TRILINEAR FORMS AND TRIPLE PRODUCT EPSILON FACTORS

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Abstract. We give a short and simple proof of a theorem of Dipendra Prasad on the existence and non-existence of invariant trilinear forms on a triple of irreducible representations of $GL_2(F)$ or $D^\times$, where F is a non-archimedean local field of zero or odd characteristic and D is the unique quaternion division F-algebra. The answer is controlled by the central value of the triple product epsilon factor. Our proof works uniformly for all representations and without restriction on residual characteristic. It also gives an analogous theorem for any separable cubic F-algebra.

1. Introduction

In his Harvard Ph.D. thesis [P1], D. Prasad proved the following beautiful theorem:

Theorem 1.1. *Let F be a non-archimedean local field of zero or odd characteristic and let D be the unique quaternion division F-algebra. Let π_1, π_2 and π_3 be irreducible infinite dimensional representations of $GL_2(F)$ such that $\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} = 1$ and let π_i^D be the Jacquet-Langlands lift of π_i to $D^\times$. Then we have:*

(i) $\dim \mathrm{Hom}_{GL_2}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) + \dim \mathrm{Hom}_{D^\times}(\pi_1^D \otimes \pi_2^D \otimes \pi_3^D, \mathbb{C}) = 1.$

(ii) If the residue characteristic of F is odd, then $\mathrm{Hom}_{GL_2}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) \neq 0$ *if and only if* $\epsilon(1/2, \pi_1 \times \pi_2 \times \pi_3) = 1$.

Let us make a few remarks:

Remarks:

(a) In (ii), the triple product ϵ-factor can a priori be defined in 3 possible ways:

• as the ϵ-factor of Artin type defined using the tensor product of the Langlands parameters of the representations π_i, à la Deligne and Langlands.

• by the Langlands-Shahidi method;

• by the theory of local zeta integrals à la Garrett [G], as developed by Piatetski-Shapiro and Rallis [PSR] as well as Ikeda [I].

However, a result of Ramakrishnan [R, Thm. 4.4.1] shows that these 3 definitions agree.

(b) There is a version of the theorem with $GL_2(F)^3$ replaced by $GL_2(E)$ for any separable cubic F-algebra E, which was investigated in [P2] but only completely resolved later in [PSP]. We will state the precise result later on.

(c) There is a well-known global analog of the theorem which was known as Jacquet's conjecture and which has been proven by Harris and Kudla [HK]. This relates the non-vanishing of the triple product period integral for some inner form of $GL(2)$ with the non-vanishing of the central critical value of the triple product L-function. The proof of this global result uses (i) of the local theorem above, but does not use (ii).

The statement (i) of the theorem has an entirely satisfactory proof, based on character theory and generalities. However, as Prasad has remarked in [P3] and to this author on several occasions, the proof of (ii) seems to be less satisfactory as it involves some case-by-case considerations and brute force computations. Moreover, it does not cover some supercuspidal cases when the residue characteristic of F is equal to 2; this accounts for the assumption of odd residue characteristic in (ii).

In the recent paper [P3], Prasad discovered a proof of (ii) for supercuspidal representations using a global-to-local argument, starting with the global theorem of Harris-Kudla [HK]. Besides removing the assumption of odd residual characteristic, this proof has the advantage that it takes care of any separable cubic F-algebra, using the extension of the Harris-Kudla theorem to this setting by Prasad and Schulze-Pillot [PSP].

The purpose of this paper is to give a simple and purely local proof of the above theorem of Prasad (for any separable cubic algebra E and any residue characteristic).

We are led to this proof by the following considerations. The proof of Jacquet's conjecture by Harris-Kudla makes use of the method of global theta correspondence and the integral representation of the triple product L-function of Garrett. Now theta correspondence makes sense locally as well, and the local theory of Garrett's integral representation is well-developed. It is then natural to raise the following simple-minded question:

Question: Is there an analog of the global proof of Harris-Kudla in the local setting?

