Introduction to L-functions I: Tate's Thesis

References:

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- S. Kudla, *Tate's thesis*, in *An introduction to the Langlands program (Jerusalem, 2001)*, 109–131, Birkhuser Boston, Boston, MA, 2003.

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<u>Introduction</u>

What is an L-funcrtion?

It is a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad s \in \mathbb{C}$$

which typically converges when $Re(s) \gg 0$.

Simplest example: With $a_n = 1$ for all n, get the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \quad Re(s) > 1.$$

This enjoys some nice properties:

- $\zeta(s)$ has meromorphic continuation to $\mathbb{C}$, with a pole at $s = 1$;

- $\zeta(s)$ can be expressed as a product over primes:

  $$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, \quad Re(s) > 1.$$

  Such an expression is called an Euler product.

- $\zeta(s)$ satisfies a functional equation relating $s \leftrightarrow 1 - s$. More precisely, if we set

  $$\Lambda(s) = \pi^{-s/2} \cdot \Gamma(\frac{s}{2}) \cdot \zeta(s)$$

  where $\Gamma(s)$ is the gamma function, then

  $$\Lambda(s) = \Lambda(1 - s).$$

  $\Lambda(s)$ is sometimes called the complete zeta function of $\mathbb{Q}$

What are some other ”nice” L-funcitons which are known?

- **Dirichlet L-functions**: given a character

  $$\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \longrightarrow \mathbb{C}^{\times},$$

  set

  $$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}$$

  when $Re(s) > 1$.

  This is an L-function of degree 1: the factor at p in the Euler product has the from

  $$\frac{1}{P(p^{-s})}$$

  where P is a polynomial with constant term 1 with $deg(P) = 1$.

- **Hecke L-functions**: these are associated to ”Hecke characters”, and are generalizations of Dirichlet. In particular, they are L-functions of degree 1. These are precisely the L-functions (re)-treated in Tate's thesis.

- **Modular forms**: given a holomorphic modular form of weifght k:

$$f(z) = \sum_{n \geq 0} a_n e^{2\pi i z},$$

set

$$L(s, f) = \sum_{n > 0} \frac{a_n}{n^s}.$$

If f is a "normalized cuspidal Hecke eigenform", then

$$L(s, f) = \prod_p \frac{1}{1 - a_p p^{-s} + p^{k+1-2s}}.$$

These are degree 2 L-functions.

- **Artin L-functions**: these are L-functions associated to Galois representations over $\mathbb{C}$. Namely, given a continuous

$$\rho : \mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathsf{GL}_n(\mathbb{C}) = \mathsf{GL}(V),$$

set

$$L(s,\rho) = \prod_p L_p(s,\rho)$$

where

$$L_p(s,\rho) = \frac{1}{\det(1 - p^{-s}\rho(Frob_p)|V^{I_p})}$$

where I_p is the inertia group at p and $Frob_p$ is a Frobenius element at p. This is a degree n L-function, but of a very special type.

Question: What is a natural source of ”nice” L-functions of degree n?

Answer: (cuspidal) automorphic representations of GL(n).

$n = 1$: Hecke characters

$n = 2$: modular forms

It is conjectured (by Langlands) that every Artin L-function actually belongs to this class of ”automorphic” L-funcitons.

This Lecture: the theory for $n = 1$.

Some notations:

- F = number field;
- F_v = local field attached to a place v of F;
- $\mathcal{O}_v$ = ring of integers of F_v,
- ϖ_v = a uniformizer of F_v;
- q_v = cardinality of residue field of F_v;
- $\mathbb{A} = \prod'_v F_v$, ring of adèles of F;
- $\mathbb{A}^\times = \prod'_v F_v^\times$, group of idèles;
- Absolute value $|-| = \prod_v |-|_v : \mathbb{A}^\times \to \mathbb{R}_+^\times$.

Definition: A Hecke character is a continuous homomophism

$$\chi : F^\times \backslash \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$$

Say that χ is unitary if it takes values in unit circle S^1.

Every Hecke character χ is of form

$$\chi = \chi_0 \cdot |-|^s$$

with χ_0 unitary and $s \in \mathbb{R}$. So no harm in assuming χ unitary henceforth.

