${}^\sigma\pi_v$.

The above local observation says that the problem of arithmeticity of automorphic periods is a consequence of a positive solution of the classical local-to-global problem: 'If τ is an automorphic representation, and suppose at every place v, τ_v is (H_v, χ_v)-distinguished, then does τ have a nonzero global (H, χ)-period?' Here, take τ to be ${}^\sigma\pi$.

Given a cuspidal automorphic representation π of $G(\mathbb{A})$, and a $\sigma \in \mathrm{Aut}(\mathbb{C})$ we need to discuss when the representation ${}^\sigma\pi$ makes sense. This will be possible when the representation π contributes to the cohomology of a locally symmetric space of G with coefficients in a sheaf attached to a finite-dimensional coefficient system. In Section 2 we briefly discuss the appropriate cohomological preliminaries needed to talk about the Galois-conjugated representation ${}^\sigma\pi$ for a general G, and in Section 3 we explicate the case of $\mathrm{GL}(n)$ and recall the basic Theorem 3.1 due to Clozel which says that ${}^\sigma\pi$ is also a cuspidal cohomological representation. In Theorem 9.5, we give the natural generalization of Clozel's theorem to certain classical groups, exploiting the recent results of Arthur [2].

In Section 4, we begin by looking at two of the easiest nontrivial examples when the ambient group $G = \mathrm{GL}_2/F$. In particular, in the GL_2 context, we look at the question of arithmeticity for Whittaker periods which boils down to every cuspidal automorphic representation being globally generic and that the space of cuspidal cohomology having a rational structure; indeed, the same ingredients give arithmeticity for Whittaker periods when $G = \mathrm{GL}_n/F$. The other GL_2 example we analyze is (GL_1, χ)-periods for Hecke characters χ; here, arithmeticity is a consequence of Manin's and Shimura's classical algebraicity results on the critical values of L-functions for GL_2.

In the rest of the paper we analyze the following situations for arithmeticity problems, which are various generalizations of the GL_2 cases considered in Section 4:

1. Shalika period integrals for representations of GL_{2n}. See Theorem 5.3. The nonvanishing of Shalika period integrals is characterized in terms of functorial transfers from $\mathrm{GSpin}(2n + 1)$.

2. $\mathrm{GL}(n)/F$-periods for representations of $\mathrm{GL}(n)/E$, for a quadratic extension E/F. See Theorem 6.3. The nonvanishing of such period integrals is characterized in terms of functorial transfers from the unitary groups $\mathrm{U}(n)$.

3. $\mathrm{GL}(n-1)$-periods for representations of $\mathrm{GL}(n) \times \mathrm{GL}(n-1)$. See Theorem 7.1. These are the Gross-Prasad periods for the general linear groups.

4. $\mathrm{GL}(n) \times \mathrm{GL}(n)$-periods for representations of $\mathrm{GL}(2n)$ over a totally real field. See Theorem 8.1.

5. Whittaker and Gross-Prasad periods for classical groups. See Theorem 10.1.

It is clear that these examples are pointing toward some general motivic interpretation of period integrals. Automorphic representations with a nonzero (H, χ)-period are usually characterized in terms of functorial transfers and/or in terms of some L-function attached to π having a pole or (not having) a zero at a certain point. In terms of L-values, the situation is very similar to a conjecture of Gross on motivic L-functions; see [10, Conjecture 2.7 (ii)]. This says that for a critical motive M, the order of vanishing of the critical L-value $L(\sigma, M, 0)$ is independent of the conjugating automorphism σ. In our situation, suppose π corresponds to a motive, and suppose having a non-vanishing (H, χ)-period corresponds to the (non-)vanishing of an L-value attached to π which happens to be a critical L-value, then Gross's conjecture would predict the validity of the dictum. For example, the situation in (3), respectively (4), above exactly ties up with critical L-values of the underlying Rankin-Selberg L-function, respectively the standard L-function.

2   Cohomological Preliminaries

We will often talk about a 'cohomological cuspidal automorphic representation'. In this section, we will briefly review its definition and discuss some of its very basic properties that we need later.

Let $G/\mathbb{Q}$ be a connected reductive algebraic group over $\mathbb{Q}$ and let S be the maximal $\mathbb{Q}$-split torus in the center Z of G. Let C_∞ be a maximal compact subgroup of $G(\mathbb{R})$ and let $K_\infty = C_\infty S(\mathbb{R})$. The connected component of the identity of K_∞ is denoted K_∞°. For any open-compact subgroup $K_f \subset G(\mathbb{A}_f)$, define the locally symmetric space

$$S_{K_f}^G := G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty^\circ K_f.$$

Allowing K_f to vary, one has an inverse system $S^G = \mathrm{proj}\lim S_{K_f}^G$ of locally symmetric spaces.

Fix a maximal torus $T/\mathbb{Q} \subset G/\mathbb{Q}$, with associated absolute Weyl group $W(G,T) = (N_G(T)/T)(\mathbb{C})$. Set $X(T) = \mathrm{Hom}_\mathbb{C}(T, \mathbb{G}_m)$, so that $W(G,T)$ acts naturally on $X(T)$. By the theory of highest weight, each $W(G,T)$-orbit μ in $X(T)$ corresponds to an irreducible algebraic representation E_μ of $G(\mathbb{C})$ with highest weight μ. Let $\mathcal{E}_\mu$ be the associated (inverse system of) sheaf on S^G. We are interested in the sheaf cohomology groups

$$H^\bullet(S^G, \mathcal{E}_\mu) := \mathrm{inj}\lim_{K_f} H^\bullet(S_{K_f}^G, \mathcal{E}_\mu)$$

on which the finite adelic group $G(\mathbb{A}_f)$ acts naturally.

We can compute the above sheaf cohomology via the de Rham complex, and then reinterpreting the de Rham complex in terms of the complex computing relative Lie algebra cohomology, we get the isomorphism:

$$H^\bullet(S^G, \mathcal{E}_\mu) \simeq H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ;\ C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E_\mu).$$

The inclusion $C_\mathrm{cusp}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \hookrightarrow C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$ of the space of smooth cusp forms in the space of all smooth functions induces, via well-known results of Borel [5], an injection in cohomology; this defines cuspidal cohomology:

$$
\begin{array}{ccc}
H^\bullet(S^G, \mathcal{E}_\mu) & \longrightarrow & H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E_\mu) \\
\Big\uparrow & & \Big\uparrow \\
H_\mathrm{cusp}^\bullet(S^G, \mathcal{E}_\mu) & \longrightarrow & H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; C_\mathrm{cusp}^\infty(G(\mathbb{Q})\backslash G(\mathbb{A})) \otimes E_\mu)
\end{array}
$$

Using the usual decomposition of the space of cusp forms into a direct sum of cuspidal automorphic representations, we get the following fundamental decomposition

$$H_\mathrm{cusp}^\bullet(S^G, \mathcal{E}_\mu) = \bigoplus_\Pi H^\bullet(\mathfrak{g}_\infty, K_\infty^\circ; \Pi_\infty \otimes E_\mu) \otimes \Pi_f \qquad (2.1)$$

of $\pi_0(G_\infty) \times G(\mathbb{A}_f)$-modules.

We say that Π *is a cohomological cuspidal automorphic representation* if Π has a nonzero contribution to the above decomposition for some μ, or equivalently, if Π is a cuspidal automorphic representation whose representation at infinity Π_∞, after twisting by E_μ, has nontrivial relative Lie algebra cohomology. In this situation, we write $\Pi \in \mathrm{Coh}(G, \mu)$.

One may also consider cohomology with compact supports $H_c^\bullet(S^G, \mathcal{E}_\mu)$. *Inner cohomology* is defined as the image of compactly supported cohomology in global cohomology:

$$H_!^\bullet(S^G, \mathcal{E}_\mu) := \mathrm{Image}\left(H_c^\bullet(S^G, \mathcal{E}_\mu) \longrightarrow H^\bullet(S^G, \mathcal{E}_\mu)\right).$$

In the literature, inner cohomology is also called *interior* or *parabolic* cohomology. It is a fundamental fact (which comes from analyzing the long-exact sequence arising from the Borel-Serre compactification; see, for example, Li–Schwermer [32]) that

$$H_{\mathrm{cusp}}^\bullet(S^G, \mathcal{E}_\mu) \subset H_!^\bullet(S^G, \mathcal{E}_\mu). \tag{2.2}$$

On the other hand, since any compactly supported function is square-integrable, we also have that inner cohomology sits inside the cohomology group whose elements are represented by square-integrable automorphic forms, i.e.,

$$H_!^\bullet(S^G, \mathcal{E}_\mu) \subset H_{(2)}^\bullet(S^G, \mathcal{E}_\mu). \tag{2.3}$$

Now let us briefly recall the action of $\mathrm{Aut}(\mathbb{C})$ on the various objects introduced above. First, observe that $\mathrm{Aut}(\mathbb{C})$ acts naturally on $X(T)$ by

$$(^\sigma\chi)(t) = \sigma(\chi(\sigma^{-1}(t)), \quad \text{for } \sigma \in \mathrm{Aut}(\mathbb{C}), \, t \in T(\mathbb{C}) \text{ and } \chi \in X(T).$$

Similarly, $\mathrm{Aut}(\mathbb{C})$ acts naturally on the set of equivalence classes of irreducible algebraic representations (E, ρ) of $G(\mathbb{C})$ (where $\rho : G \to \mathrm{GL}(E)$) by

$$^\sigma\rho(g) = \sigma(\rho(\sigma^{-1}(g)) \quad \text{for } g \in G(\mathbb{C}),$$

where, the $\mathrm{Aut}(\mathbb{C})$-action on $\mathrm{GL}(E) \cong \mathrm{GL}_n(\mathbb{C})$ is with respect to the canonical Chevalley structure over $\mathbb{Q}$. In particular, it follows that if $E = E_\mu$ has highest weight μ, then $^\sigma E_\mu$ has highest weight $^\sigma\mu$.

Another description of the σ-conjugated algebraic representation $(^\sigma E, {}^\sigma\rho)$ is as follows. Set $E_\sigma = E \otimes_{\mathbb{C},\sigma} \mathbb{C}$. Then $(^\sigma E, {}^\sigma\rho)$ is isomorphic to the representation of $G(\mathbb{C})$ on E_σ with $g \in G(\mathbb{C})$ acting by

$$v \mapsto \rho(\sigma^{-1}(g))(v).$$

In particular, when restricted to $G(\mathbb{Q})$, $^\sigma E$ is realized on $E_\sigma = E \otimes_{\mathbb{C},\sigma} \mathbb{C}$ with $G(\mathbb{Q})$ acting via its action on the first component E of the tensor product. Thus, there is a natural σ-linear, $G(\mathbb{Q})$-equivariant map

$$E \longrightarrow {}^\sigma E.$$

In an analogous way, $\mathrm{Aut}(\mathbb{C})$ acts naturally on the smooth representations (W, Π_f) of $G(\mathbb{A}_f)$. Namely, we define $^\sigma \Pi_f$ to be the action of $G(\mathbb{A}_f)$ on $W \otimes_{\mathbb{C},\sigma} \mathbb{C}$ with $G(\mathbb{A}_f)$ acting on the first component W of the tensor product.

Now, passing to sheaves on the locally symmetric space S^G, the above considerations lead to a commutative diagram, where the horizontal arrows are σ-linear $G(\mathbb{A}_f)$-equivariant isomorphisms

$$
\begin{array}{ccc}
H^\bullet(S^G, \mathcal{E}_\mu) & \longrightarrow & H^\bullet(S^G, {}^\sigma\mathcal{E}_\mu) \\
\uparrow & & \uparrow \\
H_!^\bullet(S^G, \mathcal{E}_\mu) & \longrightarrow & H_!^\bullet(S^G, {}^\sigma\mathcal{E}_\mu)
\end{array}
$$

Thus we have the following

Proposition 2.1 *Suppose that π is a cohomological cuspidal automorphic representation of $G(\mathbb{A})$. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, there exists an automorphic representation τ_σ appearing in the automorphic discrete spectrum of $G(\mathbb{A})$ such that:*

- *$\tau_{\sigma,f} = {}^\sigma\pi_f$;*
- *$\tau_{\sigma,\infty}$ has nonzero Lie algebra cohomology with respect to $^\sigma\mu$.*

Proof By assumption, π_f occurrs as a $G(\mathbb{A}_f)$-summand in $H_{\mathrm{cusp}}^\bullet(S^G, \mathcal{E}_\mu)$ for some $\mathcal{E}_\mu$. We deduce by (2.2), (2.3) and the above commutative diagram that $^\sigma\pi_f$ occurs in $H_!^\bullet(S^G, {}^\sigma\mathcal{E}_\mu)$ and hence in $H_{(2)}^\bullet(S^G, {}^\sigma\mathcal{E}_\mu)$. This proves the proposition.

$\square$

Let us now suppose that G is a connected reductive group defined over a number field F with ring of adeles $\mathbb{A}_F$ and let $T \subset G$ be a maximal torus defined over F. We may apply the above discussion to the reductive group $G_0 := \mathrm{Res}_{F/\mathbb{Q}}G$ over $\mathbb{Q}$ containing the torus $T_0 = \mathrm{Res}_{F/\mathbb{Q}}T$. In this case, one has

$$G_0 \times_\mathbb{Q} \mathbb{C} \cong \prod_{\tau \in \mathrm{Hom}(F,\mathbb{C})} G_\tau \qquad \text{with} \qquad G_\tau := G \times_{F,\tau} \mathbb{C}$$

and

$$T_0 \times_{\mathbb{Q}} \mathbb{C} \cong \prod_{\tau \in \text{Hom}(F, \mathbb{C})} T_\tau \qquad \text{with} \qquad T_\tau := T \times_{F, \tau} \mathbb{C},$$

so that

$$X(T_0) \cong \bigoplus_{\tau \in \text{Hom}(F, \mathbb{C})} X(T_\tau), \quad \text{with} \qquad X(T_\tau) = \text{Hom}_{\mathbb{C}}(T_\tau, \mathbb{G}_m).$$

Note that $X(T_\tau)$ comes equipped with a natural action of $\text{Aut}(\mathbb{C}/\tau(F))$. Thus, an irreducible algebraic representation E of $G_0(\mathbb{C})$ is of the form $E = \bigotimes_\tau E_{\mu_\tau}$ where E_{μ_τ} is an irreducible algebraic representation of G_τ.

