\[ s^2 Y(s) + 2s Y(s) + Y(s) - 2s - 3 = \frac{4}{s + 1}. \]

Solving for \( Y(s) \), the transform of the solution is
\[ Y(s) = \frac{4}{(s + 1)^2} + \frac{2s + 3}{(s + 1)^2}. \]

First write
\[ \frac{2s + 3}{(s + 1)^2} = \frac{2(s + 1) + 1}{(s + 1)^2} = \frac{2}{s + 1} + \frac{1}{(s + 1)^2}. \]

We note that
\[ \mathcal{L}^{-1} \left[ \frac{4}{\xi^3} + \frac{2}{\xi} + \frac{1}{\xi^2} \right] = 2t^2 + 2 + t. \]

So based on the translation property of the Laplace transform, the solution of the IVP is
\[ y(t) = 2t^2 e^{-t} + t e^{-t} + 2 e^{-t}. \]

25. Let \( f(t) \) be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain
\[ s^2 Y(s) - s y(0) - y'(0) + Y(s) = \mathcal{L}[f(t)]. \]

Applying the initial conditions,
\[ s^2 Y(s) + Y(s) = \mathcal{L}[f(t)]. \]

Based on the definition of the Laplace transform,
\[ \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt \]
\[ = \int_0^1 t e^{-st} dt \]
\[ = \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}. \]

Solving for the transform,
\[ Y(s) = \frac{1}{s^2(s^2 + 1)} - e^{-s} \frac{s + 1}{s^2(s^2 + 1)}. \]

Using partial fractions,
\[ \frac{1}{s^2(s^2 + 1)} = \frac{1}{s^2} - \frac{1}{s^2 + 1}. \]
and
\[ \frac{s}{s^2(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}. \]

We find, by inspection, that
\[ \mathcal{L}^{-1} \left[ \frac{1}{s^2(s^2 + 1)} \right] = t - \sin t. \]

Referring to Line 13, in Table 6.2.1,
\[ \mathcal{L}[u_c(t)f(t - c)] = e^{-cs}\mathcal{L}[f(t)]. \]

Let
\[ \mathcal{L}[g(t)] = \frac{s + 1}{s^2(s^2 + 1)} = \frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}. \]

Then \( g(t) = 1 + t - \cos t - \sin t. \) It follows, therefore, that
\[ \mathcal{L}^{-1} \left[ e^{-s} \cdot \frac{s + 1}{s^2(s^2 + 1)} \right] = u_1(t)[1 + (t - 1) - \cos(t - 1) - \sin(t - 1)]. \]

Combining the above, the solution of the IVP is
\[ y(t) = t - \sin t - u_1(t)[1 + (t - 1) - \cos(t - 1) - \sin(t - 1)]. \]

26. Let \( f(t) \) be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain
\[ s^2 Y(s) - sy(0) - y'(0) + 4Y(s) = \mathcal{L}[f(t)]. \]

Applying the initial conditions,
\[ s^2 Y(s) + 4Y(s) = \mathcal{L}[f(t)]. \]

Based on the definition of the Laplace transform,
\[ \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt \]
\[ = \int_0^1 t e^{-st} dt + \int_1^{\infty} e^{-st} dt \]
\[ = \frac{1}{s^2} - \frac{e^{-s}}{s^2}. \]

Solving for the transform,
\[ Y(s) = \frac{1}{s^2(s^2 + 4)} - \frac{e^{-s}}{s^2(s^2 + 4)}. \]
Using partial fractions,
\[
\frac{1}{s^2(s^2 + 4)} = \frac{1}{4} \left[ \frac{1}{s^2} - \frac{1}{s^2 + 4} \right].
\]

We find that
\[
\mathcal{L}^{-1} \left[ \frac{1}{s^2(s^2 + 4)} \right] = \frac{1}{4} t - \frac{1}{8} \sin t.
\]