Lemma:

(i) $\chi = \prod_v \chi_v$, where

$$\chi_v : F_v^\times \to \mathbb{C}^\times$$

is defined by

$$\chi_v(a) = \chi(1, ..., a, ...),$$

with $a \in F_v^\times$ in the v-th position.

(ii) For almost all v, χ_v is trivial on $\mathcal{O}_v^\times$.

Indeed, (i) is clear and (ii) follows from continuity of χ.

Definition: Say that χ_v is **unramified** if χ_v is trivial on $\mathcal{O}_v^\times$. Such a χ_v is completely determined by

$$\chi_v(\varpi_v) \in \mathbb{C}^\times.$$

In particular, any unramified χ_v is of the form

$$\chi_v(a) = |a|_v^s$$

for some $s \in \mathbb{C}$.

Thus, a Hecke character $\chi = \prod_v \chi_v$ is almost "everywhere unramified".

Goal: Given a Hecke character $\chi = \prod_v \chi_v$, want to define an associated L-function $L(s, \chi)$ with Euler product

$$L(s, \chi) = \prod_v L(s, \chi_v),$$

and show it is nice.

This suggests that one should first treat:

Local problem: Given $\chi_v : F_v^\times \longrightarrow \mathbb{C}^\times$, define an associated Euler factor or local L-factor $L(s, \chi_v)$.

Definition: Set

$$L(s, \chi_v) = \begin{cases} \frac{1}{1-\chi_v(\varpi_v) q_v^{-s}}, \text{if } \chi_v \text{ unramified;} \\ \\ 1, \text{otherwise.} \end{cases}$$

This seems a bit arbitrary, but it is informed by the situation of Dirichlet's characters. More importantly, it is the definition which is compatible with local class field theory.

Interaction with local class field theory

The Artin reciprocity map of local class field theory

$$r : F_v^\times \longrightarrow \mathrm{Gal}(\overline{F}_v/F_v)^{ab}$$

has dense image, inducing an injection r^*

$$\{\text{characters of } \mathrm{Gal}(\overline{F}_v/F_v)\} \hookrightarrow \{\text{characters of } F_v^\times\}$$

If

$$r^*(\rho_v) = \chi_v,$$

then the definition of $L(s, \chi_v)$ given above is such that

$$L(s, \rho_v) = L(s, \chi_v)$$

where the L-factor on LHS is the local Artin L-factor.

Problem with Definition

The above definition of $L(s, \chi_v)$ is simple and direct, and allows us to define the global L-function associated to a Hecke character by:

$$L(s, \chi) = \prod_v L(s, \chi_v),$$

at least when $Re(s) \gg 0$.

However, it is not clear at all why this L-function is nice.

What we need: a framework in which these local L-factors arise naturally and which provides means of verifying the niceness of the associated global L-function .

This is what Tate's thesis achieved.

Local Theory

We suppress v from the notations. Fix unitary $\chi : F^\times \to \mathbb{C}^\times$.

Let $S(F)$ denote the space of Schwarz-Bruhat funcitons on F

$$= \begin{cases} \text{locally constant, compactly supported functions;} \\ \text{rapidly decreasing functions,} \end{cases}$$

in the finite or archimedean case resp.

Local Zeta Integrals: For $\phi \in S(F)$, set

$$Z(s, \phi, \chi) = \int_{F^\times} \phi(x) \cdot \chi(x) \cdot |x|^s \, d^\times x$$

where $d^\times x$ is a Haar measure on $F^\times$. Assume for simplicity that

$$\int_{\mathcal{O}^\times} d^\times x = 1.$$

Convergence?

Let's examine finite case.

Since

$$Z(s, \phi, \chi \cdot |-|^t) = Z(s+t, \phi, \chi),$$

no harm in assuming χ unitary.