Let us explicate the action of $\text{Aut}(\mathbb{C})$ on $X(T_0)$. For $\sigma \in \text{Aut}(\mathbb{C})$ and $\tau \in \text{Hom}(F, \mathbb{C})$, with $\tau' = \sigma \circ \tau$, the automorphism σ induces:

(a) a natural isomorphism

$$\text{Aut}(\mathbb{C}/\tau(F)) \longrightarrow \text{Aut}(\mathbb{C}/\tau'(F))$$

sending ϕ to $\sigma \circ \phi \circ \sigma^{-1}$;

(b) a natural equivariant isomorphism

$$\sigma_* : X(T_\tau) \longrightarrow X(T_{\tau'})$$

given by:

$$\sigma_* : \chi \mapsto \sigma \circ \chi \circ p_\sigma,$$

where

$$p_\sigma : T_{\tau'} = T_\tau \times_{\mathbb{C}, \sigma} \mathbb{C} \longrightarrow T_\tau$$

is the natural projection. Here, the equivariance of σ_* is with respect to the action of $\text{Aut}(\mathbb{C}/\tau(F))$ on the source and the action of $\text{Aut}(\mathbb{C}/\tau'(F))$ on the target, via the isomorphism in (a).

Then the action of $\sigma \in \text{Aut}(\mathbb{C})$ on $X(T_0)$ is via the bijections σ_* described above. In particular, for any fixed $\tau \in \text{Hom}(F, \mathbb{C})$,

$$X(T_0) \cong \text{Ind}_{\text{Aut}(\mathbb{C}/\tau(F))}^{\text{Aut}(\mathbb{C})} X(T_\tau)$$

as $\text{Aut}(\mathbb{C})$-modules.

As examples, consider the following cases:

- when G is F-split, the action of $\text{Aut}(\mathbb{C}/\tau(F))$ on $X(T_\tau)$ is trivial, so that $\sigma_* : X(T_\tau) \longrightarrow X(T_{\tau'})$ is independent of σ (subject to $\sigma \circ \tau = \tau'$). In other words, one has a canonical identification $X(T_\tau) \leftrightarrow X(T_{\tau'})$. Thus, in this case, the action of $\text{Aut}(\mathbb{C})$ on $X(T_0)$ is via the permutation of the components $X(T_\tau)$, so that if $E = \bigotimes_\tau E_{\mu_\tau}$, then

$$^\sigma E = \bigotimes_\tau E_{\mu_{\sigma^{-1} \circ \tau}}.$$

- when G is an inner form of a split group over F, the action of $\mathrm{Aut}(\mathbb{C}/\tau(F))$ on $X(T_\tau)$ factors through the action of the Weyl group $W_\tau := W(G_\tau, T_\tau)$. Thus, one still has a canonical bijection $X(T_\tau)/W_\tau \leftrightarrow X(T_{\tau'})/W_{\tau'}$, i.e., between the sets of highest weights. As in the split case, one still has

$$^\sigma E = \bigotimes_\tau E_{\mu_{\sigma^{-1} \circ \tau}}.$$

Finally, note that the set S_∞ of infinite places of F is the set of orbits of complex conjugation on $\mathrm{Hom}(F, \mathbb{C})$. If we write $G_\infty = \mathrm{Res}_{F/\mathbb{Q}}(G)(\mathbb{R}) = \prod_{v \in S_\infty} G_v$, then $\mu \in X(T_0)$ defines a finite dimensional representation $E = \otimes_v E_{\mu_v}$ via:

- if $v = \tau \in \mathrm{Hom}(F, \mathbb{R})$ is real, then $E_{\mu_v} = E_{\mu_\tau}$;
- if $v = \{\tau, \overline{\tau}\} \subset \mathrm{Hom}(F, \mathbb{C})$ is complex, then $E_{\mu_v} = E_{\mu_\tau} \otimes E_{\mu_{\overline{\tau}}}$ as a representation of $G_v = \{(g, \overline{g}) : g \in G_\tau\} \subset G_\tau \times G_{\overline{\tau}}$.

3   Cohomological Representations of GL_n

In this section, we discuss in greater depth the case when $G = \mathrm{GL}_n/F$ and recall a fundamental result due to Clozel [7, Théorème 3.13], which refines Proposition 2.1.

Theorem 3.1 (Clozel) *Let Π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\Pi \in \mathrm{Coh}(\mathrm{GL}_n/F, \mu)$. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, there is a cuspidal automorphic representation $^\sigma \Pi \in \mathrm{Coh}(\mathrm{GL}_n/F, {}^\sigma \mu)$ whose finite part is $^\sigma \Pi_f$.*

Thus, the extra information contained here is that the representation τ_σ in Proposition 2.1 is cuspidal. For the precursor to this result of Clozel, see Shimura [44, Section 2] for the classical situation of Hilbert modular forms; for representations of GL_2, see Harder [21] and Waldspurger [47]. We also remark that in [7], Clozel works with the notion of the "infinity type $p(\Pi)$" of a cohomological cuspidal representation Π, which consists of the exponents appearing in the Langlands parameter of Π_v for $v \in S_\infty$. The infinity type $p(\Pi) = \{p_\tau : \tau \in \mathrm{Hom}(F, \mathbb{C})\}$ is basically the infinitesimal character of the finite-dimensional representation $E_\mu = \otimes_{\tau \in \mathrm{Hom}(F, \mathbb{C})} E_{\mu_\tau}$ with highest weight μ. More precisely, if we assume that μ is dominant, then for each $\tau \in \mathrm{Hom}(F, \mathbb{C})$,

$$\mu_\tau + \rho_n = p_\tau + \left(\frac{n-1}{2}, \ldots, \frac{n-1}{2} \right),$$

where $\rho_n = (\frac{n-1}{2}, \frac{n-3}{2}, \cdots, -\frac{n-1}{2})$ is the usual half sum of positive roots for GL_n. We also refer the reader to [7, Defn. 3.6 on p. 107] where the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ on the infinity type $p(\Pi)$ is defined; it agrees with the action of σ on μ explicated in the previous section.

Note that in Theorem 3.1, the archimedean component of $^{\sigma}\Pi$ is not precisely specified: one only knows the finite dimensional representation $E_{\sigma \mu_v}$ with respect to which $^{\sigma}\Pi_v$ has nonzero relative Lie algebra cohomology. Thus, one is naturally led to ask: to what extent does the highest weight μ_v determine the cohomological representation Π_v? Let us examine this issue more closely for GL_n.

Assume first that $v = \{\tau, \overline{\tau}\}$ is a complex place. Then suppose that

$$\mu_\tau + \rho_n = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad \mu_{\overline{\tau}} + \rho_n = (b_1, b_2, \ldots, b_n),$$

with a_i and b_i decreasing. By the purity lemma of Clozel [7, Lemma 4.9], there is a $\mathsf{w} \in \mathbb{Z}$ (independent of v) such that $a_i + b_{n+1-i} = \mathsf{w}$ for all i. Then

$$\Pi_v = \mathrm{Ind}_B^{\mathrm{GL}_n(\mathbb{C})} z^{a_1} \overline{z}^{b_n} \times \cdots \times z^{a_n} \overline{z}^{b_1}.$$

In other words, Π_v is completely determined by $\mu_v = \{\mu_\tau, \mu_{\overline{\tau}}\}$ if v is complex.

Assume now that v is a real place. Then $\mu_v = (\mu_{v,1}, \ldots, \mu_{v,n})$ where $\mu_{v,j} \in \mathbb{Z}$ and $\mu_{v,1} \geq \mu_{v,2} \geq \cdots \geq \mu_{v,n}$. Further, one knows that the highest weight μ is pure (because only pure weights support nonzero cuspidal cohomology); namely there exists a $\mathsf{w} \in \mathbb{Z}$ (independent of v) such that

$$\mu_{v,j} + \mu_{v,n-j+1} = \mathsf{w}.$$

Note that if n is odd, then $\mathsf{w} = 2\mu_{v,(n-1)/2}$ is even. Now put

$$\ell_v = 2\mu_v - \mathsf{w} + 2\rho_n.$$

Then

$$\ell_{v,j} \;=\; 2\mu_{v,j} - \mathsf{w} + n - 2j + 1 \;=\; \mu_{v,j} - \mu_{v,n-j+1} + n - 2j + 1.$$

Observe that:
$$\begin{cases} \ell_{v,1} > \cdots > \ell_{v,[n/2]} > 0; \\ \ell_{v,n-j+1} = -\ell_{v,j}; \\ \ell_{v,j} \equiv \mathsf{w} + n - 1 \pmod 2. \end{cases}$$

In particular, $\ell_{v,j} \equiv 0 \pmod 2$ if n is odd.

We define an irreducible representation J_{μ_v} as the representation induced from the $(2, \ldots, 2)$-parabolic if $n = 2m$ is even:

$$J_{\mu_v} = D_{\ell_{v,1}} |\;|^{-\mathsf{w}/2} \times \cdots \times D_{\ell_{v,m}} |\;|^{-\mathsf{w}/2}, \tag{3.1}$$

and if $n = 2m + 1$ is odd then it is induced from the $(2, \ldots, 2, 1)$-parabolic subgroup:

$$J_{\mu_v} = D_{\ell_{v,1}} |\ |^{-\mathsf{w}/2} \times \cdots \times D_{\ell_{v,m}} |\ |^{-\mathsf{w}/2} \times |\ |^{-\mathsf{w}/2}, \qquad (3.2)$$

where D_l is the 'discrete series' representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest weight $l + 1$ and central character sgn^{l+1}. Given $\Pi \in \mathrm{Coh}(\mathrm{GL}_n/F, \mu)$, we know, when n is even, that

$$\Pi_v \ \simeq \ J_{\mu_v}, \qquad (3.3)$$

and when n is odd, we know that

$$\Pi_v \ \simeq \ J_{\mu_v} \otimes \mathrm{sgn}^{\epsilon(\Pi_v)}, \qquad (3.4)$$

where $\epsilon(\Pi_v) \in \{0, 1\}$ is defined by

$$(-1)^{\epsilon(\Pi_v)} = (-1)^{(n-1)/2} \omega_{\Pi_v}(-1).$$

(See, for example, [39, Section 5.1].) Thus, when v is real, μ_v completely determines Π_v when n is even; however, when n is odd, we need not only μ_v but also the parity of the central character at the real place v to pin down Π_v.

Now we bring in the $\mathrm{Aut}(\mathbb{C})$-action. As we have noted in the previous section, for $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have ${}^\sigma\mu_\tau = \mu_{\sigma^{-1}\tau}$ for $\tau \in \mathrm{Hom}(F, \mathbb{C})$. The above discussion implies that when n is even or when v is complex, the local component ${}^\sigma\Pi_v$ is completely determined by ${}^\sigma\mu_v$. We explicate the situation in two cases:

Proposition 3.2   (i) *Assume first that F is a totally complex quadratic extension of a totally real F^+. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$${}^\sigma\Pi_v = \Pi_{\sigma^{-1}v}.$$

(ii) *Assume that F is totally real. When n is even, we have:*

$${}^\sigma\Pi_\infty \ = \ \bigotimes_{v \in S_\infty} \Pi_{\sigma^{-1}v} \ = \ \bigotimes_{v \in S_\infty} J_{\mu_{\sigma^{-1}v}}. \qquad (3.5)$$

When n is odd, we have:

$${}^\sigma\Pi_\infty \ = \ \bigotimes_{v \in S_\infty} \left(J_{\mu_{\sigma^{-1}v}} \otimes \mathrm{sgn}^{\epsilon(\Pi_v)} \right), \qquad (3.6)$$

where $\epsilon(\Pi_v)$ is as defined in (3.4). In particular, if the sign $\omega_{\Pi_v}(-1)$ is independent of the infinite place v, then

$${}^\sigma\Pi_\infty \ = \ \bigotimes_{v \in S_\infty} \Pi_{\sigma^{-1}v} \ = \ \bigotimes_{v \in S_\infty} J_{\mu_{\sigma^{-1}v}}$$

as in the case when n is even.

Proof (i) For each place v^+ of F^+, let $v = \{\tau, \overline{\tau}\}$ be the place of F over v^+, so that τ and $\overline{\tau}$ are the two elements of $\mathrm{Hom}(F, \mathbb{C})$ whose restriction to F^+ is v^+. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, $\sigma^{-1} \circ v := \{\sigma^{-1} \circ \tau, \sigma^{-1} \circ \overline{\tau}\}$ are the two elements which restrict to $\sigma^{-1} \circ v^+$. Thus, $^{\sigma}\mu_v = \mu_{\sigma^{-1}.v}$ and so we have:

$$^{\sigma}\Pi_v = \Pi_{\sigma^{-1} \circ v}.$$

(ii) Now assume that F is totally real, so that $S_\infty = \mathrm{Hom}(F, \mathbb{C})$. The case when n is even follows from our discussion above. When n is odd, the situation is a little more tricky and we need to consider central characters. Let ω_Π be the global central character of Π. It is of the form:

$$\omega_\Pi = \omega_\Pi^\circ \otimes | \; |^{-n\mathsf{w}/2}$$

with ω_Π° a Hecke character of finite order. Let us simplify notations and write this as: $\omega = \omega^\circ | \; |^{\mathsf{m}}$ where ω° is a finite-order Hecke character and $\mathsf{m} \in \mathbb{Z}$. Then, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have:

$$(^{\sigma}\omega)_v = \omega_v, \quad \forall v \in S_\infty. \tag{3.7}$$

This follows from the following two observations:

- $(^{\sigma}\omega^\circ) = \sigma \circ \omega^\circ$; this is because the latter is still a continuous character of $F^\times \backslash \mathbb{A}_F^\times$ as ω° takes value in a finite group. In particular, for real v, $\sigma \circ \omega_v^\circ = \omega_v^\circ$ since ω_v° takes value in $\{\pm 1\}$.
- $\sigma | \; | = | \; |$; this is because at all finite places w, $| \; |_w$ takes value in $\mathbb{Q}$ and so $\sigma | \; |_w = | \; |_w$ (and a Hecke character is determined by almost all its local components, by weak approximation).

Next, the global central character of $^{\sigma}\Pi$ satisfies:

$$\omega_{\sigma\Pi} = {}^{\sigma}\omega_\Pi,$$

which can be seen by checking equality of local characters at all finite unramified places. Hence the parity that is needed in pinning down the representations at infinity as in (3.4) is given by $\epsilon(^{\sigma}\Pi_v) = \epsilon(\Pi_v)$, since

$$(-1)^{\epsilon(^{\sigma}\Pi_v)} = (-1)^{(n-1)/2}\omega_{\sigma\Pi_v}(-1) \text{ and } \omega_{\sigma\Pi_v}(-1) = {}^{\sigma}\omega_{\Pi_v}(-1) = \omega_{\Pi_v}(-1)$$

by (3.7). This proves the proposition. $\qquad \square$

Remark 3.3 When F is totally real and n is odd, the hypothesis in the above theorem that the sign $\omega_{\Pi_v}(-1)$ is independent of v is arithmetically interesting because it is a necessary condition for the standard L-function

of Π to have a critical point. This will also be the case if the rank n Grothendieck motive $M = M_\Pi$ over F that is conjecturally attached to Π is *special*, i.e., has the property that complex conjugation acts via the 'same' scalar on the middle Hodge type for every real embedding v of F; see, for example, Blasius [6, M3].

4 The GL_2-examples

After the preliminary discussions of the previous two sections, we can now begin the consideration of periods. In this section, we illustrate the question we will study for the case of GL_2. Let us, for the sake of simplicity, take $G = \mathrm{GL}_2/\mathbb{Q}$, although everything discussed in this section works for GL_2 over any number field.

