ENDOSCOPIC LIFTS FROM PGL_3 TO G_2

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Abstract. We determine essentially completely the theta correspondence arising from the dual pair $PGL_3 \times G_2 \subset E_6$ over a p-adic field. Our first result determines the theta lift of any non-supercuspidal representation of PGL_3 and shows that the lifting respects Langlands functoriality. Our second result shows that the theta lift $\theta(\pi)$ of a (non-self-dual) supercuspidal representation π of PGL_3 is an irreducible generic supercuspidal representation of G_2; we also determine $\theta(\pi)$ explicitly when π has depth zero.

1. Introduction

Let k be a non-archimedean local field of characteristic zero and residue characteristic p. Let $\underline{H}$ be a split adjoint linear algebraic group of type E_6 over k, and let $H = \underline{H}(k)$ be its group of k-points. There is a dual reductive pair:

$$PGL_3 \times G_2 \hookrightarrow H,$$

and one can thus consider the representation correspondence induced by the restriction of the minimal representation Π of H to this dual pair.

For an irreducible admissible representation π of PGL_3, we let $\Theta(\pi)$ denote the set of irreducible admissible representations π' of G_2, counted with multiplicities, such that

$$\mathrm{Hom}_{PGL_3 \times G_2}(\Pi, \pi \otimes \pi') \neq 0.$$

In [GS], Gross and Savin have given a precise conjecture regarding the determination of the set $\Theta(\pi)$ in terms of π. For the convenience of the reader, we review the conjecture briefly.

The irreducible admissible representations of PGL_3 are known to be parametrized by L-parameters, which are admissible homomorphisms

$$\varphi : W_k \times SL_2(\mathbb{C}) \to SL_3(\mathbb{C}),$$

where W_k denotes the Weil group of k. Let A_φ be the component group of the centralizer of the image of φ. Then

$$A_\varphi = \begin{cases} 1, & \text{if } \varphi \text{ is reducible;} \\ \mu_3, & \text{if } \varphi \text{ is irreducible.} \end{cases}$$

If π is supercuspidal, then its corresponding L-parameter φ is irreducible, so that $A_\varphi = \mu_3$.

On the other hand, the irreducible admissible representations of G_2 are *conjecturally* parametrized by pairs (φ', χ'), where

$$\varphi' : W_k \times SL_2(\mathbb{C}) \to G_2(\mathbb{C})$$

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is an L-parameter for G_2, and χ' is an irreducible character of the finite component group $A_{\varphi'}$. This conjectured parametrization is known for non-supercuspidal representations but not for supercuspidal representations in general.

Now, up to conjugacy, there is a natural inclusion of dual groups

$$i : SL_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C}).$$

Composing φ with this natural inclusion, we obtain an L-parameter $\varphi' = i \circ \varphi$ for the group G_2. The inclusion i also induces an injection $i_* : A_\varphi \to A_{\varphi'}$ [GS, Proposition 2.7], which identifies A_φ as a subgroup of index 1 or 2.

Consider the restriction of the minimal representation Π of H to the dual pair $PGL_3 \times G_2$. The maximal π-isotypic quotient

$$\Pi / \bigcap_{\phi \in \mathrm{Hom}_{PGL_3}(\Pi,\pi)} \mathrm{Ker}(\phi)$$

of Π can be expressed as $\pi \otimes \theta(\pi)$, for some smooth representation $\theta(\pi)$ of G_2 [MVW, Lemme III.4], and by definition, there is a $PGL_3 \times G_2$-equivariant surjection $\Pi \to \pi \otimes \theta(\pi)$. Then $\Theta(\pi)$ is simply the set of irreducible quotients of $\theta(\pi)$, counted with multiplicities. The following is the conjecture from [GS]:

Conjecture: The correspondence is functorial with respect to the inclusion of dual groups. More precisely, $\Theta(\pi)$ is the set of π' whose parameter (φ', χ') satisfies:

$$\begin{cases} \varphi' = i \circ \varphi_\pi, \\ \chi' \circ i_* = 1. \end{cases}$$

This conjecture was verified in [MS] for π a tempered spherical representation, and in [GS] for π a generalized principal series representation. The first main result of this paper is the complete determination of the set $\Theta(\pi)$ when π is non-supercuspidal, given in Theorems 11, 12 and 13. As a result, we have:

Theorem 1. *The conjecture is true for non-supercuspidal* π.

We highlight here an important special case, given by our Theorem 14. If $\pi = St$ is the Steinberg representation of PGL_3, then its L-parameter φ is trivial on W_k and $\varphi|_{SL_2(\mathbb{C})}$ is the homomorphism given by the adjoint representation of $SL_2(\mathbb{C})$. Thus, $A_\varphi \cong \mu_3$ and $A_{\varphi'} \cong S_3$. The symmetric group S_3 has three irreducible characters: 1, ρ and ϵ, where ρ is a 2-dimensional character, and ϵ is the sign character. We show that

$$\Theta(St) = \{\pi(1)', \pi_{sc}[1]\},$$

where $\pi(1)'$ is a generic discrete series representation of G_2 corresponding to the parameter $(\varphi', 1)$, and $\pi_{sc}[1]$ is a non-generic supercuspidal representation corresponding to the parameter (φ', ϵ).

Since the local Langlands conjecture for supercuspidal representations of G_2 is not known, the conjecture of Gross-Savin cannot be verified literally for supercuspidal representations of

PGL_3. However, if π is supercuspidal with L-parameter φ, then

$$A_{i\circ\varphi} = \begin{cases} \mu_3, & \text{if } \pi \text{ is not self-dual;} \\ S_3, & \text{if } \pi \text{ is self-dual.} \end{cases}$$

In particular, the conjecture predicts that:

$$\#\Theta(\pi) = \begin{cases} 1, & \text{if } \pi \text{ is not self-dual;} \\ 2, & \text{if } \pi \text{ is self-dual.} \end{cases}$$

Our second main result verifies the above prediction when π is not self-dual:

Theorem 2. *Let π be a supercuspidal representation of PGL_3 which is not self-dual. Then $\theta(\pi)$ is an irreducible generic supercuspidal representation of G_2.*

We remind the reader that, if $p \neq 2$, there are no self-dual supercuspidal representations of PGL_3, so that the above result holds for all supercuspidal representations.

The two theorems above give essentially complete information about the theta lifting from PGL_3 to G_2 (when $p \neq 2$), except for the determination of the irreducible supercuspidal representation $\theta(\pi)$ when π is supercuspidal. It may be possible to give a characterization of $\theta(\pi)$ in terms of π without referring to the local Langlands conjecture. In the final section, we do this for π of depth zero, using the construction of supercuspidal representations by induction from open compact subgroups.

Note that PGL_3 is an endoscopic group of G_2, and the above results show that the theta correspondence for $PGL_3 \times G_2$ can be used to construct elements of the corresponding endoscopic L-packets of G_2. To obtain every element of such a packet, one would need to consider the dual pair correspondence arising from $PD^\times \times G_2$, where D is a degree 3 division algebra. This dual pair is an inner twist of the one considered in this paper and was studied in [S3]. These local results are necessary preliminaries for the construction of global endoscopic L-packets using global theta liftings from PGL_3 and $PD^\times$. We plan to pursue this global question in a future paper.

2. Representations of l-Groups

In this section, we establish some notations and discuss some basic facts on the representation theory of p-adic groups that are required later.