Once this question is asked, it is not surprising that the answer is in the affirmative (modulo some details) and one obtains a short and sweet proof of the following theorem:

Theorem 1.2. *Let F be a non-archimedean local field of zero or odd characteristic, E a separable cubic F-algebra and D the unique quaternion division F-algebra. Let Π_E be an irreducible generic representation of $GL_2(E)$ whose central character ω_{Π_E} is trivial on $F^\times$. Let Π_E^D be the Jacquet-Langlands lift of Π_E to $D_E^\times = (D \otimes_F E)^\times$. Then we have:*

(i) $\dim \mathrm{Hom}_{GL_2}(\Pi_E, \mathbb{C}) + \dim \mathrm{Hom}_{D^\times}(\Pi_E^D, \mathbb{C}) = 1.$

(ii) $\mathrm{Hom}_{GL_2}(\Pi_E, \mathbb{C}) \neq 0$ *if and only if*

$$\epsilon(1/2, \Pi_E, \rho) \cdot \omega_{K/F}(-1) = 1.$$

Here K is the quadratic discriminant algebra of E, and the epsilon factor $\epsilon(s, \Pi_E, \rho)$ is the one defined by the local theory of the triple product zeta integral, as developed in [PSR] and [I].

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2. **A See-Saw Identity**

Let E be a separable cubic algebra over F, with associated trace map $\mathrm{Tr}_{E/F}$ and norm map $N_{E/F}$. Let B be either $M_2(F)$ or D, and set $B_E = B \otimes_F E$. Let V_B be the rank 4 quadratic space given by the reduced norm N_B of B, so that

$$GSO(V_B) \cong (B^\times \times B^\times)/\Delta F^\times.$$

Similarly, we may consider the rank 12 quadratic space

$$V_{B,E} = Res_{E/F}(V_B \otimes E) = Res_{E/F} B_E$$

given by the quadratic form $\mathrm{Tr}_{E/F} \circ N_{B_E}$. Then

$$GSO(V_{B,E}) \cong (B_E^\times \times B_E^\times)/\Delta E^\times.$$

We regard $GSO(V_{B,E})$ as an algebraic group over F and observe that there is a natural embedding

$$GSO(V_B) \hookrightarrow GSO(V_{B,E}).$$

In fact, the image of this embedding is contained in the subgroup

$$GSO(V_{B,E})^0 = \{g \in GSO(V_{B,E}) : \mathrm{sim}(g) \in F^\times\},$$

where sim is the similitude character.

Analogously, let $W = Fe_1 \oplus Fe_2$ be equipped with the standard alternating form with $\langle e_1, e_2 \rangle = 1$, so that its associated symplectic similitude group is $G = GL_2(F)$. Let $(W_E, \langle -, - \rangle_E)$ be the symplectic space over E obtained from W by extension of scalars, with similitude group $G_E = GL_2(E)$. Then we have the natural embedding

$$\Delta : G \hookrightarrow G_E$$

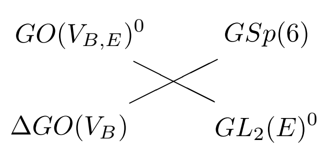
whose image is contained in the subgroup

$$G_E^0 = \{g \in G_E : \det(g) \in F^\times\}.$$

One may consider the 6-dimensional F-vector space $Res_{E/F}(W_E)$, equipped with the alternating form $\mathrm{Tr}_{E/F} \circ \langle -, - \rangle_E$. Denoting the associated symplectic similitude group by $GSp(6)$, one has a natural embedding

$$G_E^0 \hookrightarrow GSp(6).$$

The set up of the Harris-Kudla proof is the following see saw diagram:



Given an irreducible representation Π_E of $GL_2(E)$, the see-saw identity then gives:

$$\mathrm{Hom}_{GO(V_B)}(\Theta_{B_E}(\Pi_E), \mathbb{C}) \cong \mathrm{Hom}_{G_E^0}(\Theta_B(1), \Pi_E),$$

where

$$\begin{cases} \Theta_{B_E}(\Pi_E) = \text{the big theta lift of } \Pi_E \text{ to } GO(V_{B,E}); \\ \Theta_B(1) = \text{the big theta lift of the trivial representation of } \Delta GO(V_B) \text{ to } GSp(6). \end{cases}$$

Now we note the following propositions.