4.1 Whittaker periods

For the subgroup H we take

$$H = U_2 = U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \; : \; x \in \mathbb{G}_a \right\},$$

i.e., U is the unipotent radical of the standard Borel subgroup of upper triangular matrices in G. Fix a nontrivial additive character $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$. Then, as usual, ψ gives a character $\psi : U(\mathbb{Q})\backslash U(\mathbb{A}) \to \mathbb{C}^\times$ by $\psi\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = \psi(x)$. Using the same symbol ψ for both the characters will cause no confusion. In this situation, the linear functional ℓ_ψ defined in (1.1) is called a global Whittaker functional.

Given a cuspidal automorphic representation (π, V_π) of $\mathrm{GL}_2(\mathbb{A})$ we can define for each $\phi \in V_\pi$ the associated Whittaker vector

$$W_\phi(g) \; := \; \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \phi(ug)\psi(u)^{-1} \, du. \tag{4.1}$$

Observe that $W_\phi(1) = \ell_\psi(\phi)$. Using the action of $\mathrm{GL}_2(\mathbb{A})$ we see that $\ell_\psi(\phi) \neq 0$ for some ϕ if and only if $W_\phi \neq 0$ for some ϕ. A fundamental fact at the heart of the GL_2-theory of automorphic forms is that W_ϕ determines ϕ. (See, for example, [8, Lecture 4, Section 1].) Indeed, we have a Fourier expansion of the form

$$\phi(g) \; = \; \sum_{\gamma \in \mathbb{Q}^\times} W_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right), \tag{4.2}$$

In particular, every cuspidal automorphic representation has a nonvanishing Whittaker period.

Now let us suppose that π is a cohomological cuspidal automorphic representation of $GL_2(\mathbb{A})$, and in particular, π has nonvanishing Whittaker periods. For any $\sigma \in \mathrm{Aut}(\mathbb{C})$ Theorem 3.1 says that $^\sigma\pi$ is also a (cohomological) cuspidal automorphic representation of $GL_2(\mathbb{A})$. Hence, by the above discussion once again, $^\sigma\pi$ also has nonvanishing Whittaker periods, i.e, we have arithmeticity for Whittaker periods for GL_2.

The main ingredients in arithmeticity for Whittaker periods are (4.2) and Theorem 3.1. Both these ingredients, which are nontrivial assertions, are valid for GL_n/F over any number field F after suitable modification; for example, the Fourier expansion takes the form:

$$\phi(g) \;=\; \sum_{\gamma \in GL_{n-1}(F)/U_{n-1}(F)} W_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right), \qquad (4.3)$$

(See, for example, [8, loc. cit.].) Hence we get arithmeticity for Whittaker periods for GL_n/F. In Section 10 we consider the case of classical groups, especially split where the analysis is far more complicated.

Reverting to $GL_2/\mathbb{Q}$, let us go through the analysis for arithmeticity for Whittaker periods in the classical context of modular forms. Fix a positive integer N and consider the space $S_k(N)$ consisting of all holomorphic cusp forms of weight k on the upper half plane for the discrete subgroup $\Gamma_1(N)$ of $SL_2(\mathbb{R})$. A cusp form $\varphi \in S_k(N)$ has a Fourier expansion

$$\varphi(z) = \sum_{n=1}^{\infty} a_n(\varphi) e^{2\pi i n z}.$$

Now define $S_k(N, \mathbb{Q})$ to be the $\mathbb{Q}$-subspace of the $\mathbb{C}$-vector space $S_k(N)$ consisting of all φ such that $a_n(\varphi) \in \mathbb{Q}$ for all $n \geq 1$. One has the following *nontrivial* fact:

$$S_k(N) \;=\; S_k(N, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}. \qquad (4.4)$$

(See, for example, Shimura [45, Theorem 3.52].) This may be stated as the fact that the space of cusp forms of weight k and level N has a basis of cusp forms all of whose Fourier coefficients are in $\mathbb{Q}$. Indeed, there is a deeper integrality statement which says that the above is true with $\mathbb{Z}$ instead of $\mathbb{Q}$; however, for our purposes, a $\mathbb{Q}$-basis is sufficient. Let us note that (4.4) is the classical analogue of the statement (see Clozel [7, Théorème 3.19]) that cuspidal cohomology for GL_n/F admits a suitable rational structure. Now, given $\varphi \in S_k(N)$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$ we can define a function $^\sigma\varphi$ via a

q-expansion.

$$^\sigma \varphi(z) \ := \ \sum_{n=1}^{\infty} \sigma(a_n(\varphi)) e^{2\pi i n z}.$$

It follows from (4.4) that $^\sigma \varphi \in S_k(N)$. This is the classical analogue of Theorem 3.1. Arithmeticity for Whittaker models takes the form:

$$a_n(\varphi) \neq 0 \implies a_n(^\sigma \varphi) \neq 0,$$

which is built into the definition of the Galois conjugate $^\sigma \varphi$. The depth of the phenomenon lies in the rationality statement in (4.4).

4.2 GL$_1$-periods

We continue with $G = \mathrm{GL}_2/\mathbb{Q}$ and now we take

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \ : \ x \in \mathbb{G}_m \right\} \simeq \mathrm{GL}_1.$$

Take a Hecke character $\chi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$, which gives a character $\chi : H(\mathbb{Q}) \backslash H(\mathbb{A}) \to \mathbb{C}^\times$ by $\chi \left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix} \right) = \chi(x)$. Using the same symbol χ for both the characters will cause no confusion. Consider a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$. Suppose there is a $\phi \in V_\pi$ such that

$$\ell_\chi(\phi) \ = \ \int_{x \in \mathbb{Q}^\times \backslash \mathbb{A}^\times} \phi \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) \chi(x)\, dx \ \neq \ 0.$$

To analyze these integrals, and to relate them to L-values, following Jacquet-Langlands [24], fix a nontrivial additive character ψ as in the previous subsection and consider the Whittaker model $\mathcal{W}(\pi) = \mathcal{W}(\pi, \psi)$ of π. Let the cusp form ϕ correspond to $W_\phi \in \mathcal{W}(\pi)$ where W_ϕ is defined in (4.1). Then for a complex variable s such that $\Re(s) \gg 0$, the classical unfolding argument gives:

$$\begin{aligned} \ell(s, \phi, \chi) \ &:= \ \int_{x \in \mathbb{Q}^\times \backslash \mathbb{A}^\times} \phi \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) \chi(x) |x|^{s-\frac{1}{2}}\, dx \\ &= \ \int_{x \in \mathbb{A}^\times} W_\phi \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) \chi(x) |x|^{s-\frac{1}{2}}\, dx. \end{aligned}$$

Denote the *zeta integral* on the right hand side by $Z(s, W_\phi, \chi)$, and note that all the ingredients in that integral are factorizable. Changing notation if necessary, there is a cusp form ϕ so that $\ell_\chi(\phi) \neq 0$ and the associated Whittaker vector W_ϕ is a pure-tensor $W_\phi = \otimes_p W_p$. Outside a finite set

of primes S containing the infinite prime and all the primes where π or χ is ramified, one knows that W_p is the spherical vector normalized so that $W_p(1) = 1$, and the local zeta-integral computes the local L-function:

$$Z(s, W_p, \chi_p) = \int\limits_{x_p \in \mathbb{Q}_p^\times} W_p\left(\begin{pmatrix} x_p & 0 \\ 0 & 1 \end{pmatrix}\right) \chi_p(x_p)|x_p|_p^{s-\frac{1}{2}} \, dx_p \; = \; L(s, \pi_p \otimes \chi_p).$$

(See, for example, Gelbart [15, Prop. 6.17].) Let

$$L^S(s, \pi \otimes \chi) := \prod_{p \notin S} L_p(s, \pi_p \otimes \chi_p)$$

denote the partial L-function. So far we have:

$$\ell(s, \phi, \chi) = Z(s, W_\phi, \chi) = \left(\prod_{p \in S} Z(s, W_p, \chi_p)\right) \cdot L^S(s, \pi \otimes \chi).$$

Now multiply and divide the right hand side by the local L-factors at $p \in S$ to get:

$$\ell(s, \phi, \chi) = \left(\prod_{p \in S} \frac{Z(s, W_p, \chi_p)}{L_p(s, \pi_p \otimes \chi_p)}\right) \cdot L(s, \pi \otimes \chi). \qquad (4.5)$$

The integral $\ell(s, \phi, \chi)$ converges for all s since ϕ is rapidly decreasing. On the right hand side, one knows from Jacquet-Langlands that each of the ratios $Z(s, W_p, \chi_p)/L_p(s, \pi_p \otimes \chi_p)$, *a priori* defined only for $\Re(s) \gg 0$, in fact have an analytic continuation to all of s (see, for example, [15, Theorem 6.12 (ii)]), and that the completed L-function $L(s, \pi \otimes \chi)$ is an entire function of s (see, for example, [15, Theorem 6.18]). We can now prove the following characterization of the existence of (GL_1, χ)-periods and nonvanishing of a certain L-value:

Proposition 4.1 *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$, and χ a Hecke character of $\mathbb{Q}$. Then, the following are equivalent:*

1. *There exists a cusp form $\phi \in V_\pi$ such that $\ell_\chi(\phi) \neq 0$.*
2. *$L(\frac{1}{2}, \pi \otimes \chi) \neq 0$.*

Proof For (1) $\Rightarrow$ (2), put $s = 1/2$ in (4.5) to get:

$$0 \neq \ell_\chi(\phi) = r \cdot L(\tfrac{1}{2}, \pi \otimes \chi)$$

where r is an ad-hoc notation for the product

$$\prod_{p \in S} Z(\tfrac{1}{2}, W_p, \chi_p)/L_p(\tfrac{1}{2}, \pi_p \otimes \chi_p).$$

Hence the right hand side is not zero.

For $(2) \Rightarrow (1)$, given $L(\tfrac{1}{2}, \pi \otimes \chi) \neq 0$, to construct a cusp form ϕ with non-vanishing period, we construct the associated Whittaker vector W_ϕ as a pure-tensor. Outside a finite set S as above, take W_p to be the normalized spherical vector. For places in S, given any s_0 (for us $s_0 = 1/2$), we are always guaranteed the existence of a Whittaker vector W_p such that the ratio $Z_p(s_0, W_p, \chi_p)/L(s_0, \pi_p \otimes \chi_p) \neq 0$. See [15, (6.29)]. (**Note:** Indeed, for GL_2 there is a W_p for each place p so that the local zeta integral computes the local L-factor, and so the ratio is in fact 1. We deliberately stated it in a weaker form of just nonvanishing of that ratio as that is the way it will generalize to $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$.) Now put all the local Whittaker vectors to get a global Whittaker vector, and take ϕ to be the associated cusp form. The proof follows again from (4.5) at $s = 1/2$.

$\square$

Remark 4.2 Observe that it is possible for $L(\tfrac{1}{2}, \pi \otimes \chi) \neq 0$ and yet $L_f(\tfrac{1}{2}, \pi \otimes \chi) = 0$. (We use $L(s, \dots)$ for the completed L-function, and $L_f(s, \dots)$ for the finite part.) Such a phenomenon will happen when the infinite part $L_\infty(s, \pi \otimes \chi)$ has a pole at $s = 1/2$. Here is an easy example: Let $\Delta \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ be the Ramanujan Δ-function which is a weight 12 cusp form of full level. Let $\pi := \pi(\Delta) \otimes |\ |^{-6}$ and take χ to be the trivial character. (For us, cuspidal automorphic representations need not be unitary, and indeed, π is not unitary.) Anyway, let $L(s, \pi)$ be the Jacquet-Langlands L-function, and $L(s, \Delta)$ be the classical Hecke L-function; then $L(s, \pi) = L(s - 6, \pi(\Delta)) = L(s - 1/2, \Delta)$. Using the classical functional equation $L(s, \Delta) = L(12 - s, \Delta)$ we get

$$L(\tfrac{1}{2}, \pi) \; = \; L(0, \Delta) \; = \; L(12, \Delta) \; \neq \; 0,$$

The L-factor at infinity is given by:

$$L_\infty(s, \pi_\infty) \; = \; L_\infty(s - 6, \pi(\Delta)_\infty) \; = \; 2\,(2\pi)^{-s+\frac{1}{2}}\,\Gamma\left(s - \tfrac{1}{2}\right)$$

(For the last equation, see, for example, [40, 4.4]; the presence of an additional factor of 2 makes no difference to the discussion at hand.) Hence, $L_\infty(s, \pi_\infty)$ has a pole at $s = 1/2$, in other words, *nonvanishing of the global L-function at a (seemingly interesting) point does not guarantee that the point is a critical point.*

Now, given a cuspidal representation π as above with a nonvanishing (GL_1, χ)-period, and consequently with $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$, we want to analyze the dictum of arithmeticity, whence we take π to be of cohomological type. But, even if π is of cohomological type with $L(\frac{1}{2}, \pi \otimes \chi) \neq 0$, it is not guaranteed that $s = 1/2$ is a critical point. The same counterexample as in the above remark will work for this. Henceforth, we assume in addition that $s = 1/2$ is a critical point; i.e., by definition, $L_\infty(s, \pi \otimes \chi)$ and $L_\infty(s, \pi^\vee \otimes \chi^{-1})$ are regular at $s = 1/2$. Since local L-values are always nonzero, under the additional assumption of criticality of $s = 1/2$, we get

$$L(\tfrac{1}{2}, \pi \otimes \chi) \neq 0 \iff L_f(\tfrac{1}{2}, \pi \otimes \chi) \neq 0. \tag{4.6}$$

Now one can prove arithmeticity, for which we need the following algebraicity theorem due to Manin [34] in certain special cases, and more generally due to Shimura [43]; for the version stated below, see [40].

Proposition 4.3 *Let π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. There exists two nonzero complex number $p^\pm(\pi)$ such that if $s = \frac{1}{2}$ is critical then for any algebraic Hecke character χ, and any $\sigma \in \mathrm{Aut}(\mathbb{C})$ we have*

$$\sigma\left(\frac{L_f(\frac{1}{2}, \pi \otimes \chi)}{p^{\epsilon_\chi}(\pi)\mathcal{G}(\chi)(2\boldsymbol{\pi}i)^{d_\infty}} \right) = \frac{L_f(\frac{1}{2}, {}^\sigma\pi \otimes {}^\sigma\chi)}{p^{\epsilon_\chi}({}^\sigma\pi)\mathcal{G}({}^\sigma\chi)(2\boldsymbol{\pi}i)^{d_\infty}},$$

where $\mathcal{G}(\chi)$ is the Gauß sum attached to χ, ϵ_χ is a sign keeping track of the parity of χ, and d_∞ is an integer determined entirely by π_∞. (For more details see [40].)