Let G be the k-points of a connected reductive algebraic group $\underline{G}$ over k. Then recall that G is an l-group in the terminology of [BZ]. Let $Alg(G)$ be the category of smooth representations of G. For $\pi \in Alg(G)$, we shall let π^* denote the full linear dual of π; it is a representation of G but is usually not smooth. Its subspace of smooth vectors is the contragredient representation $\pi^\vee$.

Recall that if $P = MN$ is a parabolic subgroup of G, then we have an exact functor (normalized parabolic induction)

$$I_P : Alg(M) \longrightarrow Alg(G).$$

We also have the normalized Jacquet functor

$$R_P : Alg(G) \longrightarrow Alg(M).$$

Let $\bar{P}$ be the opposite parabolic subgroup. Then we have the following adjointness properties:

$$\begin{cases} Hom_G(\pi, I_P(\sigma)) = Hom_M(R_P(\pi), \sigma), \\ Hom_G(I_P(\sigma), \pi) = Hom_M(\sigma, R_{\bar{P}}(\pi)). \end{cases}$$

A complete proof of the second property has now been given in [B].

Sometimes, we may wish to talk about unnormalized induction and unnormalized Jacquet functor. We shall use Ind and ind to denote unnormalized smooth and compact induction respectively, and we shall write $\pi \mapsto \pi_N$ (coinvariants) for unnormalized Jacquet functor. More generally, if ψ is a (unitary) character of N, $\pi_{N,\psi}$ will denote the maximal quotient of π on which N acts by ψ.

An irreducible representation π is said to belong to the discrete series or is square-integrable if π has unitary central character and its matrix coefficients are square-integrable modulo center. An example is the so-called Steinberg representation. A representation $\pi \in Alg(G)$ is called supercuspidal if $R_P(\pi) = 0$ for any proper parabolic subgroup P. If π is an irreducible unitarizable supercuspidal representation, then it is square-integrable. For any $\pi \in Alg(G)$, we have a decomposition (see [BZ])

$$\pi = \pi_c \oplus \pi_i$$

where π_c is the supercuspidal part of π and π_i has no non-zero supercuspidal subquotient.

Let us also recall the notion of a "generic representation" of G. For simplicity, assume that $\underline{G}$ is split and has connected center. Let $B = T \cdot U$ be a Borel subgroup with unipotent radical U and maximal split torus T. A character ψ of U is said to be in general position if its stabilizer in T is equal to the center of G. The group T acts transitively on the set of such characters of U and an irreducible representation π of G is said to be generic if $\pi_{U,\psi} \neq 0$ for some (and hence any) ψ in general position. By Frobenius reciprocity, this is equivalent to saying that π admits a non-zero G-equivariant map into $Ind_U^G \psi$. In this case, $dim(\pi_{U,\psi}) = 1$.

Finally, on several occassions, we shall use the following simple observation. Let

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

be an exact sequence of G-modules. If an element of the Bernstein center of G acts on V_1 and V_3 by different eigenvalues, then V_1 is a quotient of V_2, so that $V_2 \cong V_1 \oplus V_3$.

3. **Structure of Maximal Parabolics**

Let $\tilde{Q}_1$ and $\tilde{Q}_2$ be the two non-conjugate maximal parabolic subgroups of $GL_3(k) = GL(W_3)$ stabilizing a 1-dimensional subspace W_1 and a 2-dimensional subspace W_2 of W_3 respectively. We fix $W_1 \subset W_2$. Their Levi factors are $GL(W_1) \times GL(W_1^\perp)$ and $GL(W_2) \times GL(W_2^\perp)$ respectively, where $W_1^\perp$ and $W_2^\perp$ are annihilators of W_1 and W_2 in W_3^*. The corresponding maximal parabolic subgroups in PGL_3 will be denoted by $Q_1 = L_1U_1$ and $Q_2 = L_2U_2$. We have isomorphisms

$$\begin{cases} L_1 \cong GL(W_1^\perp) \\ L_2 \cong GL(W_2). \end{cases}$$

Using this identifications, the modular characters of L_1 and L_2 are

$$\rho_1(g) = |\det g|^{1/2} \text{ and } \rho_2(g) = |\det g|^{1/2}.$$

Further $Q_0 = Q_1 \cap Q_2$ is a Borel subgroup of PGL_3.

The maximal parabolic subgroups of G_2 can be defined as the stabilizers of non-trivial nil subalgebras of the octonion k-algebra $\mathbb{O}$. A nil subalgebra of $\mathbb{O}$ is a subspace consisting of trace zero elements with trivial multiplication (i.e. the product of any two elements is 0). The possible dimensions are 1 and 2. Fix a pair of nil-subalgebras $V_1 \subset V_2$. Then the stabilizers $P_1 = M_1N_1$ and $P_2 = M_2N_2$ of V_1 and V_2 respectively are two non-conjugate maximal parabolic subgroups of G_2, with $P_0 = P_1 \cap P_2$ a Borel subgroup. Let

$$V_3 = \{x \in \mathbb{O} \mid \bar{x} = -x, \text{ and } x \cdot V_1 = 0\}.$$

We have isomorphisms

$$\begin{cases} M_1 \cong GL(V_3/V_1) \\ M_2 \cong GL(V_2). \end{cases}$$

The action of the Levi factor of P_1 on V_1 is given by det, and the modular characters are

$$\rho_1'(g) = |\det(g)|^{5/2} \text{ and } \rho_2'(g) = |\det(g)|^{3/2}.$$

4. **Representations of** PGL_3

In this section we shall describe the irreducible smooth representations of PGL_3. It will be convenient to describe the representations in terms of discrete series representations of Levi factors. First of all, we have the discrete series representations: the supercuspidal representations, the Steinberg representation St and its twists

$$St_{\chi_E} \quad \text{and} \quad St_{\chi_E^2} \cong St_{\chi_E}^{\vee}$$

by cubic characters χ_E - each corresponding to a Galois cubic extension E of the base field k.

Next, let δ denote a (unitarizable) discrete series representation of GL_2. Then δ is either a supersuspidal representation or a twist st_μ of the Steinberg representation st by a unitary character μ. Let δ_s be the twist of δ by $|det|^s$ and write $I_{Q_k}(\delta, s)$ for $I_{Q_k}(\delta_s)$ ($k = 1$ or 2). For $s \geq 0$, let

$$J_k(\delta, s) = \text{ the } \textit{unique} \text{ irreducible quotient of } I_{Q_k}(\delta, s).$$

Here we have identified the Levi factors with GL_2 as in the previous section. The following will be very useful for our calculations (here $i \neq j$):

$$\begin{cases} J_i(\delta, s) \cong J_j(\delta, s)^{\vee} \\ J_i(\delta, s) \subseteq I_{Q_j}(\delta^{\vee}, -s). \end{cases}$$

Finally we have the representations $J(\chi_1, \chi_2, \chi_3)$ which are the unique irreducible quotient of representations obtained by (normalized) parabolic induction from the Borel subgroup

$Q_0 = Q_1 \cap Q_2$. Here the χ_i's are three characters such that $\chi_1\chi_2\chi_3 = 1$. Furthermore, if we write each character χ_i as $\chi_i = \mu_i \cdot |\cdot|^{s_i}$, where μ_i is unitary and $s_i \in \mathbb{R}$, then

$$\begin{cases} s_1 \geq s_2 \geq s_3, \\ s_1 + s_2 + s_3 = 0. \end{cases}$$

In terms of intermediate induction, the representation $J(\chi_1, \chi_2, \chi_3)$ can be realized as a submodule

$$J(\chi_1, \chi_2, \chi_3) \subset I_{Q_2}(\pi(\chi_3, \chi_2)).$$

Since $J(\chi_1, \chi_2, \chi_3)^\vee \cong J(\chi_3^{-1}, \chi_2^{-1}, \chi_1^{-1})$, we also have

$$J(\chi_1, \chi_2, \chi_3)^\vee \subset I_{Q_2}(\pi(\chi_1^{-1}, \chi_2^{-1}))$$

The following proposition is very useful for later purposes and can be read off from the above discussion.