(1) If $\phi(0) = 0$, then absolute convergence for all $s \in \mathbb{C}$, since integration is over an annulus $\{x : a < |x| < b\}$ which is compact. Then integral becomes a finite sum, and

$$Z(s, \phi, \chi) \in \mathbb{C}[q^s, q^{-s}].$$

Since any ϕ can be expressed as:

$$\phi = a \cdot \phi_1 + \phi_2$$

with

$$\phi_1 = \text{ characteristic function of a nbd of 0}$$

and

$$\phi_2(0) = 0,$$

we are reduced to examining:

(2) If $\phi = \phi_0 =$ characteristic function of $\mathcal{O}$, then

$$Z(s, \phi_0, \chi_0)$$

$$= \int_{\mathcal{O}-\{0\}} \chi(x) \cdot |x|^s \, d^\times x$$

$$= \sum_{n \geq 0} \int_{\mathcal{O}^\times} \chi(x \cdot \varpi^n) \cdot q^{-ns} d^\times x$$

$$= \left(\sum_{n \geq 0} \chi(\varpi)^n \cdot q^{-ns} \right) \cdot \int_{\mathcal{O}^\times} \chi(x) \, d^\times x$$

$$= \begin{cases} \frac{1}{1-\chi(\varpi)q^{-s}} \text{ if } \chi \text{ unramified;} \\ \\ 0 \text{ if } \chi \text{ ramified} \end{cases}$$

when $Re(s) > 0$.

Proposition:

(i) If χ is unitary, then $Z(s,\phi,\chi)$ converges absolutely when $Re(s) > 0$. It is equal to a rational function in q^{-s} and hence has meromorphic continuation to $\mathbb{C}$.

(ii) If χ is ramified, then $Z(s,\phi,\chi)$ is entire.

(iii) There exists $\phi \in S(F)$ such that

$$Z(s,\phi,\chi) = L(s,\chi).$$

(iv) For all $\phi \in S(F)$, the ratio

$$Z(s,\phi,\chi)/L(s,\chi)$$

is entire.

The properties (iii) and (iv) are often summarized as:

" $L(s,\chi)$ is a GCD of the family of zeta integrals $Z(s,\phi,\chi)$" .

Distributions on F

The next topic is the functional equation satisfied by the local zeta integrals. But first need to introduce some new objects.

A continuous linear functionals

$$Z : S(F) \longrightarrow \mathbb{C}$$

is called a distribution on F. Let $D(F)$ denote the space of distributions on F.

Corollary: For fixed $s \in \mathbb{C}$, the map

$$Z(s,\chi) : \phi \mapsto \frac{Z(s,\phi,\chi)}{L(s,\chi)}$$

is a nonzero distribution on F.

Action of $F^\times$

Now $F^\times$ acts on F by multiplication, and thus acts on $S(F)$ and, by duality, on$D(F)$:

$$(t \cdot \phi)(x) = \phi(xt), \quad \phi \in S(F)$$

$$(t \cdot Z)(\phi) = Z(t^{-1} \cdot \phi) \quad Z \in D(F)$$

Given a character χ of $F^\times$, let

$$D(F)_\chi = \{Z \in D(F) : t \cdot Z = \chi(t) \cdot Z \text{ for } t \in F^\times\}$$

be the χ-eigenspace in $D(F)$.

Exercise: Check that $Z(s, \chi)$ lies in the $\chi|-|^s$-eigenspace of $D(F)$.

In particular, for any χ, $D(F)_\chi$ is nonero.

Fourier transform

Given an additive character ψ of F and a Haar measure dx of F, one can define the Fourier transform

$$\mathcal{F} : S(F) \longrightarrow S(F)$$

$$\phi \mapsto \widehat{\phi}$$

given by

$$\widehat{\phi}(y) = \int_F \phi(x) \cdot \psi(-xy)\, dx.$$

One has the Fourier inversion formula

$$\widehat{\widehat{\phi}}(x) = c \cdot \phi(-x)$$

for some c. By adjusting dx, may assume $c = 1$, in which case say that dx is self-dual with respect to ψ.

By composition, $\mathcal{F}$ acts on $D(F)$:

$$\mathcal{F}(Z) = Z \circ \mathcal{F}.$$

Exercise: Check that if $Z \in D(F)_\chi$, then $\mathcal{F}(Z)$ lies in the $|-| \cdot \chi^{-1}$-eigenspace.

In particular,

$$Z(s,\chi) \circ \mathcal{F} \in D(F)_{\chi^{-1} \cdot |-|^{1-s}}$$

Equivalently, $Z(1-s,\chi^{-1}) \circ \mathcal{F}$ lies in the $\chi|-|^s$-eigenspace of $D(F)$.