Proposition 2.1. *Let Π_E be an irreducible generic representation of $Gl_2(E)$ and let Π_E^B be the Jacquet-Langlands lift of Π_E to $B_E^\times$. Then under the theta correspondence for $GL_2(E) \times GO(V_{B,E})$,*

$$\Theta_{B_E}(\Pi_E) = \Pi_E^B \boxtimes \Pi_E^{B^\vee}$$

as a representation of $GSO(V_{B,E})$.

Proof. This is well-known. □

Proposition 2.2. *If P is the Siegel parabolic subgroup of $GSp(6)$ and $I_P(0)$ denotes the degenerate principal series representation unitarily induced from the trivial character of P, then*

$$\Theta_B(1) \hookrightarrow I_P(0).$$

Further, $\Theta_B(1)$ is irreducible and

$$I_P(0) = \Theta_{M_2(F)}(1) \oplus \Theta_D(1).$$

Moreover, if sign is the unique order 2 character of $GO(V_B)$ which is trivial on $GSO(V_B)$, then $\Theta_B(sign) = 0$.

Proof. This follows immediately from [KR, Cor. 3.7]. □

One consequence of the above two propositions is that we may replace GO by GSO in the theta correspondences and see-saw diagram under consideration. The see-saw identity, together with Props. 2.1 and 2.2, now gives:

$$\begin{aligned}
&\dim \mathrm{Hom}_{GL_2}(\Pi_E, \mathbb{C}) + \dim \mathrm{Hom}_{D^\times}(\Pi_E^D, \mathbb{C}) \\
\le &(\dim \mathrm{Hom}_{GL_2}(\Pi_E, \mathbb{C}))^2 + (\dim \mathrm{Hom}_{D^\times}(\Pi_E^D, \mathbb{C}))^2 \\
= &\dim \mathrm{Hom}_{G_E^0}(\Theta_{M_2(F)}(1), \Pi_E) + \dim \mathrm{Hom}_{G_E^0}(\Theta_D(1), \Pi_E^D) \\
= &\dim \mathrm{Hom}_{G_E^0}(I_P(0), \Pi_E).
\end{aligned}$$

Thus, the identity

$$\dim \mathrm{Hom}_{GL_2}(\Pi_E, \mathbb{C}) + \dim \mathrm{Hom}_{D^\times}(\Pi_E^D, \mathbb{C}) = 1$$

of Thm. 1.2(i) would follow if we could show that

$$\dim \mathrm{Hom}_{G_E^0}(I_P(0), \Pi_E) = 1.$$

The following proposition, which is the main result of this section, gives the crucial upper bound:

Proposition 2.3. *For any irreducible generic representation Π_E of $GL_2(E)$,*

$$\dim \mathrm{Hom}_{G_E^0}(I_P(0), \Pi_E) \le 1.$$

If Π_E is supercuspidal, then the above dimension is 1.

The rest of the section is devoted to the proof of this proposition. Before that, we remark that in this paper, if H is a subgroup of G, we shall write I_H to denote the normalized (smooth) induction functor from H to G, whereas we shall write Ind_H^G and ind_H^G to denote the unnormalized (smooth) induction functor and the unnormalized (smooth) induction functor with compact support modulo H.

Not surprisingly, the proof of Prop. 2.3 is by Mackey theory. In [PSR, Lemma 1.1 and Prop. 3.1], this was carried out to a certain extent, but to obtain the result above requires a more refined analysis.

In [PSR, Lemma 1.1] and its corollary, Piatetski-Shapiro and Rallis determined the (finitely many) orbits for the action of G_E^0 on $P\backslash GSp(6)$ and the stabilizer of a point in each orbit. Using this, one sees that as a G_E^0-module $I_P(0)$ has a certain natural filtration associated to the orbit structure. The following lemma describes the successive quotients of this filtration.

Lemma 2.4. *As a G_E^0-module, $I_P(0)$ has a filtration*

$$0 \subset A_0 \subset A_1 \subset A_2 = I_P(0)$$

whose successive quotients are given as follows.

(i) The top quotient is:

$$A_2/A_1 \cong Ind_{B_E^0}^{G_E^0} \delta_{B_E^0},$$

where B_E is the standard Borel subgroup of G_E and the induction is unnormalized.