Proposition 3. *Let π be an ireducible smooth representation of PGL_3 which is not square integrable. Then either π or its contragredient $\pi^\vee$ is the unique irreducible submodule of an induced representation $I_{Q_2}(\tau^\vee)$, where τ is an irreducible representation of L_2 described as follows:*

(i) If $\pi = J_i(\delta, s)$, then $\tau = \delta_s$.

(ii) If $\pi = J(\chi_1, \chi_2, \chi_3)$, we may assume that $\chi_2 = \mu_2|\cdot|^{s_2}$ with $s_2 \geq 0$ (by possibly replacing π by $\pi^\vee$). There are now two cases:

(a) If $\chi_1/\chi_2 \neq |\cdot|$, then $\tau = \pi(\chi_1, \chi_2)$.

(b) Otherwise, suppose that $\chi_1 = \mu|\cdot|^{s+1}$ and $\chi_2 = \mu|\cdot|^s$ for some unitary character μ. Then $\tau = \mu(det)_{s+\frac{1}{2}}$.

In each case, the central character χ_τ of τ satisfies $|\chi_\tau| = |\cdot|^t$ with $t \geq 0$. Further, whenever $t = 0$, $I_{Q_2}(\tau^\vee)$ is irreducible.

5. **Minimal Representation**

There is a notion of a "minimal representation" of a p-adic reductive group; we refer the reader to [GaS] for the definition. For a split simply-laced group H of type A_{2n-1}, D or E, it is known that a minimal representation exists and is unique up to twisting by (one dimensional) characters of H. For the purpose of this paper, we need to specify exactly what we mean by the minimal representation of H.

Since the algebraic group $\underline{H}$ is split, we may assume that $\underline{H}$ is defined over $\mathbb{Z}$. If A is the ring of integers of k, then $K = \underline{H}(A)$ is a maximal compact subgroup of H. Recall that an irreducible representation of H is said to be unramified if it has a non-zero K-fixed vector. The set of isomorphism classes of unramified representations is in natural bijection with semisimple conjugacy classes in the Langlands dual group $H^\vee(\mathbb{C})$. Given an unramified representation, the corresponding semisimple class in $H^\vee(\mathbb{C})$ is called its Satake parameter.

The minimal representation Π of H is an unramified representation, and its Satake parameter is described as follows. Let

$$\iota : SL_2(\mathbb{C}) \longrightarrow H^\vee(\mathbb{C})$$

be a homomorphism corresponding to the subregular unipotent orbit of $H^\vee(\mathbb{C})$ by the Jacobson-Morrozov theorem. Then the Satake parameter of Π is the conjugacy class of

$$s_\Pi = \iota \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix},$$

where q is the order of the residue field of k. The fact that this unramified representation is minimal was shown in [S1].

Henceforth, H will denote the split adjoint group of type E_6. We shall need various important properties of the minimal representation; these will be reviewed in the following two sections. We conclude this section by observing that any automorphism of H fixes the isomorphism class of Π since it preserves the corresponding Satake parameter. Furthermore, there is an automorphism of H which stabilizes the subgroup $PGL_3 \times G_2$, acts trivially on G_2 and maps to the non-trivial outer automorphism of PGL_3. As a consequence of this, we see that for any irreducible representation π of PGL_3,

$$\Theta(\pi) = \Theta(\pi^\vee).$$

Hence, for the purpose of determining $\Theta(\pi)$, we can restrict ourselves to representations π satisfying the conditions of Prop. 3.

6. Jacquet Modules for PGL_3

In this section, we describe the $L_2 \times G_2$-module $R_{Q_2}(\Pi)$. The result is given by [MS; Theorem 4.3]. To state it, we need some additional notations. There exists a maximal parabolic $\mathfrak{Q}_2 = \mathfrak{L}_2\mathfrak{U}_2$ in H whose Levi factor $\mathfrak{L}_2$ is of type D_5, and such that

$$\begin{cases} (PGL_3 \times G_2) \cap \mathfrak{L}_2 = L_2 \times G_2 \\ PGL_3 \cap \mathfrak{U}_2 = U_2. \end{cases}$$

Let B be the Borel subgroup of $GL(W_2)$ stabilizing the line W_1. Here is the result:

Proposition 4. *The $GL_2 \times G_2$-module $R_{Q_2}(\Pi)$ has a filtration*

$$0 = \Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \Pi_3 = \Pi_{U_2}$$

such that

$$\Pi_1/\Pi_0 \cong |\det|^{3/2} \otimes Ind_{GL_2 \times P_2}^{GL_2 \times G_2}(C_c^\infty(GL_2))$$

$$\Pi_2/\Pi_1 \cong |\det|^{3/2} \otimes Ind_{B \times P_1}^{GL_2 \times G_2}(C_c^\infty(GL_1))$$

$$\Pi_3/\Pi_2 = |det|^{-1/2} \cdot \Pi_{\mathfrak{U}_2} \cong |\det|^{1/2} \otimes \Pi(\mathfrak{L}_2) + |\det|^{3/2} \otimes 1$$

Here det *is the determinant map on* $GL(W_2)$. *Moreover,* $C_c^\infty(GL_2)$ *is the regular representation of*

$$GL(W_2) \times GL(V_2).$$

and $C_c^\infty(GL_1)$ *is the regular representation of*

$$GL(W_1) \times GL(V_1).$$

Finally, $\Pi(\mathfrak{L}_2)$ *is the minimal representation of* $\mathfrak{L}_2$. *The center of* $\mathfrak{L}_2$, *which coincides with the center of* $GL(W_2)$, *acts trivially on* $\Pi(\mathfrak{L}_2)$.

Analogously, there is a maximal parabolic subgroup $\mathfrak{Q}_1 = \mathfrak{L}_1\mathfrak{U}_1$ of H whose Levi factor $\mathfrak{L}_1$ has type D_5 and such that

$$\begin{cases} (PGL_3 \times G_2) \cap \mathfrak{L}_1 = L_1 \times G_2 \\ PGL_3 \cap \mathfrak{U}_1 = U_1. \end{cases}$$

The two parabolic subgroups $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are not conjugate in H, but are conjugate under a non-inner automorphism fo H. We have the obvious analog of Prop. 4 for $R_{Q_1}(\Pi)$.

Now $\mathfrak{Q}_0 = \mathfrak{Q}_1 \cap \mathfrak{Q}_2$ is a parabolic subgroup of H whose Levi factor $\mathfrak{L}_0$ is of type D_4. Its intersection with PGL_3 is equal to the Borel subgroup $Q_0 = Q_1 \cap Q_2$ and the center of $\mathfrak{L}_0$ is precisely the Levi factor L_0 of Q_0. We have the following proposition:

Proposition 5. *Let* $\Pi(D_4)$ *be the minimal representation of* $\mathfrak{L}_1 \cap \mathfrak{L}_2$ *with the center* L_0 *acting trivially. Then the* $L_0 \times G_2$*-module* $R_{Q_0}(\Pi)$ *has* $\rho_0 \otimes \Pi(D_4)$ *as a quotient. Here,* ρ_0 *is the modulus character of* Q_0.