Thus, there is a chance that it is equal to $Z(s,\chi)$

A Multiplicity One result

Proposition:

For any character χ, $\dim D(F)_\chi = 1$.

Corollary: There is a meromorphic function $\epsilon(s, \chi, \psi)$ such that

$$\frac{Z(1-s, \widehat{\phi}, \chi^{-1})}{L(1-s, \chi^{-1})} = \epsilon(s, \chi, \psi) \cdot \frac{Z(s, \phi, \chi)}{L(s, \chi)}$$

for all $\phi \in S(F)$.

The function $\epsilon(s, \chi, \psi)$ is called the **local epsilon factor** associated to χ (and ψ). Since both the fractions in the functional eqn above are entire, one deduces:

Corollary: The local epsilon factor $\epsilon(s, \chi, \psi)$ is a rational function in q^{-s} which is entire with no zeros. Thus it is of the form $a \cdot q^{bs}$.

Local Epsilon and Gamma factors

The local fuctional eqn is sometimes written as

$$Z(1-s,\widehat{\phi},\chi^{-1}) = \gamma(s,\chi,\psi)\cdot Z(s,\phi,\chi)$$

with

$$\gamma(s,\chi,\psi) = \epsilon(s,\chi,\psi)\cdot\frac{L(1-s,\chi^{-1})}{L(s,\chi)}.$$

The function $\gamma(s,\chi,\psi)$ is called the local gamma factor associated to (χ,ψ).

Computation of epsilon factor

Since we have the functional eqn

$$\frac{Z(1-s,\widehat{\phi},\chi^{-1})}{L(1-s,\chi^{-1})} = \epsilon(s,\chi,\psi)\cdot\frac{Z(s,\phi,\chi)}{L(s,\chi)},$$

and we know what is $L(s,\chi)$, in order to compute $\epsilon(s,\chi,\psi)$, it suffices to pick a suitable Schwarz funciton ϕ for which we can calculate both the local zeta integrals.

Exercise: Suppose that

- χ is unramified,
- ψ has conductor $\mathcal{O}$, i.e. χ is trivial on $\mathcal{O}$ but not on $\varpi^{-1}\mathcal{O}$, and
- $\phi =$ characteristic function of $\mathcal{O}$.

Then $\epsilon(s,\chi,\psi) = 1$.

Proof of Multiplicity One Result

One has an exact sequence:

$$0 \longrightarrow S(F^\times) \longrightarrow S(F) \xrightarrow{ev_0} \mathbb{C} \longrightarrow 0$$

Dualizing gives:

$$0 \longrightarrow \mathbb{C}\cdot\delta_0 \longrightarrow D(F) \longrightarrow D(F^\times) \longrightarrow 0$$

where $\delta_0 =$ Dirac delta.

Taking the χ-eigen-subspace, get:

$$0 \longrightarrow (\mathbb{C}\cdot\delta_0)_\chi \longrightarrow D(F)_\chi \longrightarrow D(F^\times)_\chi$$

Now note:

$$(\mathbb{C}\cdot\delta_0)_\chi = \begin{cases}\mathbb{C}, \text{ if } \chi \text{ trivial;}\\ 0, \text{ else,}\end{cases}$$

and

$$D(F^\times)_\chi = \mathbb{C} \quad \text{for any } \chi.$$

generated by: $\phi \mapsto \int_{F^\times} \phi(x)\chi(x)d^\times x$.

So we deduce:

$$0 \le \dim D(F)_\chi \le 1 \quad \text{if } \chi \text{ non-trivial;}$$

and

$$1 \le \dim D(F)_\chi \le 2 \qquad \text{if } \chi \text{ trivial.}$$

But $D(F)_\chi \neq 0$: $Z(0,\chi)/L(0,\chi)$ is a nonzero element in $D(F)_\chi$. This proves the proposition when χ is non-trivial.

Exercise: when χ is trivial, show that the unique $F^\times$-invariant distribution on $F^\times$ does not extend to F.