(ii) The structure of the middle quotient depends on the type of the cubic algebra E. There are 3 different cases:

(a) *if $E = F \times F \times F$, so that $G_E = GL_2(F) \times GL_2(F) \times GL_2(F)$, then*

$$A_1/A_0 \cong A_{100} \oplus A_{010} \oplus A_{001},$$

where

$$A_{100} \cong ind_{(B \times \Delta GL_2)^0}^{G_E^0} \delta_B \boxtimes 1.$$

Here, B is the Borel subgroup of the first factor of GL_2 in G_E and ΔGL_2 is "diagonally" embedded into the other two copies of GL_2. Moreover, A_{010} and A_{001} are similarly defined, by permuting the factor with the Borel subgroup.

(b) *if $E = F \times K$ with K a quadratic extension of F, so that $G_E = GL_2(F) \times GL_2(K)$, then*

$$A_1/A_0 \cong ind_{(B \times \Delta GL_2)^0}^{G_E^0} \delta_B \boxtimes 1.$$

Here, B is the Borel subgroup of the $GL_2(F)$ factor in G_E and ΔGL_2 is given by an embedding of $GL_2(F)$ into $GL_2(K)$.

(c) *if E is a field, then $A_1/A_0 = 0$.*

(iii) The submodule A_0 sits in a short exact sequence:

$$0 \longrightarrow ind_{\Delta Z \cdot U_E}^{G_E^0}(1 \boxtimes \psi) \longrightarrow A_0 \longrightarrow Ind_{B_E^0}^{G_E^0} \delta_{B_E^0}^{2/3} \cdot C_c^\infty(T_E^0/\Delta T) \longrightarrow 0.$$

Here, ΔT and ΔZ denote the maximal (diagonal) torus and the center of $GL_2(F)$, naturally embedded into G_E^0, and U_E is the unipotent radical of the Borel subgroup B_E^0 of G_E^0. Moreover, ψ is a character of U_E in general position, so that

$$ind_{\Delta Z \cdot U_E}^{G_E^0}(1 \boxtimes \psi)$$

is the Gelfand-Graev module of G_E^0 (with ΔZ acting trivially).

The following corollary can immediately be deduced from the lemma:

Corollary 2.5. *Suppose that Σ_E is an irreducible representation of G_E.*

(i) If $\mathrm{Hom}_{G_E^0}(A_2/A_1, \Sigma_E) \neq 0$, then Σ_E is a 1-dimensional character.

(ii) If $\mathrm{Hom}_{G_E^0}(A_1/A_0, \Sigma_E) \neq 0$ *(so that E is not a field), then we have*

$$\Sigma_E = \begin{cases} \chi \boxtimes \chi^{-1}\pi^\vee \boxtimes \pi, & \text{if } E = F \times F \times F; \\ \chi \boxtimes \pi_K, & \text{if } E = F \times K. \end{cases}$$

for some character χ *of* $F^\times$ *and a representation* π *(resp.* π_K*) of* $GL(2)$ *(resp.* $GL_2(K)$*).*

(iii) For an irreducible generic representation Π_E*, we have*

$$\mathrm{Hom}_{G_E^0}(A_2/A_1, \Pi_E) = \mathrm{Hom}_{G_E^0}(A_1/A_0, \Pi_E) = 0.$$

Moreover,

$$\dim \mathrm{Hom}_{G_E^0}(A_0, \Pi_E) \leq 1,$$

except possibly when

$$\Pi_E \text{ is a quotient of } I_{B_E}(\chi_E)$$

with

$$\chi_E|_{\Delta T} = \delta_B^{1/2}.$$

Here, $I_{B_E}(\chi_E)$ *is the principal series representation of* G_E *unitarily induced from the character* χ_E *of the maximal torus* T_E*.*

In particular, outside of the exceptional situation in (iii), the corollary immediately implies Prop. 2.3. Observe that when one is in the exceptional situation of (iii), then

- Π_E is not a discrete series representation. This is because if $I_{B_E}(\chi_E)$ has a twisted Steinberg representation as a quotient, then the condition $\chi_E|_{\Delta T} = \delta_B^{1/2}$ cannot hold.
- Π_E is the unique irreducible quotient of $I_{B_E}(\chi_E)$. This is simply because any principal series representation of G_E has a unique irreducible quotient.