Referring to Line 13, in Table 6.2.1,
\[
\mathcal{L} \left[ u_c(t)f(t - c) \right] = e^{-cs} \mathcal{L}[f(t)].
\]

It follows that
\[
\mathcal{L}^{-1} \left[ e^{-s} \cdot \frac{1}{s^2(s^2 + 4)} \right] = u_1(t) \left[ \frac{1}{4}(t - 1) - \frac{1}{8} \sin(t - 1) \right].
\]

Combining the above, the solution of the IVP is
\[
y(t) = \frac{1}{4} t - \frac{1}{8} \sin t - u_1(t) \left[ \frac{1}{4}(t - 1) - \frac{1}{8} \sin(t - 1) \right].
\]

28(a). Assuming that the conditions of Theorem 6.2.1 are satisfied,
\[
F'(s) = \frac{d}{ds} \int_0^\infty e^{-st} f(t) dt
\]
\[
= \int_0^\infty \frac{\partial}{\partial s} [e^{-st} f(t)] dt
\]
\[
= \int_0^\infty [-t e^{-st} f(t)] dt
\]
\[
= \int_0^\infty e^{-st} [-tf(t)] dt.
\]

(b). Using mathematical induction, suppose that for some \( k \geq 1 \),
\[
F^{(k)}(s) = \int_0^\infty e^{-st} \left[ (-t)^k f(t) \right] dt.
\]

Differentiating both sides,
\[ F^{(k+1)}(s) = \frac{d}{ds} \int_0^\infty e^{-st} \left[ (-t)^k f(t) \right] dt \]
\[ = \int_0^\infty \frac{\partial}{\partial s} \left[ e^{-st} (-t)^k f(t) \right] dt \]
\[ = \int_0^\infty \left[ -te^{-st} (-t)^k f(t) \right] dt \]
\[ = \int_0^\infty e^{-st} \left[ (-t)^{k+1} f(t) \right] dt. \]

29. We know that
\[ \mathcal{L}[e^{at}] = \frac{1}{s - a}. \]

Based on Prob. 28,
\[ \mathcal{L}[-te^{at}] = \frac{d}{ds} \left[ \frac{1}{s - a} \right]. \]

Therefore,
\[ \mathcal{L}[te^{at}] = \frac{1}{(s - a)^2}. \]

31. Based on Prob. 28,
\[ \mathcal{L}[-t^n] = \frac{d^n}{ds^n} \mathcal{L}[1] \]
\[ = \frac{d^n}{ds^n} \left[ \frac{1}{s} \right]. \]

Therefore,
\[ \mathcal{L}[t^n] = (-1)^n \frac{(-1)^n n!}{s^{n+1}} \]
\[ = \frac{n!}{s^{n+1}}. \]

33. Using the translation property of the Laplace transform,
\[ \mathcal{L}[e^{at} \sin bt] = \frac{b}{(s - a)^2 + b^2}. \]

Therefore,
7. Using the Heaviside function, we can write

\[ f(t) = (t - 2)^2 u_2(t) . \]

The Laplace transform has the property that

\[ \mathcal{L}[u_c(t) f(t - c)] = e^{-cs} \mathcal{L}[f(t)] . \]

Hence

\[ \mathcal{L}[(t - 2)^2 u_2(t)] = \frac{2e^{-2s}}{s^2} . \]

9. The function can be expressed as

\[ f(t) = (t - \pi)[u_\pi(t) - u_{2\pi}(t)] . \]

Before invoking the translation property of the transform, write the function as

\[ f(t) = (t - \pi) u_\pi(t) - (t - 2\pi) u_{2\pi}(t) - \pi u_{2\pi}(t) . \]

It follows that

\[ \mathcal{L}[f(t)] = \frac{e^{-\pi s}}{s^2} - \frac{e^{-2\pi s}}{s^2} - \frac{\pi e^{-2\pi s}}{s} . \]