Proof. The exponents of Π are known (see [S1]). In particular, it can be shown that $\rho_0^2 \otimes \Pi(D_4)$ is a summand of $\Pi_{\mathfrak{U}_0}$, regarded as a $L_0 \times G_2$-module. Since $\rho_0^{-1} \otimes \Pi_{\mathfrak{U}_0}$ is a quotient of $R_{Q_0}(\Pi)$, the lemma easily follows. ☐

7. Jacquet Modules for G_2

In this section, we describe the structure of the $PGL_2 \times M_2$-module $R_{P_2}(\Pi)$. The result is given in [MS, Theorem 7.6]. As in the previous section, there is a maximal parabolic subgroup $\mathfrak{P}_2 = \mathfrak{M}_2\mathfrak{N}_2$ of H whose Levi factor $\mathfrak{M}_2$ is of type A_5 and such that

$$\begin{cases} (PGL_3 \times G_2) \cap \mathfrak{M}_2 = PGL_3 \times M_2 \\ G_2 \cap \mathfrak{N}_2 = N_2. \end{cases}$$

Let B' be the Borel subgroup of $M_2 = GL(V_2)$ stabilizing the line V_1.

Proposition 6. *The* $PGL_3 \times GL_2$*-module* $R_{P_2}(\Pi)$ *has a filtration*

$$0 = \Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \Pi_3 = R_{P_2}(\Pi)$$

such that

$$\Pi_1/\Pi_0 \cong |det|^{\frac{1}{2}} \otimes Ind_{Q_1 \times GL_2}^{PGL_3 \times GL_2} C_c^\infty(GL_2) \bigoplus |det|^{\frac{1}{2}} \otimes Ind_{Q_2 \times GL_2}^{PGL_3 \times GL_2} C_c^\infty(GL_2);$$

$$\Pi_2/\Pi_1 \cong |det|^{\frac{1}{2}} \otimes Ind_{Q_0\times B'}^{PGL_3\times GL_2} C_c^\infty(GL_1),$$

$$\Pi_3/\Pi_2 \cong \Pi(\mathfrak{M}_2) \bigoplus 1 \otimes |det|^{\frac{1}{2}}.$$

Here det *is the usual determinant on* $GL(V_2)$*. Moreover,* $C_c^\infty(GL_2)$ *is the regular representation of*

$$GL(W_1^\perp)\times GL(V_2) \quad \textit{and} \quad GL(W_2)\times GL(V_2)$$

respectively, and $C_c^\infty(GL_1)$ *is the regular representation of*

$$GL(W_1\otimes W_2^\perp)\times GL(V_2/V_1).$$

Finally, $\Pi(\mathfrak{M}_2)$ *is the minimal representation of* $\mathfrak{M}_2$*. The center of* $\mathfrak{M}_2$*, which coincides with the center of* $GL(V_2)$*, acts trivially on* $\Pi(\mathfrak{M}_2) = \Pi(PGL_6)$*.*

8. **Non-Square-Integrable Representations**

In this section, we determine the set $\Theta(\pi)$ for non-square-integrable π and verify the conjecture of Gross-Savin for such π. As we remarked before, there is no loss of generality in assuming that π satisfies the conditions of Proposition 3. In particular, we have an irreducible representation τ of L_2 according to that proposition, and

$$\pi \hookrightarrow I_{Q_2}(\tau^\vee).$$

We define the non-negative number t by $|\chi_\tau| = |\cdot|^t$.

Proposition 7. *As representations of* G_2*,*

$$Hom_{PGL_3}(\Pi, I_{Q_2}(\tau^\vee)) \cong I_{P_2}(\tau)^*.$$

Proof. By Frobenius reciprocity, we have:

$$Hom_{PGL_3}(\Pi, I_{Q_2}(\tau^\vee)) = Hom_{L_2}(R_{Q_2}(\Pi), \tau^\vee).$$

The structure of $R_{Q_2}(\Pi)$ as a representation of $L_2\times G_2$ has been determined in Prop. 4. Using the notations of that proposition, we now note the following lemma:

Lemma 8.

$$Hom_{L_2}(R_{Q_2}(\Pi), \tau^\vee) = Hom_{L_2}(\Pi_1, \tau^\vee).$$

Proof. The central characters of L_2 on Π/Π_2 are $|\cdot|$ and $|\cdot|^3$ and these are different form $|\chi_{\tau^\vee}| = |\cdot|^{-t}$. Thus, we have

$$Hom_{L_2}(R_{Q_2}(\Pi), \tau^\vee) = Hom_{L_2}(\Pi_2, \tau^\vee).$$

Next, for any $\tau^\vee$ as in Proposition 3, we can find an element z in the Bernstein center of GL_2 such that $z(\tau^\vee) = 1$ and z vanishes for all representations of GL_2 with inducing parameters of type $(\chi, |\cdot|^2)$ (cf. [MS, Pg. 104-105]). Such a z vanishes on Π_2/Π_1 and thus we have

$$Hom(R_{Q_2}(\Pi), \tau^\vee) = Hom(\Pi_1, \tau^\vee).$$

The lemma is proved. □

We now return to the proof of the proposition. By the lemma, we need to identify the G_2-module

$$Hom_{L_2}(Ind_{L_2\times P_2}^{L_2\times G_2}(C_c^\infty(GL_2)), \tau^\vee_{-3/2}).$$

Now if we express the maximal $\tau^\vee_{-3/2}$-isotypic quotient of $Ind_{L_2\times P_2}^{L_2\times G_2}(C_c^\infty(GL_2))$ as $V\otimes\tau^\vee_{-3/2}$ for some smooth G_2-module V, then

$$Hom_{L_2}(Ind_{L_2\times P_2}^{L_2\times G_2}(C_c^\infty(GL_2)), \tau^\vee_{-3/2}) \cong V^*.$$

It remains to prove that $V\cong I_{P_2}(\tau)$.

By [MVW, Lemme, Pg. 59], the maximal $\tau^\vee_{-3/2}$-isotypic quotient of $C_c^\infty(GL_2)$ is isomorphic to $\tau_{3/2}\otimes\tau^\vee_{-3/2}$. Hence that of $Ind_{P_2\times L_2}^{G_2\times L_2}(C_c^\infty(GL_2))$ is $I_{P_2}(\tau)\otimes\tau^\vee_{-3/2}$. In other words, $V\cong I_{P_2}(\tau)$ and the proposition is proved. □

Corollary 9. *Assume that* $\pi\hookrightarrow I_{Q_2}(\tau^\vee)$ *as above. Then*
(i) $\theta(\pi)$ *is a quotient of* $I_{P_2}(\tau)$. *In particular, it has finite length and*

$$\Theta(\pi)\subset\{\textit{the irreducible quotients of } I_{P_2}(\tau)\}.$$

(ii) For any irreducible quotient σ *of* $I_{P_2}(\tau)$, *there is a non-zero* $PGL_3\times G_2$*-equivariant map*

$$\Pi\longrightarrow I_{Q_2}(\tau^\vee)\otimes\sigma.$$

Proof. (i) Observe that

$$\theta(\pi)^* = Hom_{PGL_3}(\Pi,\pi)\subset Hom_{PGL_3}(\Pi, I_{Q_2}(\tau^\vee)) = I_{P_2}(\tau)^*$$

so that $\theta(\pi)^\vee\subset I_{P_2}(\tau)^\vee$. Hence $\theta(\pi)^\vee$ is admissible, and thus so is $\theta(\pi)$. The result follows.