Archimedean case

- $F = \mathbb{R}$. Any χ has the form

$$\chi = |-|^s \quad \text{or} \quad \mathrm{sign} \cdot |-|^s.$$

One has

$$L(s, \mathbf{1}) = \pi^{-s/2} \cdot \Gamma(s/2)$$

and

$$L(s, \mathrm{sign}) = L(s+1, \mathbf{1}) = \pi^{-(s+1)/2} \cdot \Gamma(\frac{s+1}{2}).$$

- $F = \mathbb{C}$. Any χ has the form

$$\chi(z) = \chi_n(z) \cdot (z \cdot \bar{z})^s$$

with

$$\chi_n(z) = e^{in\theta} \quad \text{if } z = re^{i\theta}.$$

One has

$$L(s, \chi_n) = (2\pi)^{1-s} \cdot \Gamma(s + \frac{|n|}{2}).$$

Summary:

For each local field F, we considered a family of local zeta integrals

$$\{Z(s, \phi, \chi) : \phi \in S(F)\}$$

and obtained the local L-factor $L(s, \chi)$ as a GCD of this family,

The local zeta integrals satisfy a local functional eqn relating

$$Z(s, \phi, \chi) \leftrightarrow Z(1 - s, \widehat{\phi}, \chi^{-1}).$$

The constant of proportionality gives the local gamma factor $\gamma(s, \chi, \psi)$ or equivalently the local epsilon factor $\epsilon(s, \chi, \psi)$.

Global Theory

Facts about $\mathbb{A}$:

- Let

  $$\mathbb{A}^1 = \text{kernel of } |-| : \mathbb{A}^\times \to \mathbb{R}_+^\times$$

  Then

  $$\mathbb{A}^\times \cong \mathbb{R}_+^\times \times \mathbb{A}^1$$

- $F^\times \subset \mathbb{A}^1$ (product formula) as a discrete subgroup such that

  $$\mathbb{A}^1/F \text{ is compact}$$

  Let $E \subset \mathbb{A}^1$ be a fundamental domain for $\mathbb{A}^1/F^\times$.

Additive Characters of $\mathbb{A}/F$

Let

$$\psi : \mathbb{A}/F \longrightarrow S^1$$

be a nontrivial additive character. Then any other such ψ' is of form

$$\psi'(x) = \psi(ax)$$

for some $a \in F^\times$. Moreover,

$$\psi = \prod_v \psi_v.$$

Let $dx = \prod_v dx_v$ be Haar measure on $\mathbb{A}$ so that dx_v is self-dual wrt ψ_v for all v.

Start with a Hecke character

$$\chi : \mathbb{A}^\times / F^\times \longrightarrow \mathbb{C}^\times$$

and recall that

$$\chi = \prod_v \chi_v$$

with χ_v unramified for almost all v. Assume wlog that χ is unitary.

We have defined $L(s, \chi_v)$ and $\epsilon(s, \chi_v, \psi_v)$ for all v. So we may define:

Definition:

$$L(s, \chi) = \prod_{v<\infty} L(s, \chi_v).$$

$$\Lambda(s, \chi) = \prod_v L(s, \chi_v)$$

$$\epsilon(s, \chi) = \prod_v \epsilon(s, \chi_v, \psi_v)$$

The first two products converge absolutely if $Re(s) > 1$.

The third product is a finite one, since

$$\epsilon(s, \chi_v, \psi_v) = 1$$

for almost all v. Moreover, it is independent of ψ.

For any finite set S of places of F, also set

$$L^S(s, \chi) = \prod_{v \notin S} L(s, \chi_v).$$

Goal: Show that $\Lambda(s, \chi)$ has meromorphic continuation to $\mathbb{C}$ and satisfies a functional eqn $s \leftrightarrow 1 - s$:

$$\epsilon(s, \chi) \cdot \Lambda(1 - s, \chi^{-1}) = \Lambda(s, \chi).$$

Schwarz space on $\mathbb{A}$

We will imitate the local situation by considering "global zeta integrals".

Let $S(\mathbb{A})$ denote the space of Schwarz-Bruhat funcitons on $\mathbb{A}$. Then

$$S(\mathbb{A}) = S(F \otimes_{\mathbb{Q}} \mathbb{R}) \otimes (\otimes'_{v<\infty} S(F_v))$$

where $\otimes'_v$ stands for the restrictied tensor product.

More concretely, a function in $S(\mathbb{A})$ is a finite linear combination of functions of the form

$$f(x) = f_\infty(x_\infty) \cdot \prod_{v<\infty} f_v(x_v)$$

with

$$f_v = \text{characteristic function } \phi_{0,v} \text{ of } \mathcal{O}_v$$

for almost all v.