A trivial corollary to the above deep proposition is that

$$L_f(\tfrac{1}{2}, \pi \otimes \chi) \neq 0 \iff L_f(\tfrac{1}{2}, {}^\sigma\pi \otimes {}^\sigma\chi) \neq 0. \tag{4.7}$$

The reader should compare this with Gross's conjecture mentioned in the introduction.

Theorem 4.4 (Arithmeticity for (GL_1, χ)-periods for GL_2) *Let π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Suppose that π has a nonvanishing (GL_1, χ)-period for an algebraic Hecke character of $\mathbb{Q}$. Suppose, further, that $s = 1/2$ is a critical point for the L-function $L(s, \pi \otimes \chi)$. Then ${}^\sigma\pi$ has a nonvanishing $(\mathrm{GL}_1, {}^\sigma\chi)$-period.*

Proof Follows from Proposition 4.1, (4.6) and (4.7).

$\square$

Before closing this section, let us note that the above discussion is equivalent to taking:

$$G = \mathrm{GL}_2 \times \mathrm{GL}_1, \quad \text{and} \quad H = \Delta\mathrm{GL}_1 := \left\{ \left(x, \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \right) \; : \; x \in \mathrm{GL}_1 \right\}.$$

It is from this perspective that it generalizes readily to the context of GL_n and GL_{n-1} which we discuss in Section 7.

5   Arithmeticity of Shalika Models for GL_{2n}

In the remainder of the paper, we shall consider various generalizations of the results in the previous section. One generalization of the Whittaker model for GL_2 to GL_n is the so-called Shalika model. We will first define the notion of a *Shalika model* of a cuspidal automorphic representation Π of $\mathrm{GL}_{2n}(\mathbb{A})$ where $\mathbb{A} = \mathbb{A}_F$ is the adele ring of a number field F; this particular situation was our original motivation to consider arithmeticity questions for periods. Let

$$\mathcal{S} := \left\{ s = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \; \middle| \; \begin{array}{l} h \in \mathrm{GL}_n \\ X \in \mathrm{M}_n \end{array} \right\} \subset \mathrm{GL}_{2n} =: G.$$

It is traditional to call $\mathcal{S}$ the Shalika subgroup of G. A character $\eta : F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ and a character $\psi : F\backslash\mathbb{A} \to \mathbb{C}^\times$ can be extended to a character of $\mathcal{S}(\mathbb{A})$:

$$s = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mapsto (\eta \otimes \psi)(s) := \eta(\det(h))\psi(Tr(X)).$$

We will also denote $\eta(s) = \eta(\det(h))$ and $\psi(s) = \psi(Tr(X))$. For a cusp form $\varphi \in \Pi$, and a character η with $\eta^n = \omega_\Pi$, consider the integral

$$\mathcal{S}_\psi^\eta(\varphi)(g) := \int_{Z_G(\mathbb{A})\mathcal{S}(F)\backslash\mathcal{S}(\mathbb{A})} (\Pi(g) \cdot \varphi)(s)\eta^{-1}(s)\psi^{-1}(s)ds, \quad g \in \mathrm{GL}_{2n}(\mathbb{A}).$$

When $n = 1$, observe that S_ψ^η is simply the ψ-Whittaker period of $\mathrm{GL}(2)$, since η is forced to be the central character of Π.

The following theorem, due to the works of many people (Jacquet–Shalika [26], Asgari–Shahidi [3, 4], Hundley–Sayag [23]) gives a necessary and sufficient condition for $\mathcal{S}_\psi^\eta$ to be non–zero.

Theorem 5.1 *Let Π be a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$. For a pair of characters (η, ψ), the following are equivalent:*

(i) *There is a $\varphi \in \Pi$ and $g \in G(\mathbb{A})$ such that $\mathcal{S}_\psi^\eta(\varphi)(g) \neq 0$.*

(ii) *$\mathcal{S}_\psi^\eta$ defines an injection of $G(\mathbb{A})$-modules*

$$\Pi \hookrightarrow \operatorname{Ind}_{\mathcal{S}(\mathbb{A})}^{G(\mathbb{A})}[\eta \otimes \psi].$$

(iii) *Let S be any finite set of places containing $S_{\Pi,\eta}$. The twisted partial exterior square L-function*

$$L^S(s, \Pi, \wedge^2 \otimes \eta^{-1}) := \prod_{v \notin S} L(s, \Pi_v, \wedge^2 \otimes \eta_v^{-1})$$

has a pole at $s = 1$.

(iv) *Π is the transfer of a globally generic cuspidal automorphic representation π of $\mathrm{GSpin}_{2n+1}(\mathbb{A})$ whose central character $\omega_\pi = \eta$.*

Moreover, when these conditions hold, the transfer in (iv) is strong at all archimedean places, in the sense that it respects L-parameters.

If Π satisfies any one, and hence all, of the equivalent conditions of Theorem 5.1, then we say that Π *has an (η, ψ)-Shalika model*, and we call the isomorphic image $\mathcal{S}_\psi^\eta(\Pi)$ of Π under $\mathcal{S}_\psi^\eta$ a *global (η, ψ)-Shalika model* of Π.

There is a companion theorem to Theorem 5.1 which considers the Sym^2 L-function (see [3, 4] and [23]):

Theorem 5.2 *Let Π be a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$. Then the following are equivalent:*

(i) *Let S be any finite set of places containing $S_{\Pi,\eta}$. The twisted partial symmetric square L-function*

$$L^S(s, \Pi, \mathrm{Sym}^2 \otimes \eta^{-1}) := \prod_{v \notin S} L(s, \Pi_v, \mathrm{Sym}^2 \otimes \eta_v^{-1})$$

has a pole at $s = 1$.

(ii) *Π is the transfer of a globally generic cuspidal automorphic representation π of $\mathrm{GSpin}_{2n}(\mathbb{A})$ with connected central character $\omega_\pi^0 = \eta$.*

Moreover, when these conditions hold, the transfer in (ii) is strong at all archimedean places, in the sense that it respects L-parameters.

Finally, here is the main result of this section:

Theorem 5.3 *(Arithmeticity of Shalika periods.) Suppose that F is totally real. Let Π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ which has an (η, ψ)-Shalika model. Then, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, $^\sigma\Pi$ is a cohomological cuspidal automorphic representation with a $(^\sigma\eta, \psi)$-Shalika model.*

Proof The proof is the content of the appendix of [19]; but for the sake of completeness we give a brief sketch here, elaborating upon certain important points. The hypothesis that Π is a cohomological cuspidal representation having an (η, ψ)-Shalika model imposes certain restrictions on η. In particular, we claim that η_v *is independent of* $v \in S_\infty$.

Recall from Sections 2 and 3 that the representation Π being cohomological means that there is a highest weight $\mu = (\mu_v)_{v \in S_\infty}$, with $\mu_v = (\mu_{v,1}, \ldots, \mu_{v,2n})$ where $\mu_{v,j} \in \mathbb{Z}$ and $\mu_{v,1} \geq \mu_{v,2} \geq \cdots \geq \mu_{v,2n}$, etc., and, from (3.1) and (3.3) we have

$$\Pi_v = J_{\mu_v} = D_{\ell_{v,1}}| \ |^{-\mathsf{w}/2} \times \cdots \times D_{\ell_{v,n}}| \ |^{-\mathsf{w}/2}. \tag{5.1}$$

Then the central character of Π_v is given by

$$\omega_{\Pi_v} = (\mathrm{sgn}^{\ell_{v,1}+1}| \ |^{-\mathsf{w}}) \cdots (\mathrm{sgn}^{\ell_{v,n}+1}| \ |^{-\mathsf{w}}) = \mathrm{sgn}^{n\mathsf{w}}| \ |^{-n\mathsf{w}}.$$

Now let us invoke the hypothesis that Π has an (η, ψ)-Shalika model. Then $\eta^n = \omega_\Pi$. Hence for all $v \in S_\infty$, there is an $e_v \in \{0, 1\}$ such that $\eta_v = \mathrm{sgn}^\mathsf{w}| \ |^{-\mathsf{w}}\mathrm{sgn}^{e_v}$, with $ne_v \equiv 0 \pmod 2$. In fact, a stronger statement is true: for all $v \in S_\infty$ one has

$$\eta_v \ = \ \mathrm{sgn}^\mathsf{w}| \ |^{-\mathsf{w}}. \tag{5.2}$$

To prove (5.2), note that by Theorem 5.1, Π is a Langlands functorial transfer of a cuspidal representation of $\mathrm{GSpin}_{2n+1}(\mathbb{A})$ with central character η. Moreover, the lift is strong at the archimedean places, i.e., for each archimedean place, the L-parameter ϕ_v of Π_v factors through the dual group $\mathrm{GSp}_{2n}(\mathbb{C})$ of GSpin_{2n+1} with similitude character η_v. For $v \in S_\infty$, let (ϕ_v, U_v) be the L-parameter of Π_v, i.e., ϕ_v is a representation of $W_{F_v} = W_\mathbb{R}$ on a $2n$-dimensional $\mathbb{C}$-vector space U_v. From (5.1) we know that

$$(\phi_v, U_v) \ = \ \bigoplus_{i=1}^n (\phi_{v,i}, U_{v,i}),$$

where each $\phi_{v,i}$ is an irreducible 2-dimensional representation of W_{F_v}. To say that the L-parameter ϕ_v factors through $\mathrm{GSp}(2n, \mathbb{C})$, means that there is a skew-symmetric non-degenerate bilinear form B_v on U_v such that

$$B_v \in \mathrm{Hom}_{W_{F_v}}(U_v \otimes U_v, \eta_v).$$

Further, from (5.1) one knows that

$$\phi_{v,j} \;=\; \mathrm{Ind}_{\mathbb{C}^\times}^{W_\mathbb{R}} \left(re^{i\theta} \mapsto e^{i\ell_{v,j}\theta}\right) \otimes |\;|^{-\mathsf{w}/2} \;=:\; I(\ell_{v,j}) \otimes |\;|^{-\mathsf{w}/2}.$$

In particular, the dual $\phi_{v,j}^\vee$ is $I(\ell_{v,j}) \otimes |\;|^{\mathsf{w}/2}$. Hence if $i \neq j$ then $\phi_{v,i}$ is not twist-equivalent to the dual of $\phi_{v,j}$. This implies that $B_v = \bigoplus_i B_{v,i}$ and each $B_{v,i} := B|_{U_{v,i}}$ is a non-degenerate skew-symmetric bilinear form on $U_{v,i}$:

$$B_{v,i} \in \mathrm{Hom}_{W_{F_v}}(U_{v,i} \otimes U_{v,i}, \eta_v) \;=\; \mathrm{Hom}_{W_{F_v}}(\phi_{v,i} \otimes \phi_{v,i}, \eta_v).$$

But $B_{v,i}$ is skew-symmetric, hence

$$0 \neq B_{v,i} \in \mathrm{Hom}_{W_{F_v}}(\wedge^2 U_{v,i}, \eta_v) = \mathrm{Hom}_{W_{F_v}}(\det(\phi_{v,i}), \eta_v) =$$
$$\mathrm{Hom}_{W_{F_v}}(\mathrm{sgn}^{\mathsf{w}}|\;|^{-\mathsf{w}}, \eta_v),$$

since $\det(\phi_{v,i}) = \mathrm{sgn}^{\ell_{v,i}+1}|\;|^{-\mathsf{w}} = \mathrm{sgn}^{\mathsf{w}}|\;|^{-\mathsf{w}}$. This proves (5.2). Together with (3.7), we conclude that

$$(^\sigma\eta)_v = \eta_v = \eta_{\sigma^{-1}v}$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$.

Next, it follows by Theorem 5.1 that $L^S(s, \Pi, \wedge^2 \otimes \eta^{-1})$ has a pole at $s = 1$, and thus $\Pi^\vee \cong \Pi \otimes \eta^{-1}$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we see, by checking locally almost everywhere, that

$$^\sigma\Pi^\vee \cong {}^\sigma\Pi \otimes {}^\sigma\eta^{-1},$$

and thus

$$L^S(s, {}^\sigma\Pi \otimes {}^\sigma\Pi \otimes {}^\sigma\eta^{-1}) = L^S(s, {}^\sigma\Pi, \mathrm{Sym}^2 \otimes {}^\sigma\eta^{-1}) \cdot L^S(s, {}^\sigma\Pi, \wedge^2 \otimes {}^\sigma\eta^{-1})$$

has a pole at $s = 1$. To prove the theorem, we need to show that the Sym^2 L-function does not have a pole at $s = 1$.

Let us suppose, for the sake of contradiction, that $L^S(s, {}^\sigma\Pi, \mathrm{Sym}^2 \otimes {}^\sigma\eta^{-1})$ has a pole at $s = 1$. Then by Theorem 5.2 , one knows that $^\sigma\Pi$ is a Langlands functorial transfer from a cuspidal representation of $\mathrm{GSpin}_{2n}(\mathbb{A})$ with connected central character $^\sigma\eta$, and this lift is strong at the archimedean places. Fix any $v \in S_\infty$ and put $w = \sigma^{-1}v$. The L-parameter of $^\sigma\Pi_v = \Pi_{\sigma^{-1}v} = \Pi_w$ (see (3.5)) preserves a symmetric non-degenerate bilinear form C_w with similitude character $(^\sigma\eta)_v = \eta_v = \mathrm{sgn}^{\mathsf{w}}|\;|^{-\mathsf{w}}$. In this case, we get a non-degenerate symmetric bilinear form $C_{w,i}$ on $U_{w,i}$ and W_{F_w} preserves this form up to similitude character $\mathrm{sgn}^{\mathsf{w}}|\;|^{-\mathsf{w}}$, i.e.,

$$C_{w,i} \in \mathrm{Hom}_{W_{F_w}}(U_{v,i} \otimes U_{v,i}, \mathrm{sgn}^{\mathsf{w}}|\;|^{-\mathsf{w}}).$$

But each $\phi_{w,i}$ being irreducible, by Schur's Lemma, we know that the space

$$\mathrm{Hom}_{W_{F_w}}(U_{v,i} \otimes U_{v,i},\ \mathrm{sgn}^{\mathsf{w}}|\ |^{-\mathsf{w}})\ =\ \mathrm{Hom}_{W_{F_w}}(\phi_{v,i},\ \phi_{v,i}^{\mathsf{v}} \otimes \mathrm{sgn}^{\mathsf{w}}|\ |^{-\mathsf{w}}),$$

is one-dimensional. Hence, the non-degenerate symmetric form $C_{w,i}$ is a multiple of the non-degenerate skew-symmetric form $B_{w,i}$; a contradiction!

$$\square$$

Remark 5.4 The hypothesis that F is totally real is rather artificial. One expects the arithmeticity result to hold even without this hypothesis, however, the above proof would not go through. Suppose, for example, F is an imaginary quadratic extension, then we need to consider cohomological representations of $\mathrm{GL}_{2n}(\mathbb{C})$. For the infinite place v, the parameter of the representation Π_v, which is a $2n$-dimensional representation of $W_{\mathbb{C}} = \mathbb{C}^{\times}$, is of the form:

$$\phi_v = \bigoplus_{j=1}^{2n}(z \mapsto z^{a_j}\bar{z}^{b_j}).$$

(Here the a_j and b_j are half-integers; see Clozel [7, p.112].) If Π has a Shalika model, then the image of the Langlands parameter ϕ_v is inside a split torus in $\mathrm{Sp}(2n,\mathbb{C})$. But this split torus may also be viewed as sitting inside $\mathrm{SO}(2n,\mathbb{C})$. Hence, from information of $\Pi_{\infty} = \Pi_v$ it is not possible to deduce that the parameter of ${}^{\sigma}\Pi_v$ is not of orthogonal type.