(ii) The isomorphism of the proposition implies that there is a non-zero G_2-equivariant map

$$\sigma^\vee\hookrightarrow Hom_{PGL_3}(\Pi, I_{Q_2}(\tau^\vee)).$$

Since

$$Hom_{G_2}(\sigma^\vee, Hom_{PGL_3}(\Pi, I_{Q_2}(\tau^\vee)))\cong Hom_{G_2\times PGL_3}(\Pi,\sigma\otimes I_{Q_2}(\tau^\vee)),$$

the result follows. □

For σ an irreducible smooth representation of G_2, we may define $\theta'(\sigma)$ and $\Theta'(\sigma)$ as in the case of PGL_3. One has:

Proposition 10. *Assume that* $t>0$. *Let* σ *be any irreducible quotient of* $I_{P_2}(\tau)$. *Then, ignoring multiplicities, we have:*

$$\Theta'(\sigma)\subset\{\pi,\pi^\vee\}.$$

Proof. It is not difficult to check that $I_{P_2}(\tau)$ is a quotient of an induced representation of Langlands type (i.e. a representation induced from a quasi-tempered representation of a Levi subgroup, whose central character is in the relevant positive chamber). In particular, $I_{P_2}(\tau)$ has a unique irreducible quotient σ. Further, this unique irreducible quotient σ is the image of a standard intertwining map $I_{P_2}(\tau) \to I_{P_2}(\tau^\vee)$. From this, we deduce that σ is self-contragredient and is the unique irreducible submodule of $I_{P_2}(\tau^\vee)$.

Now, for any representation δ of PGL_3, we have:

$$Hom_{PGL_3\times G_2}(\Pi, \delta\otimes\sigma) \subset Hom_{PGL_3\times G_2}(\Pi, \delta\otimes I_{P_2}(\tau^\vee)).$$

Using Prop. 6, it is not difficult to estimate the latter space. In particular, one sees that the latter space is zero unless $\delta = \pi$ or $\pi^\vee$, which gives the desired upper bound on $\Theta'(\sigma)$. We remark that one does not have the analog of Prop. 7 because the analog of Lemma 8 is not true in general. □

The following is the main result of this section.

Theorem 11. *(i) Let π be an irreducible representation of PGL_3 which is not square-integrable. Suppose that $\pi \hookrightarrow I_{P_2}(\tau^\vee)$ with τ as in Prop. 3. Then*

$$\Theta(\pi) = \Theta(\pi^\vee) = \{\textit{the irreducible quotients of } I_{P_2}(\tau)\}.$$

Further, the correspondence is functorial for π.

(ii) Suppose that $t > 0$. If σ is the unique irreducible quotient of $I_{P_2}(\tau)$, then

$$\Theta'(\sigma) = \{\pi, \pi^\vee\} \quad \textit{if } \pi \neq \pi^\vee,$$

and

$$\Theta'(\sigma) = \{\pi\} \quad \textit{if } \pi = \pi^\vee.$$

Proof. (i) After Cor. 9(i), we need to show that if σ is any irreducible quotient of $I_{P_2}(\tau)$, then $\pi\otimes\sigma$ is a quotient of Π. By Cor. 9(ii), there is a non-zero map

$$f : \Pi \longrightarrow I_{Q_2}(\tau^\vee)\otimes\sigma.$$

If $I_{Q_2}(\tau^\vee)$ is irreducible and hence equal to π, then we are done. If $I_{Q_2}(\tau^\vee)$ is reducible, then $t > 0$. But by Prop. 10, any irreducible quotient of the image of f must be $\pi\otimes\sigma$ or $\pi^\vee\otimes\sigma$, i.e. $\sigma\in\Theta(\pi) = \Theta(\pi^\vee)$. Thus the first statement of (i) is proved.

To show that the correspondence is functorial for π, we need to show that the L-parameters match up. This is straightforward except in case (ii) of Prop. 3 when $s_2 = 0$, so that the representation $\sigma\in\Theta(\pi)$ is a Langlands quotient of a representation induced from P_1. We leave the verification to the reader.

(ii) This follows from (i) and Prop. 10. □

9. **Discrete series**

In this section, we determine the set $\Theta(\pi)$ for π a non-cuspidal discrete series representation. Recall that π is either the Steinberg representation St or its twist St_χ by a cubic character χ. The pair $\{\chi, \chi^{-1}\}$ of cubic characters determine a Galois cubic extension E of F by local class field theory. The field E then determines a square-integrable representation $\pi(E)$ of G_2 (cf. [Mu, Prop. 4.2(i)] where this representation was denoted by $\pi(\chi) = \pi(\chi^{-1})$).

Theorem 12. *Let χ be a cubic character with corresponding cyclic cubic extension E. Then*

$$\Theta(St_\chi) = \Theta(St_{\chi^2}) = \{\pi(E)\}.$$

Proof. Let st denote the Steinberg representation of L_2 and let st_χ be the twisted representation $st \otimes \chi(det)$. Then St_χ is a submodule of $I_{Q_2}(st_\chi, 1/2)$. Thus,

$$\theta(St_\chi)^* \hookrightarrow Hom_{PGL_3}(\Pi, I_{Q_2}(st_\chi, 1/2)).$$

As in the proof of Prop. 7, one sees by Prop. 4 that

$$Hom_{PGL_3}(\Pi, I_{Q_2}(st_\chi, 1/2)) \cong Hom_{L_2}(\Pi_1, st_\chi \otimes |det|^{1/2}) \cong I_{P_2}(st_{\chi^2}, -1/2)^*.$$

In particular, $\theta(St_\chi)$ is a quotient of $I_{P_2}(st_{\chi^2}, -1/2)$. Since $\pi(E)$ is the unique irreducible quotient of $I_{P_2}(st_{\chi^2}, -1/2)$ [Mu, Prop. 4.2(ii)], we have $\Theta(St_\chi) \subset \{\pi(E)\}$.

Further, as in Cor. 9(ii), there is a non-zero $G_2 \times PGL_3$-equivariant map

$$f : \Pi \longrightarrow \pi(E) \otimes I_{Q_2}(st_\chi, 1/2),$$

and we claim that its image is the submodule $\pi(E) \otimes St_\chi$. If not, then $\pi(E)$ will be an element of $\Theta(J_2(st_\chi, 1/2))$. But Theorem 11 shows that $\Theta(J_2(st_\chi, 1/2)) = \{J_{P_2}(st_\chi, 1/2)\}$. With this contradiction, we see that $\Theta(St_\chi) = \{\pi(E)\}$ and the theorem is proved. □

Theorem 13.

$$\Theta(St) = \{\pi(1)', \pi_{sc}[1]\}.$$

Here, $\pi(1)'$ is a generic discrete series representation of G_2 defined in [Mu, Prop. 4.3(i)] and $\pi_{sc}[1]$ is a non-generic supercuspidal representation defined in [HMS].

This is certainly the most difficult part of this paper. The rest of the section is concerned with its proof. We need the following lemma, which will be used repeatedly:

Lemma 14. *(i) Under Θ and Θ', we have the correspondences*

$$\{1\} \leftrightarrow \{J_{P_1}(\pi(1,1), 1)\} \quad \textit{and} \quad \{J_1(st, 1/2), J_2(st, 1/2)\} \leftrightarrow \{J_{P_2}(st, 1/2)\}.$$

(ii) $J_{P_1}(st, 1/2) \notin \Theta(St)$.