10. It follows directly from the translation property of the transform that

\[ \mathcal{L}[f(t)] = \frac{e^{-s}}{s} + 2\frac{e^{-3s}}{s} - 6\frac{e^{-4s}}{s} . \]

11. Before invoking the translation property of the transform, write the function as

\[ f(t) = (t - 2) u_2(t) - u_3(t) - (t - 3) u_3(t) - u_3(t) . \]
It follows that

\[ \mathcal{L}[f(t)] = \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}. \]

12. It follows directly from the translation property of the transform that

\[ \mathcal{L}[f(t)] = \frac{1}{s^2} - \frac{e^{-s}}{s^2}. \]

13. Using the fact that \( \mathcal{L}[e^{at}f(t)] = \mathcal{L}[f(t)]|_{s \rightarrow s-a} \),

\[ \mathcal{L}^{-1}\left[\frac{3!}{(s-2)^4}\right] = t^3 e^{2t}. \]

15. First consider the function

\[ G(s) = \frac{2(s-1)}{s^2 - 2s + 2}. \]

Completing the square in the denominator,

\[ G(s) = \frac{2(s-1)}{(s-1)^2 + 1}. \]

It follows that

\[ \mathcal{L}^{-1}[G(s)] = 2e^t \cos t. \]

Hence

\[ \mathcal{L}^{-1}[e^{-2s}G(s)] = 2 e^{(t-2)} \cos (t-2) \cdot u_2(t). \]

16. The inverse transform of the function \( 2/(s^2 - 4) \) is \( f(t) = \sinh 2t \). Using the translation property of the transform,

\[ \mathcal{L}^{-1}\left[\frac{2 e^{-2s}}{s^2 - 4}\right] = \sinh 2(t-2) \cdot u_2(t). \]

17. First consider the function

\[ G(s) = \frac{(s-2)}{s^2 - 4s + 3}. \]

Completing the square in the denominator,
\[ G(s) = \frac{(s - 2)}{(s - 2)^2 - 1}. \]

It follows that
\[ \mathcal{L}^{-1}[G(s)] = e^{2t} \cosh t. \]

Hence
\[ \mathcal{L}^{-1}\left[\frac{(s - 2)e^{-s}}{s^2 - 4s + 3}\right] = e^{2(t - 1)} \cosh (t - 1) u_1(t). \]

18. Write the function as
\[ F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}. \]

It follows from the translation property of the transform, that
\[ \mathcal{L}^{-1}\left[\frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}\right] = u_1(t) + u_2(t) - u_3(t) - u_4(t). \]

19(a). By definition of the Laplace transform,
\[ \mathcal{L}[f(\xi t)] = \int_0^\infty e^{-st} f(\xi t) dt. \]

Making a change of variable, \( \tau = \xi t \), we have
\[ \mathcal{L}[f(\xi t)] = \frac{1}{\xi} \int_0^\infty e^{-s(\tau/c)} f(\tau) d\tau \]
\[ = \frac{1}{\xi} \int_0^\infty e^{-s/\xi} f(\tau) d\tau. \]

Hence \( \mathcal{L}[f(\xi t)] = \frac{1}{\xi} F\left(\frac{s}{\xi}\right) \), where \( s/\xi > a \).

(b). Using the result in Part (a),
\[ \mathcal{L}\left[ f\left(\frac{t}{k}\right) \right] = k F(ks). \]

Hence
\[ \mathcal{L}^{-1}[F(ks)] = \frac{1}{k} f\left(\frac{t}{k}\right). \]
(c). From Part (b),
\[ \mathcal{L}^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right). \]

Note that \( as + b = a(s + b/a) \). Using the fact that \( \mathcal{L}[e^{ct} f(t)] = \mathcal{L}[f(t)]_{s \rightarrow s-c} \),
\[ \mathcal{L}^{-1}[F(as + b)] = e^{-bt/a} \frac{1}{a} f\left(\frac{t}{a}\right). \]