Remark 5.5 The proof of Theorem 5.3 amounts to showing that

$L^S(s,\Pi,\wedge^2 \otimes \eta^{-1})$ has a pole at $s = 1 \Longrightarrow$
$$L^S(s,{}^{\sigma}\Pi,\wedge^2 \otimes {}^{\sigma}\eta^{-1})\ \text{has a pole at } s = 1$$

under the conditions in the theorem. The same proof shows that when F is totally real,

$L^S(s,\Pi,\mathrm{Sym}^2 \otimes \eta^{-1})$ has a pole at $s = 1 \Longrightarrow$
$$L^S(s,{}^{\sigma}\Pi,\mathrm{Sym}^2 \otimes {}^{\sigma}\eta^{-1})\ \text{has a pole at } s = 1$$

when Π is cuspidal cohomological.

6   Arithmeticity of $\mathrm{GL}(n)/F$-periods for Representations of $\mathrm{GL}(n)/E$

The argument of the previous section can be applied to prove the arithmeticity of $\mathrm{GL}_n(F)$-periods for representations of $\mathrm{GL}_n(E)$, where E is a quadratic extension of F.

More precisely, let c be the nontrivial element in $\mathrm{Gal}(E/F)$ and let $\omega_{E/F}$ be the quadratic Hecke character associated to E/F by global class field theory. Let χ be a Hecke character of $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_F^\times$ is equal to $\omega_{E/F}$. Then for $\epsilon = \pm$, we set

$$\omega_{E/F}^\epsilon = \begin{cases} 1, \text{ if } \epsilon = +; \\ \omega_{E/F} \text{ if } \epsilon = -. \end{cases} \qquad \chi^\epsilon = \begin{cases} 1 \text{ if } \epsilon = +; \\ \chi \text{ if } \epsilon = -. \end{cases}$$

For a cuspidal representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ and $\epsilon = \pm$, we shall consider the period integral

$$\mathcal{P}^\epsilon(\varphi) = \int_{Z_F(\mathbb{A}_F)\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F)} \varphi(h) \cdot \omega_{E/F}^\epsilon(\det(h))\, dh,$$

where $\varphi \in \Pi$. For the period integral $\mathcal{P}^\epsilon$ to have a chance to be nonvanishing, it is necessary that the central character ω_Π of Π is equal to $(\omega_{E/F}^\epsilon)^n$ when restricted to the center $Z_F(\mathbb{A}_F) = \mathbb{A}_F^\times$ of $\mathrm{GL}_n(\mathbb{A}_F)$.

Associated to Π is a pair of partial L-functions $L^S(s, \Pi, \mathrm{Asai}^\pm)$, known as the $\mathrm{Asai}^\pm$ (or twisted tensor) L-function (see [14, Section 7]). One has

$$L^S(s, \Pi, \mathrm{Asai}^-) = L^S(s, \Pi \otimes \chi, \mathrm{Asai}^+)$$

and

$$L^S(s, \Pi \times \Pi^c) = L^S(s, \Pi, \mathrm{Asai}^+) \cdot L^S(s, \Pi, \mathrm{Asai}^-)$$

where c acts on the representations of $\mathrm{GL}_n(\mathbb{A}_E)$ by

$$\Pi^c(g) = \Pi(c(g)).$$

The following theorem is a consequence of the works of many people (Kim-Krishnamurthy [30, 31], Flicker [11, 12], Ginzburg-Rallis-Soudry [16]).

Theorem 6.1 *For a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$, the following are equivalent:*

(i) *There is a $\varphi \in \Pi$ such that $\mathcal{P}^\epsilon(\varphi) \neq 0$.*

(ii) *For a sufficiently large finite set S of places of F, the partial Asai^ϵ L-function $L^S(s, \Pi, \mathrm{Asai}^\epsilon)$ has a pole at $s = 1$.*

(iii) *$\Pi \otimes \chi^{\epsilon \cdot (-1)^{n-1}}$ is the transfer (standard base change) of a globally generic cuspidal automorphic representation π of the quasi-split $\mathrm{U}_n(\mathbb{A})$.*

Moreover, when these conditions hold, the transfer in (iii) is strong at all archimedean places of F, in the sense that it respects L-parameters.

One has a local analog of the above global theorem, which is due to the works of many people (A. Kable [28], Anandavardhanan-Kable-Tandon [1], N. Matringe [35, 36]):

Theorem 6.2 *Let v be a non-archimedean place of F which is inert in E and let Π_v be a generic representation of $\mathrm{GL}_n(E_v)$. Then the following are equivalent:*

(i) *Π_v is $(\mathrm{GL}_n(F_v), \omega^\epsilon_{E_v/F_v})$-distinguished.*

(ii) *The local Asai L-function $L(s, \Pi_v, \mathrm{Asai}^\epsilon)$ has an "exceptional" pole at $s = 0$.*

(iii) *The L-parameter ϕ_v of Π_v is conjugate-self-dual with sign ϵ (in the sense of [14, Section 3]).*

In analogy with Theorem 5.3, one has the following theorem.

Theorem 6.3 *Let Π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ which is globally distinguished by $(\mathrm{GL}_n(\mathbb{A}_F), \omega^\epsilon_{E/F})$. Assume that one of the following conditions hold:*

(1) *n is odd; or*

(2) *E is a totally complex and F is totally real; or*

(3) *there is a finite place v of F which is inert in E where Π_v is discrete series.*

Then, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, ${}^\sigma\Pi$ is a cohomological cuspidal representation which is globally distinguished by $(\mathrm{GL}_n(\mathbb{A}_F), \omega^\epsilon_{E/F})$.

Proof This is similar to the proof of Theorem 5.3, exploiting Theorems 6.1 and 6.2:

Π is globally distinguished by $(\mathrm{GL}_n(\mathbb{A}_F), \omega^\epsilon_{E/F})$

$\Longrightarrow L^S(s, \Pi, \mathrm{Asai}^\epsilon)$ has a pole at $s = 1$

$\Longrightarrow \Pi^c \cong \Pi^\vee$

$\Longrightarrow {}^\sigma\Pi^c \cong {}^\sigma\Pi^\vee$

$\Longrightarrow L_E^S(s, {}^\sigma\Pi \times {}^\sigma\Pi^c)$ has a pole at $s = 1$

$\Longrightarrow L^S(s, {}^\sigma\Pi, \mathrm{Asai}^+)$ or $L^S(s, {}^\sigma\Pi, \mathrm{Asai}^-)$ has a pole at $s = 1$.

Suppose that $L^S(s, {}^\sigma\Pi, \mathrm{Asai}^{-\epsilon})$ has a pole at $s = 1$, rather than $L^S(s, {}^\sigma\Pi, \mathrm{Asai}^\epsilon)$, so that ${}^\sigma\Pi$ is distinguished by $(\mathrm{GL}_n(\mathbb{A}_F), \omega^{-\epsilon}_{E/F})$. We shall obtain a contradiction under one of the hypotheses (1), (2) or (3).

Under hypothesis (1), so that n is odd, we note that the central character of Π is equal to $\omega_{E/F}^{\epsilon}$ when restricted to the center of $\mathrm{GL}_n(\mathbb{A}_F)$, whereas that of $^\sigma\Pi$ is equal to $\omega_{E/F}^{-\epsilon}$. In particular, one restriction is the trivial character of $\mathbb{A}_F^{\times}$ whereas the other is the quadratic character $\omega_{E/F}$. However, at all finite places, it is clear that the central characters of Π_v and $^\sigma\Pi_v$ have the same restriction to the center of $\mathrm{GL}_n(F_v)$ since this restriction is at most a quadratic character. This gives the desired contradiction under hypothesis (1).

Now assume hypothesis (2), so that E is a totally complex extension of the totally real field F. By Theorem 6.1, $\Pi \otimes \chi^{\epsilon \cdot (-1)^n}$ is lifted from $U_n(\mathbb{A})$ and the lift is strong at archimedean places. Thus for each infinite place v of E, the L-parameter ϕ_v of Π_v is of the form

$$\phi_v = \bigoplus_i z^{a_i} \cdot \overline{z}^{-a_i}$$

with $a_1 > a_2 > \cdots > a_n$ half-integers and with each character $z \mapsto z^{a_i} \cdot \overline{z}^{-a_i}$ conjugate-self-dual with sign ϵ, i.e.,

$$2a_i = \begin{cases} \text{even, if } \epsilon = +1; \\ \text{odd, if } \epsilon = -1. \end{cases}$$

On the other hand, consider the L-parameter ϕ_v' of $^\sigma\Pi_v$. By Proposition 3.2, $^\sigma\Pi_v = \Pi_{\sigma^{-1} \cdot v}$, so that $\phi_v' = \phi_{\sigma^{-1}v}$. Thus ϕ_v' is the direct sum of characters which are conjugate-self-dual of sign ϵ. But if $^\sigma\Pi_v$ is distinguished by $(\mathrm{GL}_n(\mathbb{A}_F), \omega_{E/F}^{-\epsilon})$, Theorem 6.1 implies that ϕ_v' is the direct sum of characters which are conjugate-self-dual with sign $-\epsilon$. This gives the desired contradiction.

Finally, assume hypothesis (3). For all finite places v of F, $^\sigma\Pi_v$ is locally distinguished by $(\mathrm{GL}_n(F_v), \omega_{E_v/F_v}^{-\epsilon})$ and so its L-parameter ϕ_v' is conjugate-self-dual with sign $-\epsilon$. On the other hand, the L-parameter ϕ_v of Π_v is conjugate self-dual of sign ϵ. When Π_v is discrete series, $\phi_v' = {}^\sigma\phi_v$ up to the quadratic character $x \mapsto \sigma(|x|_{E_v}^{1/2})/|x|_{E_v}^{1/2}$ of $W_{E_v}^{ab} \cong E_v^{\times}$. Observe that this character is trivial on $F_v^{\times}$, so it is conjugate orthogonal in the sense of [14, Section 3]. In particular, ϕ_v and ϕ_v' are conjugate-self-dual of the same sign; this gives the desired contradiction when Π_v is discrete series.

$\square$

7 Arithmeticity of GL_{n-1} Periods on $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$

In this section, we consider the GL_{n-1}-period for cuspidal representations of $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ over $\mathbb{Q}$. This context is a very nice generalization of the example in subsection 4.2 where we studied (GL_1, χ)-periods for representations of GL_2. The nonvanishing of periods is equivalent to a certain central L-value being nonzero. If we further impose the condition that the central value is a critical value then an appropriate algebraicity theorem for this critical value gives arithmeticity. The situation is analogous to Gross's conjecture concerning order of vanishing of critical motivic L-values as discussed in the introduction.

Theorem 7.1 *Let Π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, say $\Pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu)$. Here $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. Similarly, let $\Sigma \in \mathrm{Coh}(\mathrm{GL}_{n-1}, \lambda)$. Suppose that $\Pi \otimes \Sigma$, as a representation of $(\mathrm{GL}_n \times \mathrm{GL}_{n-1})(\mathbb{A})$, has a non-vanishing period with respect to the diagonally embedded subgroup $\mathrm{GL}_{n-1}(\mathbb{A})$. Suppose further that the coefficient systems E_μ and E_λ satisfy:*

$$\mathrm{Hom}_{\mathrm{GL}_{n-1}}(E_\mu \otimes E_\lambda, \ \mathbb{1}) \neq 0.$$

Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the representation $^\sigma\Pi \times {}^\sigma\Sigma$ also has a non-vanishing period with respect to $\mathrm{GL}_{n-1}(\mathbb{A})$ under the assumption that [37, Hypothesis 3.10] holds.

Proof Every step of the proof is a suitable generalization of the proof of arithmeticity of (GL_1, χ)-periods for representations of GL_2 as in subsection 4.2.

To begin, the generalization of Proposition 4.1 goes like this: $\Pi \otimes \Sigma$ as a representation of $(\mathrm{GL}_n \times \mathrm{GL}_{n-1})(\mathbb{A})$ has a non-vanishing $\mathrm{GL}_{n-1}(\mathbb{A})$ period if and only if $L(\frac{1}{2}, \Pi \times \Sigma) \neq 0$. This follows from using the integrals studied by Jacquet, Piatetskii-Shapiro and Shalika [25] as follows. For cusp forms $\phi \in V_\Pi$ and $\phi' \in V_\Sigma$, define

$$\ell(s, \phi, \phi') = \int_{\mathrm{GL}_{n-1}(\mathbb{Q})Z_{n-1}(\mathbb{A})\backslash\mathrm{GL}_{n-1}(\mathbb{A})} \phi\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \phi'(g)|\det(g)|^{s-\frac{1}{2}}\, dg,$$

where Z_{n-1} is the center of GL_{n-1}. Our assumption on $\Pi \otimes \Sigma$ is that $\ell(\frac{1}{2}, \phi, \phi') \neq 0$ for some ϕ and ϕ'. Let W_ϕ and $W_{\phi'}$ be the corresponding Whittaker vectors; we may and will take ϕ and ϕ' so that W_ϕ and $W_{\phi'}$ are pure-tensors: $W_\phi = \otimes W_p$ and $W_{\phi'} = \otimes W'_p$. The unfolding argument gives

$\ell(s, \phi, \phi') = Z(s, W_\phi, W_{\phi'})$ for $\Re(s) \gg 0$, where

$$Z(s, W_\phi, W_{\phi'}) = \int_{N_{n-1}(\mathbb{A}) \backslash \mathrm{GL}_{n-1}(\mathbb{A})} W_\phi \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} W_{\phi'}(g) |\det(g)|^{s - \frac{1}{2}} \, dg;$$

here N_{n-1} is the subgroup of all upper triangular unipotent elements in GL_{n-1}. The analogue of (4.5) takes the form:

$$\ell(s, \phi, \phi') = \left(\prod_{p \in S} \frac{Z(s, W_p, W_p')}{L_p(s, \Pi_p \otimes \Sigma_p)} \right) \cdot L(s, \Pi \otimes \Sigma). \tag{7.1}$$

Using [25, Theorem 2.7] we get that both sides and especially both the factors on the right hand side are entire functions. Evaluating at $s = 1/2$ gives

$$\Pi \otimes \Sigma \text{ has a non-vanishing } \mathrm{GL}_{n-1}(\mathbb{A}) \text{ period} \iff L(\tfrac{1}{2}, \Pi \times \Sigma) \neq 0. \tag{7.2}$$