We say such f is factorizable.

Global Zeta Integrals

Analogous to the local setting, for $\phi \in S(\mathbb{A})$, we set

$$Z(s, \phi, \chi) = \int_{\mathbb{A}^\times} \phi(x) \cdot \chi(x)|x|^s \, d^\times x$$

where $d^\times x = \prod_v d^\times x_v$ is a Haar measure on $\mathbb{A}^\times$.

Observe that formally, if $\phi = \otimes_v \phi_v$ is factorizable,

$$Z(s, \phi, \chi) = \prod_v Z(s, \phi_v, \chi_v).$$

The integral defining $Z(s, \phi, \chi)$ converges at s if and only if

(i) the integral defining $Z(s, \phi_v, \chi_v)$ converges for all v;

(ii) the product $\prod_v Z(s, \phi_v, \chi_v)$ converges.

(i) holds whenever $Re(s) > 0$ (recall χ_v unitary by assumption).

Further, for almost all v, χ_v is unramified and $\phi_v = \phi_v^0$. For such v's,

$$Z(s, \phi_v, \chi_v) = L(s, \chi_v),$$

so that

$$Z(s, \phi, \chi) = L^S(s, \chi) \cdot \prod_{v \in S} Z(s, \phi_v, \chi_v).$$

So (ii) holds iff the product $\prod_v L(s, \chi_v)$ converges, which we have observed to hold when $Re(s) > 1$.

Lemma: The integral defining $Z(s, \phi, \chi)$ converges absolutely when $Re(s) > 1$.

Upshot: Proving meromorphic continuation and functional eqn of $L(s, \chi)$ is equivalent to proving meromorphic continuation and functional eqn for $Z(s, \phi, \chi)$.

Fourier Analysis on $\mathbb{A}$

Unlike the local case, the meromorphic continuation of global zeta integrals is less direct. It requires a global input: the Poisson summation formula.

For the fixed $\psi : \mathbb{A}/F \to S^1$ and dx the associated self-dual measure, one has a notion of Fourier transform

$$\widehat{\phi}(y) = \int_{\mathbb{A}} f(x) \cdot \psi(xy)^{-1}\, dx$$

It is clear that if $\phi = \otimes_v \phi_v$ is factorizable,

$$\widehat{\phi} = \otimes_v \widehat{\phi_v}.$$

Poisson Summation Formula

This is the key tool used in the global theory.

Proposition: For $\phi \in S(\mathbb{A})$, one has:

$$\sum_{x \in F} \phi(x) = \sum_{x \in F} \widehat{\phi}(x)$$

Proof: Let $F_\phi : \mathbb{A} \to \mathbb{C}$ be defined by

$$F_\phi(y) = \sum_{x \in F} \phi(x+y).$$

Then F_ϕ is a function on $\mathbb{A}/F$.

Consider Fourier expansion of F_ϕ:

$$F_\phi(y) = \sum_{a \in F} c_a(\phi) \cdot \psi(ay)$$

with

$$\begin{aligned} c_a(\phi) &= \int_{\mathbb{A}/F} F_\phi(z) \cdot \psi(az)^{-1}\, dz \\ &= \int_{\mathbb{A}/F} \sum_{x \in F} \phi(z+x) \cdot \psi(a(z+x))^{-1}\, dz \\ &= \int_{\mathbb{A}} \phi(z) \cdot \psi(az)^{-1} \\ &= \widehat{\phi}(a). \end{aligned}$$

Hence, we have

$$\sum_{x \in F} \phi(x+y) = F_\phi(y) = \sum_{a \in F} \widehat{\phi}(a) \cdot \psi(ay).$$

Now evaluate F_ϕ at $y = 0$ to get

$$\sum_{x \in F} \phi(x) = \sum_{a \in F} \widehat{\phi}(a).$$

Corollary: For any $b \in \mathbb{A}^\times$, have

$$\sum_{x \in F} \phi(bx) = \frac{1}{|b|} \cdot \sum_{x \in F} \widehat{\phi}(x/b).$$

Main Global Theorem of Tate's Thesis

(i) $Z(s, \phi, \chi)$ has meromorphic continuation to $\mathbb{C}$.