20. First write
\[ F(s) = \frac{n!}{(\frac{a}{2})^{n+1}}. \]
Let \( G(s) = n! / s^{n+1} \). Based on the results in Prob. 19,
\[ \frac{1}{2} \mathcal{L}^{-1}\left[ G\left(\frac{s}{2}\right) \right] = g(2t), \]
in which \( g(t) = t^n \). Hence
\[ \mathcal{L}^{-1}[F(s)] = 2 (2t)^n = 2^{n+1} t^n. \]

23. First write
\[ F(s) = \frac{e^{-4(s-1/2)}}{2(s - 1/2)}. \]
Now consider
\[ G(s) = \frac{e^{-2s}}{s}. \]
Using the result in Prob. 19(b),
\[ \mathcal{L}^{-1}[G(2s)] = \frac{1}{2} g\left(\frac{t}{2}\right), \]
in which \( g(t) = u_2(t) \). Hence \( \mathcal{L}^{-1}[G(2s)] = \frac{1}{2} u_2(t/2) = \frac{1}{2} u_4(t) \). It follows that
\[ \mathcal{L}^{-1}[F(s)] = \frac{1}{2} e^{t/2} u_4(t). \]

24. By definition of the Laplace transform,
Since there is no damping term, the solution follows the forcing function, after which the response is a steady oscillation about $y = 0$.

5. Let $f(t)$ be the forcing function on the right-hand-side. Taking the Laplace transform of both sides of the ODE, we obtain

$$s^2 Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \mathcal{L}[f(t)].$$

Applying the initial conditions,

$$s^2 Y(s) + 3sY(s) + 2Y(s) = \mathcal{L}[f(t)].$$

The transform of the forcing function is

$$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{e^{-10s}}{s}.$$

Solving for the transform,

$$Y(s) = \frac{1}{s(s^2 + 3s + 2)} - \frac{e^{-10s}}{s(s^2 + 3s + 2)}.$$

Using partial fractions,

$$\frac{1}{s(s^2 + 3s + 2)} = \frac{1}{2} \left[ \frac{1}{s} + \frac{1}{s + 2} - \frac{2}{s + 1} \right].$$

Hence

$$\mathcal{L}^{-1} \left[ \frac{1}{s(s^2 + 3s + 2)} \right] = \frac{1}{2} + \frac{e^{-2t}}{2} - e^{-t}.$$

Based on Theorem 6.3.1,

$$\mathcal{L}^{-1} \left[ \frac{e^{-10s}}{s(s^2 + 3s + 2)} \right] = \frac{1}{2} \left[ 1 + e^{-2(t-10)} - 2e^{-(t-10)} \right]u_{10}(t).$$

Hence the solution of the IVP is
\[ y(t) = \frac{1}{2}[1 - u_{10}(t)] + \frac{e^{-2t}}{2} - e^{-t} - \frac{1}{2}[e^{-(2t-20)} - 2e^{-(t-10)}] u_{10}(t). \]

The solution increases to a temporary steady value of \( y = 1/2 \). After the forcing ceases, the response decays exponentially to \( y = 0 \).

6. Taking the Laplace transform of both sides of the ODE, we obtain

\[ s^2 Y(s) - sy(0) - y'(0) + 3sY(s) - y(0)] + 2Y(s) = \frac{e^{-2s}}{s}. \]

Applying the initial conditions,

\[ s^2 Y(s) + 3sY(s) + 2Y(s) - 1 = \frac{e^{-2s}}{s}. \]

Solving for the transform,

\[ Y(s) = \frac{1}{s^2 + 3s + 2} + \frac{e^{-2s}}{s(s^2 + 3s + 2)}. \]

Using partial fractions,
\[
\frac{1}{s^2 + 3s + 2} = \frac{1}{s + 1} - \frac{1}{s + 2}
\]

and

\[
\frac{1}{s(s^2 + 3s + 2)} = \frac{1}{2} \left[ \frac{1}{s} + \frac{1}{s + 2} - \frac{2}{s + 1} \right]
\]

Taking the inverse transform term-by-term, the solution of the IVP is

\[
y(t) = e^{-t} - e^{-2t} + \left[ \frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)} \right] u_2(t).
\]

Due to the initial conditions, the response has a transient overshoot, followed by an exponential convergence to a steady value of \( y_s = 1/2 \).