Proof. (i) is a special case of Theorem 11. The proof of (ii) is quite involved. By the proof of [Mu, Prop. 4.3], $J_{P_1}(st, 1/2)$ is a submodule of $I_{P_2}(\pi(|\cdot|^{-1}, |\cdot|))$. Thus

$$Hom_{PGL_3 \times G_2}(\Pi, St \otimes J_{P_1}(st, 1/2)) \subset Hom_{PGL_3 \times M_2}(R_{P_2}(\Pi), St \otimes \pi(|\cdot|^{-1}, |\cdot|))$$

and it suffices to show that the latter space is zero. By Proposition 6, it suffices to show that the following spaces are zero:

$$Hom(Ind_{Q_i \times GL_2}^{PGL_3 \times GL_2} C_c^\infty(GL_2), St \otimes \pi(|\cdot|^{-3/2}, |\cdot|^{1/2})), \quad i = 1, 2;$$

$$Hom(Ind_{Q_0 \times B'}^{PGL_3 \times GL_2} C_c^\infty(GL_1), St \otimes \pi(|\cdot|^{-3/2}, |\cdot|^{1/2}));$$

$$Hom(\Pi(PGL_6), St \otimes \pi(|\cdot|^{-1}, |\cdot|)).$$

The vanishing of the first two terms above can be easily checked using Frobenius reciprocity. Indeed, the first term is equal to

$$Hom_{GL_2 \times GL_2}(C_c^\infty(GL_2), st \otimes \pi(|\cdot|^{-3/2}, |\cdot|^{1/2})) = 0,$$

and the second term is equal to

$$Hom_{Q_0 \times B'}(C_c^\infty(GL_1), 1 \otimes (|\cdot| \times |\cdot|^{-2})) \oplus Hom_{Q_0 \times B'}(C_c^\infty(GL_1), 1 \otimes (|\cdot|^{-1} \times 1)) = 0.$$

The last term can be treated with our methods. However, in order to avoid repetitive arguments, we shall relate it to a special case of the Howe correspondence as follows.

Remark: The minimal representation $\Pi(PGL_6)$ is the Howe lift of the trivial representation of GL_1. In particular, the correspondence for the dual pair $PGL_3 \times PGL_2 \subset PGL_6$ is a special case of the Howe correspondence for the dual pair $GL_3 \times GL_2$. Since a discrete series representation of GL_n does not appear in the Howe correspondence with GL_m unless $m \geq n$, the Steinberg representation (or any supercuspidal representation) of PGL_3 cannot be a quotient of $\Pi(PGL_6)$. The lemma is proved. □

We begin the proof of Theorem 13 by showing that $\Theta(St)$ contains $\pi_{sc}[1]$. By Frobenius reciprocity and Prop. 5, one sees that

$$Hom_{PGL_3 \times G_2}(\Pi, I_{Q_0}(\rho_0) \otimes \pi_{sc}[1]) \supset Hom_{G_2}(\Pi(D_4), \pi_{sc}[1]).$$

Now the restriction of $\Pi(D_4)$ to G_2 has been determined completely in [HMS]. Their result says that

$$\Pi(D_4) = J_{P_1}(\pi(1,1),1) \oplus 2 \cdot J_{P_2}(st, 1/2) \oplus \pi_{sc}[1].$$

Hence, we deduce that there is a non-zero map

$$\Pi \longrightarrow I_{Q_0}(\rho_0) \otimes \pi_{sc}[1]$$

so that $\Theta'(\pi_{sc}[1])$ contains some irreducible subquotient of $I_{Q_0}(\rho_0)$. Since the irreducible subquotients of $I_{Q_0}(\rho_0)$ are precisely St and the representations of PGL_3 mentioned in Lemma 14(i), the lemma implies that $St \in \Theta'(\pi_{sc}[1])$ and thus

$$\pi_{sc}[1] \in \Theta(St).$$

Next we find an upper bound for $\Theta(St)$. Since St is a submodule of $I_{Q_2}(st, 1/2)$, we have:

$$\theta(St)^* \subset Hom_{PGL_3}(\Pi, I_{Q_2}(st, 1/2)) = Hom_{L_2}(R_{Q_2}(\Pi), st \otimes |det|^{1/2}) = W.$$

Since $R_{Q_2}(\Pi)$ has a natural filtration by Prop. 4, this induces a filtration on W:

$$0 = W_3 \subset W_2 \subset W_1 \subset W_0 = W$$

such that

$$W_2/W_3 \cong Hom_{L_2}(\Pi_3/\Pi_2, st \otimes |det|^{1/2});$$

$$W_1/W_2 \subset Hom_{L_2}(\Pi_2/\Pi_1, st \otimes |det|^{1/2});$$

$$W_0/W_1 \subset Hom_{L_2}(\Pi_1, st \otimes |det|^{1/2}).$$

Being a subspace of W, $\theta(St)^*$ inherits a filtration as well.

Now we investigate each of the terms above. The study of the last two terms is similar to the proof of Proposition 7. We simply state the result:

Lemma 15. *(i)* $Hom_{L_2}(\Pi_1, st \otimes |det|^{1/2}) = I_{P_2}(st, -1/2)^*$.

(ii) $W_1/W_2 = Hom_{L_2}(\Pi_2/\Pi_1, st \otimes |det|^{1/2}) = 0$.

We now consider the space

$$Hom_{L_2}(\Pi_3/\Pi_2, st \otimes |det|^{1/2})) = Hom_{L_2}(\Pi(\mathfrak{L}_2), st).$$

Recall that $\mathfrak{L}_2$ has derived group D_5 and $\Pi(\mathfrak{L}_2)$ is the minimal representation of D_5 with the center of $\mathfrak{L}_2$ acting trivially. Hence we need to study the dual pair $PGL_2 \times G_2 \subset D_5$. Using the results of [MS, §2] and particularly [MS, Prop. 2.3], one gets:

Lemma 16. *There is a short exact sequence of representations of* G_2*:*

$$0 \longrightarrow W_1' \longrightarrow Hom_{L_2}(\Pi(\mathfrak{L}_2), st) \longrightarrow W_2' \longrightarrow 0$$

with

$$W_1' \subset \Pi(D_4)^* \quad \textit{and} \quad W_2' \subset I_{P_1}(|det|^{-1/2})^*.$$

Here $\Pi(D_4)$ *is the minimal representation of* D_4*.*

Since $\theta(St)^*$ is a subspace of W, the above considerations imply that there is a filtration

$$0 \subset V_3 \subset V_2 \subset V_1 = \theta(St)^\vee$$

such that

$$\begin{cases} V_3 \subset \Pi(D_4)^\vee, \\ V_2/V_3 \subset I_{P_1}(|det|^{-1/2})^\vee, \\ V_1/V_2 \subset I_{P_2}(st, -1/2)^\vee. \end{cases}$$

In particular, we see that $\theta(St)$ is of finite length. We need to identify each of these subquotients as much as possible.