(ii) It is entire unless χ is unramfied. If χ is unramified, we may assume that $\chi = 1$. Then the only possible poles are simple and occur at

- $s = 0$, with residue $-\kappa \cdot \phi(0)$;
- $s = 1$ with residue $\kappa \cdot \widehat{\phi}(0)$,

with

$$\kappa = \int_{F^\times \backslash \mathbb{A}^1} d^\times x$$

.

(iii) There is a global functional eqn:

$$Z(s, \phi, \chi) = Z(1 - s, \widehat{\phi}, \chi^{-1}).$$

Corollary:

(i) $\Lambda(s, \chi)$ has meromorphic continuation to $\mathbb{C}$

(ii) It is entire unless χ is unramified, in which case, assuming $\chi = 1$, the only poles are at $s = 0$ and $s = 1$. The identification of the residues there is the "class number formula".

(iii) There is a functional equation

$$\Lambda(1 - s, \chi^{-1}) = \epsilon(s, \chi) \cdot \Lambda(s, \chi).$$

Proof: (i) is clear and (ii) requires a precise choice of ϕ, which we omit here.

We shall discuss the proof of (iii).

(iii) The equation

$$Z(1-s,\widehat{\phi},\chi^{-1}) = Z(s,\phi,\chi)$$

implies

$$\prod_{v\in S} Z(1-s,\widehat{\phi_v},\chi_v^{-1})\cdot L^S(1-s,\chi^{-1}) =$$

$$= \prod_{v\in S} Z(s,\phi_v,\chi_v)\cdot L^S(s,\chi),$$

for some finite set S of places of F. Now use local functional eqn: for $v \in S$,

$$Z(1-s,\widehat{\phi_v},\chi_v^{-1}) = \gamma(s,\chi_v,\psi_v)\cdot Z(s,\phi_v,\chi_v).$$

Get

$$\left(\prod_{v\in S}\gamma(s,\chi_v,\psi_v\right)\cdot L^S(1-s,\chi^{-1}) = L^S(s,\chi),$$

or equivalently

$$\epsilon(s,\chi,\psi)\cdot \Lambda(1-s,\chi^{-1}) = \Lambda(s,\chi).$$

Proof of Main Global Theorem

When $Re(s) > 1$,

$$Z(s, \phi, \chi)$$

$$= \int_{\mathbb{A}^\times} \phi(x) \cdot \chi(x) |x|^s \, d^\times x$$

$$= \int_{|x| \geq 1} (...) + \int_{|x| \leq 1} (...)$$

$$= (I) + (II)$$

Now observe that the integral (I) is absolutely convergent on $\mathbb{C}$, so that it defines an entire function. Indeed, we have already noted that (I) and (II) are convergent for $Re(s) > 1$.

Assume that $Re(s) \leq 1$, and let's examine (I).

For $|x| \geq 1$, one has the following

$$\textbf{Miracle:} \quad |x|^t \leq |x|^2 \quad \text{if } t \leq 1,$$

so that the integrand

$$|\phi(x)\chi(x)|x|^s| = |\phi(x)| \cdot |x|^{Re(s)} \leq |\phi(x)| \cdot |x|^2$$

Hence, the integral converges even better!

So the main issue is the integral (II).

We shall show, using Poisson summation,

Lemma: For $Re(s) > 1$:

$$(II) = \int_{|x|\geq 1} \widehat{\phi}(x)\chi(x)^{-1}|x|^{1-s}\, d^{\times}x$$

$$+\frac{(\kappa\widehat{\phi}(0))}{s-1} - \frac{(\kappa\phi(0))}{s}$$

This gives:

$$Z(s,\phi,\chi) =$$

$$\int_{|x|\geq 1} \phi(x)\cdot\chi(x)|x|^{s}\, d^{\times}x$$

$$+\int_{|x|\geq 1} \widehat{\phi}(x)\cdot\chi(x)^{-1}|x|^{1-s}\, d^{\times}x.$$

$$+\frac{(\kappa\widehat{\phi}(0))}{s-1} - \frac{(\kappa\phi(0))}{s}$$

This gives meromorphic cont., and the functional eqn, since this expression is defined for all s and is invariant under