7. Taking the Laplace transform of both sides of the ODE, we obtain

\[
s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{e^{-3ms}}{s}.
\]

Applying the initial conditions,
\[ s^2 Y(s) + Y(s) - s = \frac{e^{-3\pi s}}{s}. \]

Solving for the transform,

\[ Y(s) = \frac{s}{s^2 + 1} + \frac{e^{-3\pi s}}{s(s^2 + 1)}. \]

Using partial fractions,

\[ \frac{1}{s(s^2 + 1)} = \frac{1}{s} - \frac{s}{s^2 + 1}. \]

Hence

\[ Y(s) = \frac{s}{s^2 + 1} + e^{-3\pi s} \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right). \]

Taking the inverse transform, the solution of the IVP is

\[ y(t) = \cos t + [1 - \cos(t - 3\pi)]u_{3\pi}(t) \]
\[ = \cos t + [1 + \cos t]u_{3\pi}(t). \]

Due to initial conditions, the solution temporarily oscillates about \( y = 0 \). After the forcing is applied, the response is a steady oscillation about \( y_m = 1 \).
18(a).

(b). The forcing function can be expressed as

\[ f_k(t) = \frac{1}{2k} [u_{4-k}(t) - u_{4+k}(t)]. \]

Taking the Laplace transform of both sides of the ODE, we obtain

\[ s^2 Y(s) - sy(0) - y'(0) + \frac{1}{3} [sY(s) - y(0)] + 4Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}. \]

Applying the initial conditions,

\[ s^2 Y(s) + \frac{1}{3} sY(s) + 4Y(s) = \frac{e^{-(4-k)s}}{2ks} - \frac{e^{-(4+k)s}}{2ks}. \]

Solving for the transform,

\[ Y(s) = \frac{3e^{-(4-k)s}}{2ks(3s^2 + s + 12)} - \frac{3e^{-(4+k)s}}{2ks(3s^2 + s + 12)}. \]

Using partial fractions,

\[ \frac{1}{s(3s^2 + s + 12)} = \frac{1}{12} \left[ \frac{1}{s} - \frac{1 + 3s}{3s^2 + s + 12} \right] = \frac{1}{12} \left[ \frac{1}{s} - \frac{1}{6} \left( \frac{1}{s + \frac{1}{6}} \right) \right]. \]

Let

\[ H(s) = \frac{1}{8k} \left[ \frac{1}{s} - \frac{\frac{1}{6}}{(s + \frac{1}{6})^2 + \frac{143}{36}} - \frac{s + \frac{1}{6}}{(s + \frac{1}{6})^2 + \frac{143}{36}} \right]. \]

It follows that
\[ h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{8k} - \frac{e^{-t/6}}{8k} \left[ \frac{1}{\sqrt{143}} \sin \left( \frac{\sqrt{143} t}{6} \right) + \cos \left( \frac{\sqrt{143} t}{6} \right) \right]. \]

Based on Theorem 6.3.1, the solution of the IVP is
\[ y(t) = h(t - 4 + k) u_{4-k}(t) - h(t - 4 - k) u_{4+k}(t). \]

As the parameter \( k \) decreases, the solution remains null for a longer period of time.
Since the magnitude of the impulsive force increases, the initial overshoot of the response also increases. The duration of the impulse decreases. All solutions eventually decay to $y = 0$.

19(a).

(c). From Part (b),

$$u(t) = 1 - \cos t + 2 \sum_{k=1}^{n} (-1)^k [1 - \cos(t - k\pi)]u_{n+}(t).$$