Together with the fact that $\pi_{sc}[1] \in \Theta(St)$, Lemma 14 implies that

$$V_3 = \pi_{sc}[1].$$

We now consider V_2/V_3. Since V_3 is supercuspidal, V_2/V_3 is actually a submodule of $\theta(St)^\vee$. If V_2/V_3 is non-zero, then some irreducible quotient of $I_{P_1}(|det|^{-1/2})$ will be an element of $\Theta(St)$. Now the irreducible constituents of $I_{P_1}(|det|^{-1/2})$ are precisely the 3 representations of G_2 in Lemma 14 [Mu, Prop. 4.3]; in fact $J_{P_1}(st, 1/2)$ is the unique irreducible quotient. Hence Lemma 14 implies that:

$$V_2/V_3 = 0.$$

By the above, we see that

$$\theta(St) = \pi_{sc}[1] \oplus (\text{a quotient of } I_{P_2}(st, -1/2)).$$

We claim that the second term is non-zero. This follows from Corollary 18 (to be proved in the next section), which implies that $\theta(St)$ has non-zero Whittaker functionals, and the fact that $\pi_{sc}[1]$ is non-generic. Since $I_{P_2}(st, -1/2)$ has $\pi[1]'$ as its unique irreducible quotient (cf. [Mu, Prop. 4.3] and its proof), we conclude that

$$\Theta(St) = \{\pi_{sc}[1], \pi[1]'\}.$$

The theorem is proved.

10. **Whittaker Vectors**

Before we come to the study of $\Theta(\pi)$ for supercuspidal π, we need to establish some results on Whittaker vectors. The results of this section are independent of those in Sections 8 and 9 and thus could have been treated after Section 7. Suppose that $G \times G' \subset H$ is one of the following dual pairs, with G of smaller dimension:

$$\begin{cases} PGL_3 \times G_2 \subset E_6 \\ G_2 \times PGSp_6 \subset E_7. \end{cases}$$

The second dual pair is not the subject matter of this paper, but will be crucially needed in the next section. Let U and U' be a maximal unipotent subgroup of G and G' respectively, and let ψ and ψ' be unitary characters of U and U' which are in general position. Also, let Π be the minimal representation of H. We have:

Proposition 17. *As representations of G,*

$$\Pi_{U',\psi'} \cong ind_U^G \psi,$$

the Gelfand-Graev representation of G.

Proof. This is similar to the proof for the finite field analogue [G, Theorem 7.1]. We shall give a sketch for the case $G \times G' = PGL_3 \times G_2$. Let $\psi'_2 = \psi'|_{N_2}$. Then ψ'_2 is a non-trivial character of $N_2/(N_2 \cap N_1) \cong \mathbb{G}_a$. Let $N_\alpha \cong \mathbb{G}_a$ be the unipotent radical of the Borel subgroup $M_2 \cap P_1$ of M_2. Then N_α normalizes N_2 and fixes the character ψ'_2. Thus Π_{N_2,ψ'_2} is naturally a representation of $N_\alpha \times PGL_3$. Let $\psi_\alpha = \psi'|_{N_\alpha}$; it is a non-trivial character of N_α and $\Pi_{U',\psi'} = (\Pi_{N_2,\psi'_2})_{N_\alpha,\psi_\alpha}$.

Using [MS, Thm. 6.1], one sees that there is an isomorphism of $N_\alpha \times PGL_3$-modules

$$\Pi_{N_2,\psi'_2} \cong C_c^\infty(\mathcal{N})$$

where $\mathcal{N}$ is the set of 3×3 nilpotent matrices. The action of PGL_3 on $C_c^\infty(\mathcal{N})$ is the action induced by conjugation whereas the action of $b \in N_\alpha$ is induced by the action

$$X \mapsto X + b \cdot X^2.$$

Now $\mathcal{N}$ possesses a natural stratification by the rank of matrices. Thus $C_c^\infty(\mathcal{N})$ has a natural filtration whose successive quotients are $C_c^\infty(\mathcal{N}_i)$, where $\mathcal{N}_i$ is the set of nilpotent matrices of rank i ($i = 0$, 1 or 2). Since N_α acts trivially on $\mathcal{N}_i$ for $i \leq 1$, we conclude that

$$\Pi_{U',\psi'} \cong C_c^\infty(\mathcal{N}_2)_{N_\alpha,\psi_\alpha}.$$

To prove the proposition, we need to identify the latter module as the Gelfand-Graev representation of PGL_3.

The action of $N_\alpha \times PGL_3$ on $\mathcal{N}_2$ is clearly transitive. Let

$$X_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{N}_2.$$

Without loss of generality, take U to be the group of upper triangular unipotent matrices in PGL_3. The stabilizer of X_0 in $N_\alpha \times PGL_3$ is contained in the subgroup $N_\alpha \times U$ and the $N_\alpha \times U$-orbit $\mathcal{N}_2'$ of X_0 consists of the elements

$$X_b = \begin{pmatrix} 0 & 1 & b \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Indeed, the action of

$$b \times \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in N_\alpha \times U$$

is via:

$$X_0 \mapsto X_{x-y-b}.$$

The above discussion implies that

$$C_c^\infty(\mathcal{N}_2) \cong ind_{N_\alpha \times U}^{N_\alpha \times PGL_3} C_c^\infty(\mathcal{N}_2')$$

where the action of $N_\alpha \times U$ on $C_c^\infty(\mathcal{N}_2')$ is induced by the geometric action above. From this, we conclude that

$$C_c^\infty(\mathcal{N}_2)_{N_\alpha,\psi_\alpha} \cong ind_U^{PGL_3}(C_c^\infty(\mathcal{N}_2')_{N_\alpha,\psi_\alpha}) \cong ind_U^{PGL_3}\psi$$

where ψ is the character of U given by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \psi_\alpha(y-x).$$

This proves the proposition. □

Corollary 18. *(i) If $\pi^\vee$ is non-generic, then $\theta(\pi)$ does not have any generic subquotients.*

(ii) If $\pi^\vee$ is generic, then $\theta(\pi)$ has a unique irreducible generic subquotient. In particular, $\theta(\pi)$ is non-zero.

11. **Supercuspidal representations**

In this section, we give the proof of Theorem 2. For convenience, let us repeat the statement of the theorem:

Theorem 19. *Let π be a supercuspidal representation of PGL_3. Then*

(i) $\theta(\pi)$ is supercuspidal and has a unique generic summand.

(ii) If π is non-self-dual, then $\theta(\pi)$ is irreducible.

Recall that we have a surjection

$$\Pi \longrightarrow \pi \otimes \theta(\pi).$$

By the exactness of the Jacquet functors, we have a surjection

$$R_{P_2}(\Pi) \longrightarrow \pi \otimes R_{P_2}(\theta(\pi)).$$

By Prop. 6, $\pi \otimes R_{P_2}(\theta(\pi))$ must be a quotient of $\Pi(PGL_6)$. But following the Remark in the proof of Lemma 14, one knows that a square-integrable representation of PGL_3 does not occur as a quotient of $\Pi(PGL_6)$. Thus $R_{P_2}(\theta(\pi)) = 0$.

We now show that $R_{P_1}(\theta(\pi)) = 0$. There is a surjection

$$R_{P_1}(\Pi) \longrightarrow \pi \otimes R_{P_1}(\theta(\pi))$$

and by the above, we know that $R_{P_1}(\theta(\pi))$ is a supercuspidal representation of M_1. If it is not zero, then it is generic and so we have a surjection

$$R_{P_1}(\Pi)_{N_\beta,\psi_\beta} \longrightarrow \pi \otimes R_{P_1}(\theta(\pi))_{N_\beta,\psi_\beta}.$$

Here, $N_\beta \cong N_2/(N_1 \cap N_2) \cong \mathbb{G}_a$ is the unipotent radical of the Borel subgroup $M_1 \cap P_2$ of M_1, and ψ_β is a non-trivial character of N_β. It remains to see that $R_{P_1}(\Pi)_{N_\beta,\psi_\beta}$ does not contain supercuspidal representations of PGL_3 as subquotients.

