APPLIED ALGEBRA:
PROBLEM SHEET 5

(1) We showed in class that the ring of symmetric functions
\[ \Lambda = \mathbb{Z}[e_1, e_2, \ldots] = \mathbb{Z}[h_1, h_2, \ldots]. \]
On the other hand, for each \( n \), we have a surjective projection map
\[ \Lambda \rightarrow \Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n} \]
given by setting \( x_{n+1} = x_{n+2} = \ldots = 0. \)

(i) By studying this projection map, show that
\[ \Lambda_n = \mathbb{Z}[e_1, \ldots, e_n] = \mathbb{Z}[h_1, \ldots, h_n], \]
where, by abuse of notation, we write \( e_1, \ldots, e_n \) for the image of \( e_1, \ldots, e_n \) under this map.

(ii) Determine the images of \( e_i \) and \( h_i \) for \( i \geq n + 1 \), by expressing them as polynomials in the variables \( e_1, \ldots, e_n \) or \( h_1, \ldots, h_n \) respectively.

(2) We defined a ring automorphism \( \omega \) of \( \Lambda \) by setting
\[ \omega(e_i) = h_i. \]

(i) Show that \( \omega(h_i) = e_i. \)

(ii) Show that \( \omega(p_k) = (-1)^{k-1} \cdot p_k, \) and deduce that for any partition \( \lambda, \) \( \omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} \cdot p_\lambda. \)

(3) Show that \( e_n = \det A, \) where \( A \) is the \( n \times n \) matrix whose \( (i, j) \)-th entry is \( h_{1-i+j} \). Here, we interpret \( h_0 = 1 \) and \( h_i = 0 \) if \( i < 0. \)

(i) Show that \( h_n = \det B \) where \( B \) is the \( n \times n \) matrix whose \( (i, j) \)-th entry is \( e_{i-j} \).

(ii) Show that \( p_n = \det C \) where
\[ C = \begin{pmatrix}
e_1 & 1 & 0 & 0 & \ldots \\2e_2 & e_1 & 1 & 0 & \ldots \\3e_3 & e_2 & e_1 & 1 & 0\ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

(4) We showed in class that \( e_n \) and \( h_n \) can be expressed as \( \mathbb{Q} \)-linear combinations of the \( p_\lambda \)'s. By exploiting the identities of generating functions shown in class:
\[ P(-t) = \frac{d}{dt} \log E(t) \quad \text{and} \quad P(t) = \frac{d}{dt} \log H(t), \]
show that
\[ e_n = \sum_{\lambda:|\lambda|=n} \frac{(-1)^n}{z_\lambda} \cdot p_\lambda \]
and
\[ h_n = \sum_{\lambda:|\lambda|=n} \frac{1}{z_\lambda} \cdot p_\lambda \]
where
\[ z_\lambda = \prod_i i^{m_i} \cdot (m_i)! \quad \text{if} \ \lambda = (1^{m_1}, 2^{m_2}, \ldots). \]

(5) Show that the coefficient of \(x_1x_2\ldots x_n\) in \(s_\lambda(x_1, \ldots, x_n)\) is
\[ \frac{n!}{\prod_i (\lambda_i + n - i)!} \cdot \prod_{i<j}(\lambda_i - \lambda_i - i + j). \]

(6) Show that
\[ a_\lambda(q^{n-1}, q^{n-2}, \ldots, q, 1) = a_0(q^{\lambda_1+n-1}, q^{\lambda_2+n-2}, \ldots, q^{\lambda_n}). \]

(7) Compute the value of \(s_\lambda(1, \ldots, 1)\)

(8) The principal specialization of a symmetric function in the variables \(\{x_1, \ldots, x_n\}\) is obtained by replacing \(x_i\) by \(q^i\) for all \(i\). Show that the Schur function specialization \(s_\lambda(q, q^2, \ldots, q^n)\) is the generating function for semistandard \(\lambda\)-tableaux with entries of size at most \(n\).

(9) Express \(s_\lambda \cdot e_1\) as a linear combination of Schur polynomials.