Next, the hypothesis that $\Pi \in \mathrm{Coh}(\mathrm{GL}_n, \mu)$ and $\Sigma \in \mathrm{Coh}(\mathrm{GL}_{n-1}, \lambda)$ puts us in an arithmetic context; however, this doesn't guarantee that $s = \frac{1}{2}$ is critical. With the foresight of wanting to appeal to algebraicity results, we impose the condition:

$$\mathrm{Hom}_{\mathrm{GL}_{n-1}}(E_\mu \otimes E_\lambda, \mathbb{1}) \neq 0.$$

It was observed by Kasten and Schmidt [29, Theorem 2.3] that this condition implies $s = 1/2$ is critical for the Rankin-Selberg L-function $L(s, \Pi \times \Sigma)$. The same condition is also needed for an algebraicity result for critical values due the second author; see [37]. We get the analogue of (4.6) which looks like

$$L(\tfrac{1}{2}, \Pi \otimes \Sigma) \neq 0 \iff L_f(\tfrac{1}{2}, \Pi \otimes \Sigma) \neq 0. \tag{7.3}$$

Under an additional nonvanishing hypothesis involving only representations at infinity as in [37, Hypothesis 3.10], the main result of that paper, [37, Theorem 1.1], says that

$$\sigma \left(\frac{L_f(\tfrac{1}{2}, \Pi \times \Sigma)}{p^\epsilon(\Pi) p^\eta(\Sigma) \mathcal{G}(\omega_{\Sigma_f}) \, p_\infty(\mu, \lambda)} \right) = \frac{L_f(\tfrac{1}{2}, \Pi^\sigma \times \Sigma^\sigma)}{p^\epsilon(\Pi^\sigma) \, p^\eta(\Sigma^\sigma) \, \mathcal{G}(\omega_{\Sigma_f^\sigma}) \, p_\infty(\mu, \lambda)},$$

where $p^\epsilon(\Pi)$ and $p^\eta(\Sigma)$ are nonzero complex numbers, $\mathcal{G}(\omega_{\Sigma_f})$ is the Gauss sum of the central character of Σ, and $p_\infty(\mu, \lambda)$ is a nonzero complex number determined by μ and λ. The analogue of (4.7) follows easily:

$$L_f(\tfrac{1}{2}, \Pi \otimes \Sigma) \neq 0 \iff L_f(\tfrac{1}{2}, {}^\sigma\Pi \otimes {}^\sigma\Sigma) \neq 0. \tag{7.4}$$

The reader may easily verify that $s = 1/2$ is critical for $L(s, \Pi \otimes \Sigma)$ if and only if $s = 1/2$ is critical for $L(s, {}^\sigma\Pi \otimes {}^\sigma\Sigma)$.

Arithmeticity follows from first applying (7.2), (7.3) and (7.4) to $\Pi \otimes \Sigma$ and then applying (7.3) and (7.2) to ${}^\sigma\Pi \otimes {}^\sigma\Sigma$.

$\square$

Remark 7.2 The hypothesis on the coefficient systems as in Theorem 7.1, which is itself a nonvanishing period like condition, is crucial for the methods of [37] to apply. Let us note that it is possible to have a pair of cohomological representations Π and Σ for which $s = 1/2$ is critical but for which that condition on the coefficients is not satisfied. For example, take $n = 3$, $\mu = (0, 0, 0)$ and $\lambda = (1, -1)$; then E_μ is the trivial representation of GL_3. Take $\Pi \in \mathrm{Coh}(\mathrm{GL}_3, \mu)$ and $\Sigma \in \mathrm{Coh}(\mathrm{GL}_2, \lambda)$. Then, we leave it to the reader to check that $s = 1/2$ is critical for $L(s, \Pi \times \Sigma)$, but $\mathrm{Hom}_{\mathrm{GL}_2}(E_\mu \otimes E_\lambda, \; \mathbb{1}) = 0$. Now in such a situation, suppose the representation $\Pi \times \Sigma$ of $\mathrm{GL}_3 \times \mathrm{GL}_2$ has a nonvanishing GL_2 period, then the above proof is not applicable; however, we still believe that one should have arithmeticity.

Remark 7.3 In a certain work in progress [38], the second author is studying algebraicity theorems for critical values of L-functions for $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ over any number field. This would then generalize Theorem 7.1 from $\mathbb{Q}$ to any number field.

Remark 7.4 The assumption [37, Hypothesis 3.10] is a certain limitation of the technique used in that paper. We note that this hypothesis is of a purely local nature and depends only the representations Π_∞ and Σ_∞ at infinity. For $n = 2$ the validity of this hypothesis follows from an explicit calclculation; see [40]; it is this calculation that gives the term $(2\pi i)^{d_\infty}$ in Proposition 4.3. For $n = 3$ the validity of the hypothesis has been proved by Kasten and Schmidt [29].

8 Arithmeticity of $\mathrm{GL}_n \times \mathrm{GL}_n$ Periods for Cusp Forms on GL_{2n}

In this section, we discuss yet another generalization of the example in subsection 4.2 where we studied (GL_1, χ)-periods for representations π of GL_2. Indeed, in that example, we could have carried through the entire discussion by replacing π by $\pi \otimes \chi$ and taking the trivial character of $H = \mathrm{GL}_1 \times \mathrm{GL}_1$ sitting as the diagonal torus in GL_2. (This imposes the condition that the central character of $\pi \otimes \chi$ is trivial.)

Now we take $G = \mathrm{GL}_{2n}$ over a totally real number field F. Take $H = \mathrm{GL}_n \times \mathrm{GL}_n$ sitting as block diagonal matrices in G. Let Π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. We would like to analyze arithmeticity for the periods:

$$\ell(\phi) := \int_{[H]} \phi \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} dg_1 dg_2, \quad \phi \in V_\Pi,$$

where $[H] = Z_G(\mathbb{A}_F)H(F)\backslash H(\mathbb{A}_F)$. Arithmeticity in this context follows from certain zeta integrals studied by Jacquet-Shalika [26] and Friedberg-Jacquet [13], and an algebraicity result due to Grobner and the second author [19].

Theorem 8.1 *Let Π be a cohomological cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ where F is a totally real number field. Suppose that*

1. *Π has a nonvanishing H-period;*
2. *the point $s = \frac{1}{2}$ is critical for $L(s, \Pi)$.*

Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the representation $^\sigma\Pi$ also has a non-vanishing H-period.

Proof Again, we follow the proof of arithmeticity of (GL_1, χ) for representations of GL_2 as in subsection 4.2.

To begin, by a theorem of Friedberg-Jacquet [13], π has nonzero H-period if and only if π admits a Shalika model and $L(1/2, \pi) \neq 0$. Now, suppose Π has a Shalika model, then Π having a nonzero H-period being equivalent to nonvanishing of $L(1/2, \Pi)$ maybe seen as follows. For a cusp form $\phi \in V_\Pi$ define

$$\ell(s, \phi) = \int_{H(F)Z_{2n}(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \phi \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \left| \frac{\det(g_1)}{\det(g_2)} \right|^{s-\frac{1}{2}} dg_1 dg_2,$$

where Z_{2n} is the center of GL_{2n}. Our assumption on Π is that $\ell(\frac{1}{2}, \phi) \neq 0$ for some ϕ. Let $\mathcal{S}_\phi$ be the corresponding vector in the Shalika model of Π; as before, we may and will take ϕ so that $\mathcal{S}_\phi$ is a pure-tensor: $\mathcal{S}_\phi = \otimes \mathcal{S}_p$. (For details concerning Shalika models and related matters we refer the reader to [19], and recommend that any serious reader of this section should have that paper by one's side.)

An unfolding argument ([19, Proposition 3.1.5]) gives $\ell(s, \phi) = Z(s, \mathcal{S}_\phi)$ where

$$Z(s, S_\phi) = \int_{\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F)} \mathcal{S}_\phi \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} |\det(g)|^{s-\frac{1}{2}} dg.$$

The analogue of (7.1) takes the form:

$$\ell(s, \phi) = \left(\prod_{p \in S} \frac{Z(s, \mathcal{S}_p)}{L_p(s, \pi_p)} \right) \cdot L(s, \pi). \qquad (8.1)$$

Using [19, Proposition 3.3.1] we get that the left hand side and both the factors on the right hand side are entire functions. Evaluating at $s = 1/2$ gives

$$\Pi \text{ has a non-vanishing } H\text{-period} \iff L(\tfrac{1}{2}, \Pi) \neq 0. \qquad (8.2)$$

Next, the hypothesis that $\Pi \in \mathrm{Coh}(\mathrm{GL}_{2n}, \mu)$ puts us in an arithmetic context, however, as before, this doesn't guarantee that $s = \frac{1}{2}$ is critical. So, we now need the assumption that $s = \frac{1}{2}$ is critical for $\check{L}(s, \Pi)$. (In the $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$ case, we needed a stronger condition on the coefficient system, but in the current context [19, Proposition 6.3.1] guarantees that.) The analogue of (7.3) is:

$$L(\tfrac{1}{2}, \Pi) \neq 0 \iff L_f(\tfrac{1}{2}, \Pi) \neq 0. \qquad (8.3)$$

Under the assumption that $\Pi \in \mathrm{Coh}(\mathrm{GL}_{2n}, \mu)$ has a Shalika model, the algebraicity result in [19, Theorem 7.1.2] says

$$\sigma \left(\frac{L_f(\tfrac{1}{2}, \Pi \otimes \chi)}{\omega^{\epsilon_\chi}(\Pi) \mathcal{G}(\chi) \, \omega_\infty(\mu)} \right) \;=\; \frac{L_f(\tfrac{1}{2}, {}^\sigma\Pi \otimes {}^\sigma\chi)}{\omega^{\epsilon_{\sigma_\chi}}({}^\sigma\Pi) \, \mathcal{G}({}^\sigma\chi) \, \omega_\infty(\mu)},$$

where χ is an algebraic Hecke character, ϵ_χ its parity, $\mathcal{G}(\chi)$ its Gauß sum, $\omega^{\epsilon_\chi}(\Pi)$ is a nonzero complex number, and $\omega_\infty(\mu)$ is a nonzero complex number determined by μ. The analogue of (7.4) follows easily:

$$L_f(\tfrac{1}{2}, \Pi) \neq 0 \iff L_f(\tfrac{1}{2}, {}^\sigma\Pi) \neq 0. \qquad (8.4)$$

The reader may easily verify that $s = 1/2$ is critical for $L(s, \Pi)$ if and only if $s = 1/2$ is critical for $L(s, {}^\sigma\Pi)$.

Arithmeticity follows from first applying (8.2), (8.3) and (8.4) to Π and then noting that ${}^\sigma\Pi$ has a Shalika model by Theorem 5.3 and then applying (8.3) and (8.2) to ${}^\sigma\Pi$.

$\square$

9 Arithmeticity for Classical Groups

In this section, we consider the possibility of extending Theorem 3.1 for $\mathrm{GL}(N)$ to the case of classical groups. By the recent work [2] of Arthur and others, one now has a classification of square-integrable automorphic

representations for quasi-split classical groups, in terms of automorphic representations of $\mathrm{GL}(N)$. In view of this, it is natural to ask if arithmeticity results for $\mathrm{GL}(N)$ can be transferred to these classical groups.

More precisely, let G be a quasi-split symplectic, special orthogonal or unitary group over the number field F and let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. By Arthur [2], one can attach a discrete A-parameter to π and this is a multiplicity-free formal sum

$$\Psi = \Pi_1 \boxtimes S_{r_1} \boxplus \cdots \boxplus \Pi_k \boxtimes S_{r_k},$$

where Π_i is a cuspidal automorphic representation of $\mathrm{GL}(n_i)$ (over F or a quadratic extension E) satisfying some symmetry conditions and S_{r_i} is the r_i-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. Moreover, the set of all π's with a given discrete A-parameter Ψ is a full near equivalence class in the automorphic discrete spectrum of G. If $r_i = 1$ for all i, Ψ is called a tempered A-parameter.

Now suppose further that π is cohomological. Recall from Proposition 2.1 that for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, there is a square-integrable automorphic representation τ_σ of $G(\mathbb{A}_F)$ such that

$$\tau_{\sigma,f} \cong {}^\sigma\pi_f.$$

Note that, for fixed $\sigma \in \mathrm{Aut}(\mathbb{C})$, τ_σ may not be uniquely determined, but any two such candidates are nearly equivalent to each other and thus have the same A-parameter. It is thus natural to ask:

Question: How is the A-parameter of τ_σ related to that of π?

In the remainder of this section, we shall consider this question for the *quasi-split* classical groups G when the A-parameter Ψ of π is *tempered*. In this case, the A-parameter Ψ of π is equal to the L-parameter of π. We may also regard Ψ as the representation

$$\Pi = \boxplus_i \Pi_i := \mathrm{Ind}_P^{\mathrm{GL}(N)}(\otimes_i \Pi_i) \quad \text{of } \mathrm{GL}(N).$$

One knows moreover that this induced representation is irreducible. In the following, we shall use Ψ and Π interchangeably for the A-parameter of π. In particular, for each v, the L-parameter of π_v is precisely the L-parameter of the generic representation $\Psi_v = \Pi_v = \Pi_{1,v} \boxplus \cdots \boxplus \Pi_{k,v}$.

Let us also explicate the symmetry condition satisfied by the summands Π_i in Ψ in the various cases:

- if $G = \mathrm{Sp}(2n)$ or $\mathrm{SO}(2n)$, then $L^S(s, \Pi_i, \mathrm{Sym}^2)$ has a pole at $s = 1$ for each i;
- if $G = \mathrm{SO}(2n+1)$, then $L^S(s, \Pi_i, \wedge^2)$ has a pole at $s = 1$;

- if $G = \mathrm{U}(n)$, then $L^S(s, \Pi_i, \mathrm{Asai}^{(-1)^{n-1}})$ has a pole at $s = 1$.

Remark 9.1 Henceforth, when $G = \mathrm{U}(n)$, we shall assume that the underlying Hermitian space is defined with respect to a totally complex quadratic extension E of the totally real base field F. Moreover, the target group of the functorial lifting is $\mathrm{GL}(n)$ over the CM field E. In the rest of this section, in order to simplify the exposition, we shall only give proofs for the symplectic and orthogonal groups, even though the results are stated for $\mathrm{U}(n)$ as well.

It is natural to first investigate if the functorial transfer of unramified representations from classical groups to $\mathrm{GL}(N)$ is $\mathrm{Aut}(\mathbb{C})$-equivariant.