To take care of the exceptional case, we suppose that

$$I_{B_E}(\chi_E) \twoheadrightarrow \Pi_E$$

with $\chi_E|_{\Delta T} = \delta_B^{1/2}$. As we observed above, Π_E is not a discrete series representation, so that $\Pi_E^D = 0$. It follows from Prop. 2.1 and the seesaw identity that

$$\dim \mathrm{Hom}_{G_E^0}(I_P(0), \Pi_E) = \dim \mathrm{Hom}_{G_E^0}(\Theta_{M_2(F)}(1), \Pi_E) = (\dim \mathrm{Hom}_{GL_2}(\Pi_E, \mathbb{C}))^2.$$

As

$$\mathrm{Hom}_{GL_2}(\Pi_E, \mathbb{C}) \subset \mathrm{Hom}_{GL_2}(I_{B_E}(\chi_E), \mathbb{C}),$$

Prop. 2.3 for the exceptional representations in Cor. 2.5 would follow if one can show that

$$\dim \mathrm{Hom}_{GL_2}(I_{B_E}(\chi_E), \mathbb{C}) \leq 1.$$

We can again understand this Hom space by Mackey theory. The GL_2-orbits on $B_E \backslash G_E$ can be easily calculated, from which one obtains:

Lemma 2.6. *As a* $GL_2(F)$*-module,* $I_{B_E}(\chi_E)$ *has a filtration*

$$0 \subset C_0 \subset C_1 \subset C_2 = I_{B_E}(\chi_E)$$

whose successive quotients can be described as follows.

(i) The top quotient is

$$C_2/C_1 \cong I_B(\delta_B \cdot \chi_E|_{\Delta T}).$$

(ii) The middle subquotient depends on the type of E:

(a) *if* $E = F \times F \times F$, *then*

$$C_1/C_0 = ind_T^{GL_2}(\delta_B^{1/2} \cdot (w_1\chi_E)|_{\Delta T}) \oplus ind_T^{GL_2}(\delta_B^{1/2} \cdot (w_2\chi_E)|_{\Delta T}) \oplus ind_T^{GL_2}(\delta_B^{1/2} \cdot (w_3\chi_E)|_{\Delta T})$$

where w_i *is the non-trivial Weyl group element in the* i*-th copy of* GL_2*.*

(b) *if* $E = F \times K$, *then*

$$C_1/C_0 \cong ind_T^{GL_2}(\delta_B^{1/2} \cdot (w_1\chi_E)|_{\Delta T})$$

where w_1 *is the non-trivial Weyl group element of the* GL_2 *factor in* $GL_2(E)$*.*

(c) *if* E *is a field, then* $C_1/C_0 = 0$.

(iii) The submodule is

$$C_0 = C_c^\infty(PGL_2).$$

The following corollary can be read off the lemma:

Corollary 2.7. *We have:*

(i) If $\mathrm{Hom}_{GL_2}(C_2/C_1, \mathbb{C}) \neq 0$, *then* $\chi_E|_{\Delta T} = \delta_B^{-1/2}$.

(ii) If $\mathrm{Hom}_{GL_2}(C_1/C_0, \mathbb{C}) \neq 0$ *(so that* E *is not a field), then* $(w_i\chi_E)|_{\Delta T} = \delta_B^{-1/2}$ *for some* $i = 1$, 2 *or* 3.

(iii) $\dim \mathrm{Hom}_{GL_2}(C_0, \mathbb{C}) = 1$.

In particular, if we are in the exceptional situation of Cor. 2.5(iii) so that $\chi_E|_{\Delta T} = \delta_B^{1/2}$ and the unique irreducible quotient of $I_{B_E}(\chi_E)$ is generic, then

$$\mathrm{Hom}_{GL_2}(C_2/C_1, \mathbb{C}) \neq 0 \Longrightarrow \delta_B = 1$$

and

$$\mathrm{Hom}_{GL_2}(C_1/C_0, \mathbb{C}) \neq 0 \Longrightarrow (\chi_E \cdot (w_i\chi_E)^{-1})|_{\Delta T} = \delta_B.$$

The first is clearly not possible, whereas the second would imply that $I_{B_E}(\chi_E)$ has a non-generic irreducible quotient, which is not the case since its unique irreducible quotient is the generic Π_E. This completes the proof of Prop. 2.3.