$$(s,\chi,\phi) \mapsto (1-s,\chi^{-1},\widehat{\phi})$$

Proof of Lemma

First use $\mathbb{A}^\times \cong \mathbb{R}_+^\times \times \mathbb{A}^1$ to break (II) into a double integral:

$$\int_{|x|\leq 1} \phi(x) \cdot \chi(x)|x|^s \, d^\times x$$

$$= \int_0^1 \int_{\mathbb{A}^1} \phi(tx) \cdot \chi(tx) t^s \; d^\times t \; d^\times x$$

$$= \int_0^1 \chi(t) \cdot t^s \cdot Z_t(\phi, \chi) \, d^\times t$$

with

$$Z_t(\phi, \chi) = \int_{\mathbb{A}^1} \phi(tx) \cdot \chi(x) \, d^\times x.$$

Now

$$Z_t(\phi, \chi)$$

$$= \int_{F^\times \backslash \mathbb{A}^1} \sum_{\gamma \in F^\times} \phi(t\gamma x) \cdot \chi(\gamma x) \; d^\times x$$

$$= \int_E \left(\sum_{\gamma \in F} \phi(t\gamma x) \right) \cdot \chi(x) \, d^\times x - \phi(0) \int_E \chi(x) d^\times x$$

$$= \int_E \left(\frac{1}{t} \cdot \sum_{\gamma \in F} \widehat{\phi}(\gamma / tx) \right) \cdot \chi(x) \, d^\times x - (A)$$

$$= \frac{1}{t} \cdot \int_{\mathbb{A}^1} \widehat{\phi}(1/tx) \chi(x) d^\times x + \frac{1}{t} \cdot \widehat{\phi}(0) \cdot \int_E \chi(x) d^\times x - (A)$$

$$= \frac{1}{t} Z_{1/t}(\widehat{\phi}, \chi^{-1}) + \frac{1}{t} \cdot (B) - (A)$$

Here,

$$(A) = \phi(0) \cdot \int_E \chi(x) d^\times x$$

$$= \begin{cases} \phi(0) \cdot Vol(E), \text{ if } \chi \text{ ``unramified''}; \\ \\ 0, \text{otherwise} \end{cases}$$

and

$$(B) = \widehat{\phi}(0) \cdot \int_E \chi(x) d^\times x$$

$$= \begin{cases} \widehat{\phi}(0) \cdot Vol(E) \text{ if } \chi \text{ ``unramified''} \\ \ 0, \text{ otherwise.} \end{cases}$$

Here, χ "unramified" means "χ is trivial on $\mathbb{A}^1$". This is stronger than the condition that $\chi = \prod_v \chi_v$ with χ_v unramified for all v, which is what one typically means by "unramified χ".

So assuming wlog that $\chi = 1$ if it is unramified, we have:

$$(II) = \int_0^1 \chi(t) \cdot t^s \cdot Z_t(\phi, \chi) \, d^\times t$$

$$= \int_0^1 \chi(t) \cdot t^{s-1} \cdot Z_{1/t}(\widehat{\phi}, \chi^{-1}) \; d^\times t$$

$$+ (B) \cdot \int_0^1 t^{s-1} \, d^\times t - (A) \cdot \int_0^1 t^s \, d^\times t$$

$$= \int_1^\infty \chi(t)^{-1} t^{1-s} Z_t(\widehat{\phi}, \chi^{-1}) d^\times t + \frac{(B)}{s-1} - \frac{(A)}{s}$$

$$= \int_{|x| \geq 1} \widehat{\phi}(x) \cdot \chi(x)^{-1} |x|^{1-s} \, d^\times x + \frac{(B)}{s-1} - \frac{(A)}{s}$$

This proves the lemma, and thus the main global theorem.

Summary:

(i) One considers a family of global zeta integrals, and express them as the product over v of local zeta integrals, at least when $Re(s) \gg 0$.

(ii) Study the local zeta integrals, and use them to define local L-factors (as GCD's) and local epsilon factors (via local functional eqn); this defines the global L-function and global epsilon factor when $Re(s) \gg 0$.

(iii) Prove meromorphic continuation of global zeta integrals and global functional equation.

(iv) Using (iii), deduce meromorphic continuation and funtional eqn of global L-functions.

In the 2nd lecture, we will follow this paradigm.