Lemma 9.2 *Let k be a p-adic field. Let G be an unramified classical group over k and for an unramified representation π of $G(k)$, let $\Sigma(\pi)$ be its functorial transfer to the appropriate $\mathrm{GL}(N)$.*

(i) *If $G = \mathrm{SO}(2n + 1)$, $\mathrm{Sp}(2n)$ or $\mathrm{U}(n)$, then*

$$^\sigma \Sigma(\pi) = \Sigma(^\sigma \pi).$$

(ii) *If $G = \mathrm{SO}(2n)$, then*

$$^\sigma (\Sigma(\pi) \otimes |\ |^{-1/2}) \otimes |\ |^{1/2} = \Sigma(^\sigma \pi).$$

Proof Let $T \subset B \subset G$ be a maximal torus contained in a Borel subgroup of G, both defined over k. Similarly, let $T^* \subset B^* \subset \mathrm{GL}(N)$, where $\mathrm{GL}(N)$ is the target of the functorial transfer from G. Now Langlands functoriality gives rise to a transfer $\chi \mapsto \chi^*$ from the set of unramified characters of T to that of T^*. Moreover, this transfer from T to T^* is $\mathrm{Aut}(\mathbb{C})$-equivariant.

Now $\pi \subset I(\chi) := \mathrm{Ind}_B^G \chi$ for some unramified character χ of T (where the induction is normalized here). The functorial transfer of $I(\chi)$ is then equal to $I(\chi^*)$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, $^\sigma \pi$ is still an unramified representation and

$$^\sigma \pi \subset {}^\sigma I(\chi) = I(^\sigma \chi \cdot \delta_{G,\sigma})$$

where $\delta_{G,\sigma} = {}^\sigma \delta_B^{1/2} / \delta_B^{1/2}$ and $\delta_B^{1/2}$ is the modulus character.

Thus, we are interested in whether

$$I(^\sigma \chi \cdot \delta_{G,\sigma}) \mapsto I(^\sigma \chi^* \cdot \delta_{\mathrm{GL}(N),\sigma})$$

under functorial transfer. Since $^\sigma \chi \mapsto {}^\sigma \chi^*$, it suffices to verify whether $\delta_{G,\sigma} \mapsto \delta_{\mathrm{GL}(N),\sigma}$.

Now we note:

- when $G = \mathrm{Sp}(2n)$, $\mathrm{SO}(2n)$, $\mathrm{U}(2n+1)$ or $\mathrm{GL}(2n+1)$, $\delta_B^{1/2}$ takes value in $\mathbb{Q}^\times$ and so $\delta_{G,\sigma} = 1$.
- when $G = \mathrm{SO}(2n+1)$, $\mathrm{U}(2n)$ or $\mathrm{GL}(2n)$, $\delta_B^{1/2}$ takes value in $q^{\frac{1}{2}\mathbb{Z}}$ where q is the size of the residue field of k, so that $\delta_{G,\sigma}$ is a quadratic character (possibly trivial).

In the context of (i), one sees that the transfer of $\delta_{G,\sigma}$ to T^* is $\delta_{\mathrm{GL}(N),\sigma}$. On the other hand, when $G = \mathrm{SO}(2n)$, $\delta_{G,\sigma}$ is trivial whereas $\delta_{\mathrm{GL}(2n),\sigma}$ is not necessarily trivial. A short computation then gives the result in (ii).

$\square$

Lemma 9.2 already implies that when $G \neq \mathrm{SO}(2n)$, the transfer (in the sense of Arthur) of $\tau_{\sigma,f}$ to $\mathrm{GL}(N)$ is nearly equivalent to the abstract representation ${}^\sigma\Pi_f$. Next, we note the following crucial proposition:

Proposition 9.3 *Let G be a classical group over a totally real F. Assume that π is a cohomological cuspidal representation of $G(\mathbb{A})$ with tempered A-parameter. Then for all infinite places v, π_v is "as close to being a discrete series representation as possible". More precisely,*

(i) *if $G(F_v)$ has discrete series representations, then π_v is discrete series. This is the case precisely when $G(F_v)$ is not of the form $\mathrm{SO}(2a+1, 2b+1)$ with $a, b \in \mathbb{Z}$.*

(ii) *If $G(F_v) = \mathrm{SO}(2a+1, 2b+1)$, then $\pi_v = \mathrm{Ind}_P^G \chi \boxtimes \pi_{0,v}$ where P is a maximal parabolic subgroup with Levi factor $\mathrm{GL}_1(F_v) \times \mathrm{SO}(2a, 2b)$, $\chi = 1$ or sgn, and $\pi_{0,v}$ is a discrete series representation of $\mathrm{SO}(2a, 2b)$.*

In particular, π_v is tempered in all cases. When $G \neq \mathrm{SO}(2n)$, the functorial transfer of π_v to $\mathrm{GL}(N)$ is a cohomological representation.

Proof We shall treat only symplectic and orthogonal groups; the case of unitary groups is similar. We make the following observations:

(a) Since π is cohomological, the infinitesimal character of π_v (for any infinite place v) is (strongly) regular and integral.

(b) Since π has tempered A-parameter, the results of Luo-Rudnick-Sarnak [33] imply that π_v is "close to being tempered".

Now we can express π_v as a quotient of an induced representation $I = \mathrm{Ind}_{P_v}^{G_v} \Sigma$, where

- P_v is a parabolic subgroup with Levi factor

$$\mathrm{GL}_1(F_v)^a \times \mathrm{GL}_2(F_v)^b \times G_{0,v}$$

where $G_{0,v}$ is a classical group of the same type as G_v;

- The representation Σ has the form

$$\Sigma = \left(\bigotimes_{i=1}^{a} \chi_i | - |^{s_i} \right) \bigotimes \left(\bigotimes_{j=1}^{b} \tau_j | - |^{t_i} \right) \bigotimes \pi_{0,v}$$

with χ_i either the trivial or the sign character, τ_j are discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ whose central character are trivial or sign, $\pi_{0,v}$ is a discrete series representation of $G_{0,v}$ and $s_i, t_j \in \mathbb{C}$.

The integrality of the infinitesimal character of π_v implies that the numbers s_i and t_j are half-integers. The "close to temperedness" of π_v in (b) above implies that the numbers s_i and t_j are close to the imaginary axis. Taken together, they imply that $s_i = t_j = 0$. In particular, we deduce that π_v is tempered.

We have yet to make use of the regularity condition in (a). Let us fix a maximal torus T_v of G_v with associated dual complexified Lie algebra $\mathfrak{t}^*$. The character group $X(T_v)$ endows $\mathfrak{t}^*$ with an integral structure. The infinitesimal character of π_v can be regarded as an element in $\mathfrak{t}^*$ up to conjugacy by the absolute Weyl group $W(G_v, T_v)$. Now we observe:

- there are no GL_2-factors in P_v.

  To see this, note that if there were a GL_2-factor in P_v, then the infinitesimal character of π_v will have the form $(\cdots, k, -k, \cdots)$, with k a half integer. But for the classical groups, there is a nontrivial element of the absolute Weyl group which fixes such an element, namely "exchanging the coordinates k and $-k$, followed by changing the signs of both coordinates". This contradicts regularity.

- there is at most one GL_1-factor in P_v. Indeed, there can be a GL_1-factor if and only if $G_v = \mathrm{SO}(2a+1, 2b+1)$.

  To see this, note that if there were two GL_1-factors in P_v, then the infinitesimal character of π_v will have the form $(\cdots, 0, 0, \cdots)$ which is fixed by a non-trivial element of the absolute Weyl group. Further, if there is a GL_1-factor, then the infinitesimal character has the form $(\cdots, 0, \cdots)$ and this is fixed by a nontrivial element of the absolute Weyl group of type B and C, namely "changing the sign a coordinate". Thus, a GL_1-factor can only occur in type D, so that $G = \mathrm{SO}(a, b)$ with $a + b$ even. If a GL_1-factor does occur, then $\mathrm{SO}(a-1, b-1)$ must have a discrete series representation, so that a and b must both be odd. Conversely, if a and b are both odd, there must be a GL_1-factor, since $\mathrm{SO}(a, b)$ does not have discrete series representations.

Summarizing the above observations, we see that unless $G_v = \mathrm{SO}(2a+1, 2b+1)$, there are no GL_1 or GL_2 factors in P_v, so that $G_{0,v} = G_v$ and π_v is discrete series. When $G_v = \mathrm{SO}(2a+1, 2b+1)$, G_v does not have discrete series and π_v is of the form given in (ii).

To prove the last assertion of the proposition, let us explicate the discrete series L-parameter Ψ_v of π_v when $G \neq \mathrm{SO}(2n)$:

- if $G = \mathrm{Sp}(2n)$, then $\Psi_v = \oplus_{j=1}^{n} \phi_j \oplus \chi$, where $\chi = 1$ or sign, and the ϕ_j's are pairwise distinct orthogonal representations of the Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$, which correspond to discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ with central character sign. Moreover, $(-1)^n \cdot \chi(-1) = 1$.
- if $G = \mathrm{SO}(2n+1)$, then $\Psi_v = \oplus_{j=1}^{n} \phi_j$ where the ϕ_j's are pairwise distinct symplectic representations of $W_{\mathbb{R}}$, which correspond to discrete series representations of $\mathrm{GL}_2(\mathbb{R})$ with trivial central character.
- if $G = \mathrm{U}(n)$, then $\Psi_v = \oplus_{j=1}^{n} \chi_j$, where the χ_j's are conjugate dual character of $W_{\mathbb{C}} = \mathbb{C}^\times$ of the form $\chi_j(z) = (z/\bar{z})^{\frac{a_j}{2}}$ with $a_j \equiv n+1$ mod 2.

From this and the description of cohomological representations of $\mathrm{GL}(N)$ given in Section 3, we observe that the representation Π_v with L-parameter Ψ_v is cohomological. The proposition is proved.

$\square$

Recall that the A-parameter of π is $\Pi := \Pi_1 \boxplus \cdots \boxplus \Pi_k$. As noted by Clozel, however, it is better to work with a Tate-twisted isobaric sum $\overset{T}{\boxplus}$:

$$\Pi = \boxplus_i \Pi_i = \overset{T}{\boxplus}_i \Sigma_i$$

where

$$\Sigma_i = \Pi_i \otimes |\ |^{\frac{n_i - N}{2}}.$$

Then the following corollary follows from Proposition 9.3:

Corollary 9.4 (i) *Suppose that $G = \mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$ or $\mathrm{U}(n)$. Then $\Pi = \overset{T}{\boxplus}_i \Sigma_i$ is a cohomological representation of $\mathrm{GL}(N)$ and for each i, Σ_i is a cohomological cuspidal representation of $\mathrm{GL}(n_i)$.*

(ii) *When $G = \mathrm{SO}(2n)$, then $\Pi \otimes |\ |^{-1/2}$ is an algebraic representation of $\mathrm{GL}(N)$ in the sense of [7], and for each i, $\Sigma_i \otimes |\ |^{-1/2}$ is an algebraic representation of $\mathrm{GL}(n_i)$.*

Now we come to the main result of this section.

Theorem 9.5 *Let F be totally real and let G be a quasi-split classical group over F with $G = \mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$ or $\mathrm{U}(n)$ (c.f. Remark 9.1). Let π be a cohomological cuspidal representation of G with a tempered A-parameter $\Psi = \boxplus_i \Pi_i = \overset{T}{\boxplus}_i \Sigma_i$. Then we have:*

(i) *For $\sigma \in \mathrm{Aut}(\mathbb{C})$, let τ_σ be a square-integrable automorphic representation such that $\tau_{\sigma,f} \cong {}^\sigma\pi_f$. Then the A-parameter of τ_σ is*

$$
{}^\sigma\Psi := {}^\sigma\Sigma_1 \overset{T}{\boxplus} \cdots \overset{T}{\boxplus} {}^\sigma\Sigma_k.
$$

(ii) *For each infinite place v, the L-parameter ${}^\sigma\Psi_v$ of $\tau_{\sigma,v}$ is equal to*

$$
\Psi_{\sigma^{-1}v} = \Pi_{1,\sigma^{-1}v} \boxplus \cdots \boxplus \Pi_{k,\sigma^{-1}v}.
$$

In particular, $\tau_{\sigma,v}$ is a discrete series representation for each infinite place v and thus τ_σ is cuspidal.

Proof   Again, we shall treat only symplectic and orthogonal groups in the proof; the case of unitary groups is similar.

(i) Let us first check that ${}^\sigma\Psi$ is actually an A-parameter for G. By Corollary 9.4 and Theorem 3.1, one has the cuspidal automorphic representations ${}^\sigma\Sigma_i$ for each i. By Remark 5.5 and its analog for $\mathrm{U}(n)$ (cf. Theorem 6.3), ${}^\sigma\Sigma_i$ has the same symmetry type as Σ_i (as detected by the poles of the twisted exterior square, symmetric square or the Asai L-function in the respective cases) and hence as Ψ. Thus ${}^\sigma\Psi$ is a bona-fide A-parameter for G and has an associated near equivalence class of square-integrable automorphic representations of G.

    To prove (i), we need to show that the representation τ_σ is contained in the near equivalence class associated to ${}^\sigma\Psi$. This follows immediately by Lemma 9.2, since

$$
{}^\sigma\Pi_v \;\cong\; \overset{T}{\boxplus}_i\, {}^\sigma\Sigma_{i,v}.
$$

(ii) In the context of the theorem, one knows by Proposition 3.2 that

$$
{}^\sigma\Psi_v \;=\; \overset{T}{\boxplus}_i\, {}^\sigma\Sigma_{i,v} \;=\; \overset{T}{\boxplus}_i\, \Sigma_{i,\sigma^{-1}v} \;=\; \boxplus_i \Pi_{i,\sigma^{-1}v}.
$$

To be more precise, we have used Proposition 3.2 to deduce that ${}^\sigma\Sigma_{i,v} = \Sigma_{i,\sigma^{-1}v}$. This follows immediately from Proposition 3.2 if n_i is even or if $F_v = \mathbb{C}$. Thus, the only issue is when n_i is odd and $F_v = \mathbb{R}$. Such a situation only occurs when $G = \mathrm{Sp}(2n)$. But it follows by Proposition 9.3

(or rather the last paragraph of its proof) that there is a unique i_0 such that n_{i_0} is odd. However, for this particular i_0, one has $(-1)^n \cdot \epsilon(\Pi_{i_0,v}) = 1$ for each infinite place v, where $\epsilon(\Pi_{i_0,v})$ is the sign in Proposition 3.2. In particular, $\epsilon(\Pi_{i_0,v})$ is independent of v, so that $^\sigma\Pi_{i_0,v} = \Pi_{i_0,\sigma^{-1}v}$.

Finally, since the archimedean component $^\sigma\Psi_v = \Psi_{\sigma^{-1}v}$ is a discrete series L-parameter for G_v and is the L-parameter of $\tau_{\sigma,v}$, we deduce that $\tau_{\sigma,v}$ is a discrete series representation. Since τ_σ is a square-integrable automorphic representation, it then follows by a well-known result of Wallach [48] that τ_σ is cuspidal. This proves (ii).