To complete the proof of (i) of Theorem 1.2, we need to produce a nonzero element in the space $\mathrm{Hom}_{G_E^0}(I_P(0), \Pi_E)$. We shall see that this can be achieved by using a local zeta integral. This is the subject of the next section.

3. **Local Zeta Integrals**

We now introduce the local theory of Garrett's integral representation of the triple product L-function, which can be used to give a definition of the local triple product L-factor and ϵ-factor. This local theory has been developed in detail in [PSR] and [I] and we shall summarize their results here.

The L-group of G_E is

$${}^LG_E = (GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) \rtimes W_F,$$

for a certain permutation action of the absolute Weil group W_F on the GL_2-factors (depending on the type of E). Hence, if Π_E is an irreducible representation of G_E, then Π_E has an associated L-parameter

$$\phi_{\Pi_E} : W_F \times SL_2(\mathbb{C}) \longrightarrow {}^LG_E.$$

As discussed in [PSR] and [I], ${}^LG_E$ has a natural 8-dimensional representation ρ, whose restriction to the identity component of ${}^LG_E$ is simply the (outer) tensor product of 3 copies of the standard representation of $GL_2(\mathbb{C})$. The action of W_F is then given by permutation of the components of this tensor product, according to the permutation action of W_F on $GL_2(\mathbb{C})^3$. Thus, one obtains a degree 8 L-factor and ϵ-factor of Artin type (i.e. on the Galois side). The local theory of Garrett's integral representation serves to give a definition of $L(s, \Pi_E, \rho)$ and $\epsilon(s, \Pi_E, \rho, \psi)$ on the representation theoretic side.

More precisely, for $f \in \Pi_E^\vee$ and a good section $\Phi_s \in I_P(s)$, one has a local zeta integral

$$Z_E(s, \Phi_s, f)$$

whose precise definition need not concern us here. The zeta integral converges when $Re(s)$ is sufficiently large and is equal to a rational function in q^{-s}. Moreover, with Φ_s ranging over all good sections of $I_P(s)$ and f ranging over all elements of $\Pi_E^\vee$, the family of local zeta integrals admits a GCD of the form $1/Q(q^{-s})$ with Q a monic polynomial. One defines this GCD to be the local triple product L-factor $L(s+1/2, \Pi_E, \rho)$. In particular, the ratio

$$Z_E^*(s, \Phi_s, f) = \frac{Z_E(s, \Phi_s, f)}{L(s+1/2, \Pi_E, \rho)}$$

is entire for any choices of data and, for each $s = s_0$, it is nonzero for some choice of data. As a consequence, we have a nonzero $GL_2(E)$-equivariant map

$$Z_E^*(0) : I_P(0) \otimes \Pi_E^\vee \longrightarrow \mathbb{C}$$

given by

$$\Phi \otimes f \mapsto Z_E^*(0, \Phi_0, f)$$

where we extend Φ to a flat section Φ_s of $I_P(s)$. Thus, $Z_E^*(0)$ defines a nonzero element of the space $\mathrm{Hom}_{G_E^0}(I_P(0), \Pi_E)$. In particular, by combining with Prop. 2.3, part (i) of Thm. 1.2 follows, i.e.

$$\dim \mathrm{Hom}_{G_E^0}(I_P(0), \Pi_E) = 1.$$

Now we come to the proof of Thm. 1.2(ii). For this, we need the local functional equation. Fix a non-trivial additive character ψ of F, which in turns fixes an additive Haar measure on F (as the one which is self dual with respect to ψ). There is a standard intertwining operator

$$M_\psi(s) : I_P(s) \longrightarrow I_P(-s),$$

whose definition requires the choice of a Haar measure on the unipotent radical of P (and hence the ultimate dependence on ψ). As in [PSR] and [I], one can define a normalized intertwining operator

$$M_\psi^*(s) : I_P(s) \longrightarrow I_P(-s)$$

such that

$$M_\psi^*(-s) \circ M_\psi^*(s) = id \quad \text{for all } s \in \mathbb{C}.$$

Note that the normalization of this intertwining operator has nothing to do with E: it is totally an issue on $GSp(6)$. Now we have:

Local Functional Equation

$$Z_E^*(-s, M_\psi^*(s)\Phi_s, f) = \epsilon(s+1/2, \Pi_E, \rho, \psi) \cdot \omega_{K/F}(-1) \cdot Z_E^*(s, \Phi_s, f).$$

Here K is the quadratic discriminant algebra of E. The function $\epsilon(s+1/2,\Pi_E,\rho,\psi)$ is of the form $A\cdot q^{Bs}$ and is defined to be the local triple product ϵ-factor. Though it depends on ψ as a function of s, its central value $\epsilon(1/2,\Pi_E,\rho,\psi)=\pm 1$ is independent of ψ.

The identity $M_\psi^*(-s)\circ M_\psi^*(s)=id$ shows that $M_\psi^*(s)$ is holomorphic at $s=0$ and also that

$$M_\psi^*(0)^2=1.$$

Since $I_P(0)$ is the direct sum of two irreducible summands by Prop. 2.2, $M_\psi^*(0)$ must act by ± 1 on each of these summands. We claim:

$$M_\psi^*(0) \text{ acts by } \begin{cases} 1 \text{ on } \Theta_{M_2(F)}(1); \\ \\ -1 \text{ on } \Theta_D(1). \end{cases}$$

To see this, suppose that $M_\psi^*(0)$ acts as $+1$ on $I_P(0)$. Then if we take $E=F\times F\times F$ and specialize to $s=0$ in the local functional equation, we would get

$$Z^*(0)=\epsilon(1/2,\pi_1\times\pi_2\times\pi_3,\psi)\cdot Z^*(0).$$

Since $Z^*(0)$ is a nonzero linear functional, this would mean that $\epsilon(1/2,\pi_1\times\pi_2\times\pi_3,\psi)=1$ for any infinite-dimensional representations π_i of GL_2. But this is not the case: if π_i is the Steinberg representation St for each i, then

$$\epsilon(1/2,St\times St\times St)=-1.$$

Similarly, if $M_\psi^*(0)$ acts as -1 on $I_P(0)$, then we would conclude that $\epsilon(1/2,\pi_1\times\pi_2\times\pi_3,\psi)=-1$ for any infinite dimensional representations π_i, which is evidently not the case. This shows that $M^*(0)$ must act by opposite signs on the two irreducible components of $I_P(0)$. It remains to argue that it acts by -1 on $\Theta_D(1)$. For this, one takes π_i to be the Steinberg representation again and note that

$$\mathrm{Hom}_{D^\times}(\pi_1^D\otimes\pi_2^D\otimes\pi_3^D,\mathbb{C})\neq 0$$

(since π_i^D is the trivial representation) so that

$$\mathrm{Hom}_{G_E^0}(\Theta_D(1),St\boxtimes St\boxtimes St)\neq 0.$$

Together with the local functional equation and the identity $\epsilon(1/2,St\times St\times St)=-1$, we see that $M_\psi^*(0)$ acts as -1 on $\Theta_D(1)$. Our claim is thus proven.

At this point, we may deduce from the local functional equation that:

$$Z_E^*(0) \text{ is non-trivial on } \Theta_B(1) \Longleftrightarrow M_\psi^*(0) \text{ acts by } \epsilon(1/2,\Pi,\rho,\psi)\cdot\omega_{K/F}(-1) \text{ on } \Theta_B(1)$$

Hence

$$Z_E^*(0) \text{ is non-trivial on } \Theta_{M_2(F)}(1) \Longrightarrow \epsilon(1/2,\Pi_E,\rho,\psi)\cdot\omega_{K/F}(-1)=1$$

whereas

$$Z_E^*(0) \text{ is non-trivial on } \Theta_D(1) \Longrightarrow \epsilon(1/2,\Pi_E,\rho,\psi)\cdot\omega_{K/F}(-1)=-1.$$

This proves (ii) of Theorem 1.2.

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