NUMBER FIELDS HW 1

(1) Which of the following are algebraic integers?
\( \frac{\sqrt{3} + \sqrt{5}}{2}, \quad \frac{\sqrt{3} + \sqrt{7}}{2}, \quad \frac{1 + \sqrt{10} + \sqrt{100}}{3} \)

(2) If \( a_i \) are algebraic integers and \( \alpha \in \mathbb{C} \) satisfies:
\[ \alpha^d + a_{d-1}\alpha^{d-1} + \ldots + a_0 = 0, \]
show that \( \alpha \) is an algebraic integer.

(3) Let \( V \) is a finite dimensional vector space over a field \( F \), and let
\[ \langle -, - \rangle : V \times V \rightarrow F \]
be a nondegenerate bilinear form. Suppose that \( \{v_i\} \) is a basis of \( V \). Show that there is a unique basis \( \{v_i^*\} \) of \( V \) so that \( \langle v_i, v_j^* \rangle = \delta_{ij} \). The basis \( \{v_i^*\} \) is called the dual basis to \( \{v_i\} \) with respect to \( \langle -, - \rangle \).

(4) Let \( \mathcal{O}_f = \mathbb{Z} + \mathbb{Z} \cdot f\sqrt{D} \). Show that \( \mathcal{O}_f \) is an order in \( \mathbb{Q}(\sqrt{D}) \). Compute \( Disc(\mathcal{O}_f) \).

(5) If \( \mathcal{O} \) is an order spanning a number field \( K \), and \( Disc(\mathcal{O}) \) is squarefree. Must \( \mathcal{O} \) be equal to the ring of integers \( \mathcal{O}_K \)?

(6) Prove Stickelberger’s criterion: if \( K \) is a number field, then \( Disc(K) = 0 \) or \( 1 \) mod 4. (Hint: use the fact that if \( \mathcal{O}_K = \oplus_i \mathbb{Z} \alpha_i \), then \( Disc(K) = |\det(\sigma_i(\alpha_j))|^2\); write the determinant as \( P - Q \), where \( P \) is the sum over even permutations and \( Q \) is the sum over odd permutations, and use \( (P - Q)^2 = (P + Q)^2 - 4PQ \)).

(7) Let \( d \) be a cubefree integer not divisible by 3 and let \( K = \mathbb{Q}(\sqrt[3]{d}) \).

(i) Show that \( \mathbb{Z}[\sqrt[3]{d}] \) has discriminant \( 27d^2 \).

(ii) If \( p \) is a prime factor of \( d \) and \( \alpha = (a + b\sqrt[3]{d} + c\sqrt[3]{d^2})/p \) lies in \( \mathcal{O}_K \) (with \( a, b, c \) rational), show by taking traces that \( p|a \). Then show by squaring that \( \alpha \in \mathbb{Z}[\sqrt[3]{d}] \).

(iii) Deduce that either \( \mathcal{O}_K = \mathbb{Z}[\sqrt[3]{d}] \) or that \( \mathcal{O}_K \) is spanned by \( \mathbb{Z}[\sqrt[3]{d}] \) and some
\[ \beta = (a + b\sqrt[3]{d} + c\sqrt[3]{d^2})/3. \]
In the second case, show by squaring that \( a \equiv 1 \) mod 3 and \( b \equiv d \) mod 3, so that we may take \( \beta \) to be
\[ \beta_0 = (1 + bd\sqrt[3]{d} + \sqrt[3]{d^2})/3. \]

(iv) Show that if \( d \neq \pm 1 \) mod 9, then \( \mathcal{O}_K = \mathbb{Z}[\sqrt[3]{d}] \), and otherwise \( \mathcal{O}_K \) is generated by \( \mathbb{Z}[\sqrt[3]{d}] \) and \( \beta_0 \).
(8) Do (the very long) Exercises 29 and 42 from Chapter 2 of D. Marcus’s “Number Fields”. These exercises determine the ring of integers of a biquadratic extension $\mathbb{Q}(\sqrt{m}, \sqrt{n})$.

(9) Do Ex. 30 from Marcus. This exercise shows that the ring of integers $\mathcal{O}_K$ is not always of the form $\mathbb{Z}[\alpha]$. 