<div align="right">□</div>

Corollary 9.6 *In the context of the theorem, suppose that $\Psi = \Pi$ is a cuspidal representation of $\mathrm{GL}(N)$. Then $^\sigma\pi := (\otimes_{v \in S_\infty} \pi_{\sigma^{-1}v}) \otimes {}^\sigma\pi_f$ is a cohomological cuspidal automorphic representation of G.*

Proof Under the hypothesis of the corollary, the A-packet associated to Ψ is stable, in the sense that every member of the abstract global A-packet is automorphic. In this case, one may replace $\tau_{\sigma,\infty}$ by $^\sigma\pi_\infty$ and thus take τ_σ to be $^\sigma\pi$.

<div align="right">□</div>

10 Arithmeticity of Periods for Classical Groups

After Theorem 9.5, it makes sense to consider the question of arithmeticity of periods for classical groups. We consider two examples here.

10.1 Whittaker periods

For a quasi-split group G over F, if we fix a maximal F-torus T contained in a Borel subgroup $B = T \cdot N$ over F, then for any generic automorphic character ψ of $N(\mathbb{A}_F)$, we may consider the ψ-Whittaker period of an automorphic representation π of G. If this period is nonzero on π, we say that π is ψ-generic.

The group $T(F)$ acts naturally on the set of generic automorphic charcaters of $N(\mathbb{A}_F)$ and the notion of being ψ-generic depends only on the $T(F)$-orbit of ψ. In particular, if $T(F)$ acts transitively on the set of generic automorphic characters of $N(\mathbb{A}_F)$, then there is no harm in suppressing ψ. This is the case when $G = \mathrm{SO}(2n+1)$ and $\mathrm{U}(2n+1)$. When $G = \mathrm{Sp}(2n)$, the set of $T(F)$-orbits of generic automorphic characters is a torsor of $F^\times/F^{\times 2}$, whereas if $G = \mathrm{U}(2n)$, it is a torsor of $F^\times/\mathbb{N}_{E/F}E^\times$.

Now we have:

Theorem 10.1 *Let F be a totally real number field, and let $G = \mathrm{SO}(2n+1)$ or $\mathrm{U}(2n+1)$. Let π be a cohomological cuspidal automorphic representation of $G(\mathbb{A}_F)$ which is globally generic. Then for any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the conjugated representation $^\sigma\pi$ (as defined in Corollary 9.6) is also a cohomological cuspidal automorphic representation of $G(\mathbb{A}_F)$ which is globally generic.*

Proof Since π is globally generic, it follows by [2] and [9] that the A-parameter Ψ of π is tempered. By Theorem 9.5, we know that $^\sigma\pi$ belongs to the global A-packet associated to the tempered parameter $^\sigma\Psi$. Moreover, $^\sigma\pi_v$ is locally $^\sigma\psi_v$-generic for all finite places v. But since all generic characters of $N(F_v)$ are in the same $T(F_v)$-orbit, we deduce that $^\sigma\pi_v$ is ψ_v-generic as well. The same holds at the infinite places, since $^\sigma\pi_v = \pi_{\sigma^{-1}v}$. Thus, we see that $^\sigma\pi$ is abstractly ψ-generic.

Now in a local L-packet of $G(F_v)$, there can be at most one ψ_v-generic representation; this is a consequence of the theory of local descent (see Jiang-Soudry [27] for the case of $G = \mathrm{SO}(2n+1)$). Thus, $^\sigma\pi$ is the only member of its A-packet which could be globally ψ-generic. However, the theory of global descent [16] says that a tempered A-packet must contain a globally ψ-generic cuspidal automorphic representation. Thus we conclude that $^\sigma\pi$ is a globally generic cohomological cuspidal representation. $\qquad\square$

Remark 10.2 When $G = \mathrm{Sp}(2n)$ or $\mathrm{U}(2n)$, it is still true that the representation $^\sigma\pi$ is abstractly generic with respect to the generic character $(\otimes_{v \in S_\infty} \psi_{\sigma^{-1}v}) \otimes (\otimes_{v \notin S_\infty} {}^\sigma\psi_v)$. However, we do not know whether the $T(\mathbb{A}_F)$-orbit of this generic character contains an automorphic character of $N(\mathbb{A}_F)$. Nevertheless, if $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}^{\mathrm{ab}})$, then $^\sigma\psi_v = \psi_v$ for all finite v, and so we conclude as in the theorem that $^\sigma\pi$ is globally ψ-generic again. One may avoid this complication by working with similitude groups, for example $\mathrm{GSp}(2n)$; however, one needs to await the generalization of Arthur's results [2] to the context of similitude groups.

10.2   Gross-Prasad period

In this speculative final subsection, we consider the Gross-Prasad periods for the classical groups. To be concrete, let us consider the Gross-Prasad period for unitary groups. Let $\pi = \pi_1 \boxtimes \pi_2$ be a tempered cuspidal representation of $G = \mathrm{U}(n) \times \mathrm{U}(n-1)$. Then a recent preprint of Wei Zhang [49] establishes the global Gross-Prasad conjecture under some local hypotheses. In particular, he shows that the period of π over the diagonally

embedded $U(n-1)$ is nonzero if and only if π_v is $\Delta U(n-1)$-distinguished for all v, and

$$L_E(\tfrac{1}{2}, \Pi_1 \times \Pi_2) \neq 0.$$

Here, Π_i denotes the transfer of π to $GL(n)$ or $GL(n-1)$ over E,

Assume now that π is stable and cohomological, say $\pi \in \mathrm{Coh}(G, \mu)$. Suppose that π has a nonvanishing period over the diagonally embedded $U(n-1)$, and that $\mathrm{Hom}_{U(n-1)}(\mu, \mathbb{C}) \neq 0$. Then by Theorem 9.5 and its corollary, one knows that $^\sigma\pi$ is also a cohomological cuspidal automorphic representation of $U(n) \times U(n-1)$. Now one may apply the same argument as in the proof of Theorem 7.1 (with the hypotheses stated there) to deduce that $^\sigma\pi$ also has nonvanishing period over the diagonally embedded $U(n-1)$. We will perhaps leave the detailed treatment of this to a future occasion.

References

[1] U. K. Anandavardhanan, A. Kable and R. Tandon, *Distinguished representations and poles of twisted tensor L-functions*, Proc. Amer. Math. Soc. **132** (2004), no. 10, 2875–2883.

[2] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, Book in preparation; preliminary version available from `http://www.claymath.org/cw/arthur/`.

[3] M. Asgari and F. Shahidi, *Generic transfer for General Spin Groups*, Duke Math. J., **132**, (2006) 137–190.

[4] M. Asgari and F. Shahidi, *Functoriality for General Spin Groups*, `arXiv:1101.3467v1`, preprint (2011).

[5] A. Borel, *Regularization theorems in Lie algebra cohomology. Applications*, Duke Math. J. **50** (1983), no. 3, 605–623.

[6] D. Blasius, *Period relations and critical values of L-functions*, Olga Taussky-Todd: in memoriam. Pacific J. Math., (1997), Special Issue, 53–83.

[7] L. Clozel, *Motifs et Formes Automorphes: Applications du Principe de Fonctorialité*. in: *Automorphic forms, Shimura varieties, and L-functions*, Vol. I, Perspect. Math., vol. 10, eds. L. Clozel and J. S. Milne, (Ann Arbor, MI, 1988) Academic Press, Boston, MA, 1990, pp. 77–159.

[8] J. Cogdell, *Lectures on L-functions, converse theorems, and functo-riality for GL_n*, Lectures on automorphic L-functions, 1–96, Fields Inst. Monogr., 20, Amer. Math. Soc., Providence, RI, 2004.

[9] J. Cogdell, H. Kim, Piatetski-Shapiro and F. Shahidi, *On lifting from classical groups to* GL_n, Publ. Math. Inst. Hautes Études Sci. No. 93 (2001), 5–30.

[10] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales. With an appendix by N. Koblitz and A. Ogus*, In: Proc. Sympos. Pure Math., Vol. XXXIII, part II, AMS, Providence, R.I., (1979) pp. 313–346.

[11] Y. Flicker, *Twisted tensors and Euler products*, Bull. Soc. Math. France **116** (1988) no. 3, 295–313.

[12] Y. Flicker, *On distinguished representations*, J. Reine Angew. Math. **418** (1991), 139–172.

[13] S. Friedberg and H. Jacquet, *Linear periods*, J. Reine Angew. Math. **443** (1993), 91–139.

[14] W.T. Gan, B.H. Gross and D. Prasad, *Symplectic local root numbers, central critical L-values and restriction problems in the representation theory of classical groups*, To appear in Asterisque.

[15] S. Gelbart, *Automorphic forms on adele groups*, Annals of Math. Studies, Number 83, Princeton University Press (1975).

[16] D. Ginzburg, S. Rallis and D. Soudry, *The descent map from auto-morphic representations of GL(n) to classical groups*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011. x+339 pp.

[17] G. Gotsbacher and H. Grobner, *On the Eisenstein Cohomology of Odd Orthogonal Groups*, To appear in Forum Math.

[18] H. Grobner and A. Raghuram, *On some arithmetic properties of auto-morphic forms of GL_m over a division algebra*, `arXiv:1102.1872v2` preprint (2011).

[19] H. Grobner and A. Raghuram, *On the arithmetic of Shalika models and the critical values of L-functions for* GL_{2n}, *with an appendix by Wee Teck Gan*, to appear in Amer. J. Math.

[20] B. Gross and M. Reeder, *From Laplace to Langlands via representa-tions of orthogonal groups*, Bulletin AMS, **43** (2006), 163–205.

[21] G. Harder, *General aspects in the theory of modular symbols.* Seminar on number theory, Paris 1981–82 (Paris, 1981/1982), 73–88, Progr. Math., 38, Birkhauser Boston, Boston, MA, 1983.

[22] G. Henniart, *Sur la conjecture de Langlands locale pour* GL_n, 21st Journées Arithmétiques (Rome, 2001). J. Théor. Nombres Bordeaux 13, no. 1, 167–187 (2001).

[23] J. Hundley and E. Sayag, *Descent construction for GSpin groups: Main results and applications*, Electronic Research Announcements in Mathematical Sciences, **16** (2009) 31–36.

[24] H. Jacquet and R. Langlands, *Automorphic forms on GL(2).* Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970.

[25] H. Jacquet, I.I. Piatetskii-Shapiro, J.A. Shalika, *Rankin–Selberg Convolutions*, American J. of Math., **105** No. 2, (1983), 367–464.

[26] H. Jacquet, J. Shalika, *Exterior square L-functions*, in: *Automorphic forms, Shimura varieties, and L-functions*, Vol. II, Perspect. Math., vol. 10, eds. L. Clozel and J. S. Milne, (Ann Arbor, MI, 1988) Academic Press, Boston, MA, 1990, 143–226.

[27] D. H. Jiang and D. Soudry, *The local converse theorem for* $\mathrm{SO}(2n{+}1)$ *and applications*, Ann. of Math. (2) **157** (2003), no. 3, 743–806.

[28] A. Kable, *Asai L-functions and Jacquet's conjecture*, Amer. J. Math. **126** (2004), no. 4, 789–820.

[29] H. Kasten and C.-G. Schmidt, *On critical values of Rankin–Selberg convolutions*, To appear in International J. of Number Theory.

[30] Henry H. Kim and M. Krishnamurthy, *Base change lift for odd unitary groups*, Functional analysis VIII, 116–125, Various Publ. Ser. (Aarhus), 47, Aarhus Univ., Aarhus, 2004.

[31] Henry H. Kim and M. Krishnamurthy, *Stable base change lift from unitary groups to* GL_n, Int. Math. Res. Pap. **2005**, no. 1, 1–52.

[32] Jian-Shu Li and Joachim Schwermer, *On the Eisenstein cohomology of arithmetic groups*, Duke Math. J. **123** (2004), no. 1, 141–169.

[33] W. Z. Luo, Z. Rudnick and P. Sarnak, *On the generalized Ramanujan conjecture for* $\mathrm{GL}(n)$, in *Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996)*, 301–310, Proc.

Sympos. Pure Math., 66, Part 2, Amer. Math. Soc., Providence, RI, 1999.

[34] Y. Manin, *Periods of cusp forms, and p-adic Hecke series*, (Russian) Mat. Sb. (N.S.) **92** (134) (1973), 378–401.

[35] N. Matringe, *Distinguished representations and exceptional poles of the Asai L-function*, Manuscripta Math. **131** (2010), no. 3-4, 415–426.

[36] N. Matringe, *Distinguished generic representations of GL(n) over p-adic fields*, Int. Math. Res. Not., 2011, no. 1, 74–95.

[37] A. Raghuram, *On the special values of certain Rankin-Selberg L-functions and applications to odd symmetric power L-functions of modular forms*, Int. Math. Res. Not., Vol. (2010), 334–372, doi:10.1093/imrn/rnp127.

[38] A. Raghuram, *On the critical values of certain Rankin-Selberg L-functions for* $\mathrm{GL}_n \times \mathrm{GL}_{n-1}$, in preparation.

[39] A. Raghuram and F. Shahidi, *Functoriality and special values of L-functions*, Eisenstein series and applications, 271–293, Progr. Math. **258**, Birkhäuser Boston, Boston, MA, 2008.

[40] A. Raghuram and N. Tanabe, *Notes on the arithmetic of Hilbert modular forms*, J. Ramanujan Math. Soc. **26** (2011), no. 3, 261–319.

[41] S. Salamanca-Riba, *On the unitary dual of real reductive groups and the* $A_{\mathfrak{q}}(\lambda)$ *modules: the strongly regular case*, Duke Math. Journal, **96** No. 3 (1998), 521–546.

[42] F. Shahidi, *Arthur packets and the Ramanujan conjecture*, Kyoto J. Math. **51** (2011), no. 1, 1–23.

[43] G. Shimura, *On the periods of modular forms*, Math. Ann. **229** no. 3, (1977), 211–221.

[44] G. Shimura, *The special values of the zeta functions associated with Hilbert modular forms*, Duke Math. J. **45** (1978), no. 3, 637–679.

[45] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Reprint of the 1971 original. Publications of the Mathematical Society of Japan, 11. Kanô Memorial Lectures, 1. Princeton University Press, Princeton, NJ, 1994.

[46] D. Soudry, *On Langlands functoriality from classical groups to* GL_n. *Automorphic forms. I*, Astérisque **298** (2005), 335–390.

[47] J.-L. Waldspurger, *Quelques propriétés arithmétiques de certaines formes automorphes sur* GL(2), Compositio Math. **54** no. 2, (1985), 121–171.

[48] N. Wallach, *On the constant term of a square integrable automorphic form*, Operator algebras and group representations, Vol. II (Neptun, 1980), Monogr. Stud. Math., 18, Pitman, Boston, MA, (1984) 227–237.

[49] W. Zhang, *Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups*, preprint, available at http://www.math.columbia.edu/∼wzhang/math/online/transfer.pdf.

Wee Teck Gan, Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road Singapore 119076

E-mail: matgwt@nus.edu.sg

A. Raghuram, Indian Institute of Science Education and Research (IISER), First floor, Central Tower, Sai Trinity Building Garware Circle, Sutarwadi, Pashan Pune, Maharashtra 411021, India.

E-mail: raghuram@iiserpune.ac.in