Next, note that $(\Pi_{N_1})_{N_\beta,\psi_\beta}$ is isomorphic to $(\Pi_{N_2,\psi_2'})_{N_\alpha}$, where ψ_2' is the character of N_2 as in the proof of Proposition 17. Since $\Pi_{N_2,\psi_2'}$ was calculated there, we see that as representations of PGL_3,

$$R_{P_1}(\theta(\pi))_{N_\beta,\psi_\beta} \cong C_c^\infty(\mathcal{N})_{N_\alpha}$$

where $\mathcal{N}$ is the space of nilpotent 3×3 matrices and N_α is the unipotent radical of the Borel subgroup $M_2 \cap P_1$ of M_2. Since $C_c^\infty(\mathcal{N})$ has a natural filtration whose successive quotients are $C_c^\infty(\mathcal{N}_i)$, where $\mathcal{N}_i$ is the set of nilpotent matrices of rank i ($i = 0, 1, 2$), it is easy to see that

$$C_c^\infty(\mathcal{N}_0)_{N_\alpha} \cong 1;$$

$$C_c^\infty(\mathcal{N}_1)_{N_\alpha} \cong Ind_{Q_0}^{PGL_3} C_c^\infty(GL_1);$$

$$C_c^\infty(\mathcal{N}_2)_{N_\alpha} \cong ind_U^{PGL_3} 1,$$

where U is the unipotent radical of Q_0. In particular, these spaces do not contain supercuspidal representations of PGL_3 as subquotients and thus $R_{P_1}(\theta(\pi)) = 0$.

We have shown that $\theta(\pi)$ is supercuspidal and thus $\theta(\pi)$ is semi-simple. Since π is supercuspidal, so is $\pi^\vee$ and thus $\pi^\vee$ is generic. Corollary 18 now implies (i) immediately.

Finally, we assume that π is not self-dual. To show that $\theta(\pi)$ is irreducible, it suffices to show that every summand of $\theta(\pi)$ is generic. The key input here is a wonderful idea of Muic-Savin which was used in [MuS] for the study of symplectic-orthogonal theta lifts. Consider the dual pair $G_2 \times PGSp_6 \subset E_7$, and recall that $PGSp_6$ has a Siegel parabolic subgroup P whose Levi subgroup is GL_3. If $\Pi(E_7)$ denotes the minimal representation of E_7, then [MS, Theorem 1.1] shows that Π is a $PGL_3 \times G_2$-quotient of $R_P(\Pi(E_7))$. Now suppose that $\pi \otimes \sigma$ is a quotient of Π, with σ irreducible supercuspidal. Then, by Frobenius reciprocity, there is a non-zero map

$$\Pi(E_7) \longrightarrow I_P(\pi) \otimes \sigma.$$

The key observation is that, when π is not self-dual, $I_P(\pi)$ is irreducible and thus generic. But by Proposition 17, applied to the dual pair $G_2 \times PGSp_6$, we deduce that $\sigma^\vee$ is generic and thus so is σ (since σ is unitarizable). The proof of the theorem is now complete.

12. **Depth Zero Lifts and a Conjecture**

The conjecture of Gross and Savin describes the theta correspondence in terms of Langlands parameters. However it is natural to ask for an alternative characterization of $\theta(\pi)$ in terms of π, since the Langlands parametrization for supercuspidal representations is not known for G_2. In this section, we determine $\theta(\pi)$ for π a depth zero supercuspidal representation and state a conjecture in the case when π has positive depth, in terms of the construction of supercuspidal representations via compact induction.

Let us now assume that $p \geq 5$, and suppose that π has depth zero. By results of Moy and Prasad [MP],

$$\pi \cong \mathrm{ind}_K^{PGL_3}\sigma,$$

where K is a hyperspecial maximal compact subgroup of PGL_3 with first principal congruence subgroup K_1 and σ is a cuspidal representation of $K/K_1 = PGL_3(\mathbb{F}_q)$. Fix an isomorphism $\mathbb{Z}_{(p)}/\mathbb{Z} \cong \bar{\mathbb{F}}_p^\times$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime p. Then, relative to this isomorphism, the irreducible generic representations of a connected reductive group G of adjoint type over $\mathbb{F}_q$ can be parametrized by semisimple conjugacy classes of the dual group $G^\vee$ [Ca]. In any case, let us write $\sigma = \sigma(s)$ and $\pi = \pi(s)$ if σ corresponds to the class of s in the above parametrization. Since σ is cuspidal, s is regular and elliptic.

Via the natural inclusion $i : SL_3 \hookrightarrow G_2$, the element $i(s)$ is still regular and elliptic in $G_2(\mathbb{F}_q)$, and hence determines a cuspidal generic representation $\sigma'(i(s))$ of $G_2(\mathbb{F}_q)$. Let

$$\pi'(i(s)) = \mathrm{ind}_{K'}^{G_2}\sigma'(i(s)),$$

where K' is a hyperspecial maximal compact subgroup of G_2. Then $\pi'(i(s))$ is an irreducible generic supercuspidal representation of G_2, and we have:

Proposition 20.

$$\theta(\pi(s)) = \pi'(i(s)).$$

Proof. The dual pair $PGL_3 \times G_2 \subset H$ can be defined over $\mathbb{Z}$, and we may take K, K' and K_H to be the hyperspecial maximal compact subgroups corresponding to the respective groups of integer points. Then $K_H \cap (PGL_3 \times G_2) = K \times K'$. Let K_1 be the first principal congruence subgroup of K_H. Then as a representation of $K_H/K_1 \cong E_6(\mathbb{F}_q)$,

$$\Pi^{K_1} \cong 1 \oplus R,$$

where R denotes the reflection representation of $E_6(\mathbb{F}_q)$ (cf. [S2] and [G]). Hence R is a K_H-equivariant quotient of Π.

It was shown in [G, Theorem 10.2] that $\sigma(s) \otimes \sigma'(i(s))$ is a $K \times K'$-equivariant quotient of R. Note that since $ind_{K\times K'}^{PGL_3\times G_2}\sigma(s) \otimes \sigma'(i(s))$ is irreducible, we have:

$$ind_{K\times K'}^{PGL_3\times G_2}\sigma(s) \otimes \sigma'(i(s)) = Ind_{K\times K'}^{PGL_3\times G_2}\sigma(s) \otimes \sigma'(i(s)).$$

Hence the result follows by Frobenius reciprocity. □

We now describe a conjecture for π of positive depth. Given a pair (T, χ), where T is an anisotropic maximal torus of PGL_3, and χ an admissible character of T, one can construct a supercuspidal representation $\pi_{T,\chi}$ of PGL_3. Moreover, every supercuspidal representation of PGL_3 arises in this way. Note that T is of the form $K^\times/k^\times$ for some cubic field extension K/k. For simplicity, let us assume that K is Galois over k.

Now there is a natural conjugacy class of embedding of T into G_2. More precisely, via the isomorphism $K^\times/k^\times \cong K_{\mathbb{N}=1}$, one can embed T naturally into SL_3, and then into G_2 via the natural conjugacy class of embedding $SL_3 \hookrightarrow G_2$. Hence, T can be regarded as an anisotropic maximal torus of G_2. By the recent results of [Yu], one can construct a supercuspidal representation $\pi'_{T,\chi}$ of G_2 out of the data (T, χ). It is natural to make the following conjecture:

Conjecture: $\theta(\pi_{T,\chi}) = \pi'_{T,\chi}$.

Using the result of [S2], it may be possible to verify the conjecture in the case when K is the unramified cubic extension of k.

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