Abstract. In this paper, we establish rigidity and vanishing theorems for Dirac operators twisted by $E_8$ bundles.

Introduction

Let $X$ be a closed smooth connected manifold which admits a nontrivial $S^1$ action. Let $P$ be an elliptic differential operator on $X$ commuting with the $S^1$ action. Then the kernel and cokernel of $P$ are finite dimensional representation of $S^1$. The equivariant index of $P$ is the virtual character of $S^1$ defined by

\begin{equation}
\text{Ind}(g, P) = \text{tr}|_g \ker P - \text{tr}|_g \text{coker} P,
\end{equation}

for $g \in S^1$. We call that $P$ is rigid with respect to this circle action if $\text{Ind}(g, P)$ is independent of $g$.

It is well known that classical operators: the signature operator for oriented manifolds, the Dolbeault operator for almost complex manifolds and the Dirac operator for spin manifolds are rigid [2]. In [30], Witten considered the indices of Dirac-like operators on the free loop space $LX$. The Landweber-Stong-Ochanine elliptic genus ([20], [28]) is just the index of one of these operators. Witten conjectured that these elliptic operators should be rigid. See [19] for a brief early history of the subject. Witten’s conjecture were first proved by Taubes [29] and Bott-Taubes [4]. Hirzebruch [13] and Krichever [15] proved Witten’s conjecture for almost complex manifold case. Various aspects of mathematics are involved in these proofs. Taubes used analysis of Fredholm operators, Krichever used cobordism, Bott-Taubes and Hirzebruch used Lefschetz fixed point formula. In [22, 23], using modularity, Liu gives simple and unified proof as well as various generalizations of the Witten conjecture. Several new vanishing theorems are also found in [22, 23]. Liu-Ma [24, 25] and Liu-Ma-Zhang [26, 27] established family versions of rigidity and vanishing theorems.

In this paper, we study rigidity and vanishing properties for Dirac operators twisted by $E_8$ bundles. Let $X$ be an even dimensional closed spin manifold and $D$ the Dirac operator on $X$. Let $P$ be an (compact-) $E_8$ principal bundle over $X$. Let $W$ be the vector bundle over $X$ associated to the complex adjoint representation $\rho$ of $E_8$. The twisted Dirac operator $D^W$ plays a prominent role in string theory and $M$ theory. In [31], the index of such twisted operator is discovered as part of the phase of the $M$-theory.
action. In [8], the partition function in M-theory, involving the index theory of an \(E_8\) bundle, is compared with the partition function in type IIA string theory described by K-theory to test M-theory/Type IIA duality. In this paper, we are interested in the equivariant index of the operator \(D^W\) and establish rigidity and vanishing theorems for this operator.

More precisely, let \(X\) be a \(2k\) dimensional closed spin manifold, which admits a nontrivial \(S^1\) action. Let \(P\) be an (compact-)\(E_8\) principal bundle over \(X\) such that the \(S^1\) action on \(X\) can be lifted to \(P\) as a left action which commutes with the free action of \(E_8\) on \(P\). Let \(W\) be the complex vector bundle associated to the complex adjoint representation of \(E_8\) mentioned above. Then the \(S^1\) action on \(P\) naturally induces an action on \(W\) by

\[
g \cdot [s, v] = [g \cdot s, v],
\]

where \([s, v]\) with \(s \in P, v \in \mathbb{C}^{248}\), is the equivalent classes defining the elements in \(W\) by the equivalent relations \((s, v) \sim (s \cdot h, \rho(h^{-1}) \cdot v)\) for \(h \in E_8\). Let \(X^{S^1}\) be the fixed point manifold and \(\pi\) be the projection from \(X^{S^1}\) to a point \(pt\). Let \(u\) be a fixed generator of \(H^2(\text{BS}^1, \mathbb{Z})\). We have the following theorem:

**Theorem 0.1.** Assume the action only has isolated fixed points and the restriction of the equivariant characteristic class \(\frac{1}{30} c_2(W)_{S^1} - p_1(TX)_{S^1}\) to \(X^{S^1}\) is equal to \(n \cdot \pi^* u^2\) for some integer \(n\).

(i) If \(n < 0\), then \(\text{Ind}(g, D^W)\) is independent of \(g\) and equal to \(-\text{Ind}(D^{TcX})\), minus the index of the Rarita-Schwinger operator. In particular, one has \(\text{Ind} D^W = -\text{Ind} D^{TcX}\) and when \(k\) is odd, i.e. \(\dim X \equiv 2 \text{ (mod 4)}\), one has \(\text{Ind}(g, D^W) \equiv 0\).

(ii) If \(n = 0\), then \(\text{Ind}(g, D^W)\) is independent of \(g\). Moreover, when \(k\) is odd, one has \(\text{Ind}(g, D^W) \equiv 0\).

(iii) If \(n = 2\) and \(k\) is odd, then \(\text{Ind}(g, D^W) \equiv 0\).

Actually we have established rigidity and vanishing results in more general settings concerning the twisted spin\(^c\) Dirac operators. See Theorem 2.1 and Theorem 2.2 for details. The above theorem is a corollary of Theorem 2.1. We prove our theorems by studying the modularity of Lefschetz numbers of certain elliptic operators involving the basic representation of the affine Kac-Moody algebra of \(E_8\). In the rest of the paper, we will first briefly review the Jacobi theta functions and the basic representation for the affine \(E_8\) by following [16] (see also [17]) as the preliminary knowledge in Section 1 and then state our theorems as well as give their proofs in Section 2.

1. **Preliminaries**

1.1. **Jacobi theta functions.** The four Jacobi theta-functions are defined as follows (cf. [5]),

\[
\theta(z, \tau) = 2q^{1/8} \sin(\pi z) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi \sqrt{-1} z} q^j)(1 - e^{-2\pi \sqrt{-1} z} q^j)],
\]
(1.2) \( \theta_1(z, \tau) = 2q^{1/8} \cos(\pi z) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\tau} \sqrt{-1}zq^j)\left(1 + e^{-2\tau} \sqrt{-1}zq^j\right)] \),

(1.3) \( \theta_2(z, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\tau} \sqrt{-1}z^{-1}q^j)\left(1 - e^{-2\tau} \sqrt{-1}z^{-1}q^j\right)] \),

(1.4) \( \theta_3(z, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\tau} \sqrt{-1}z^{-1}q^j)\left(1 + e^{-2\tau} \sqrt{-1}z^{-1}q^j\right)] \),

where \( q = e^{2\tau \sqrt{-1}}, \tau \in \mathbb{H} \), the upper half plane.

They are all holomorphic functions for \((z, \tau) \in \mathbb{C} \times \mathbb{H}\), where \( \mathbb{C} \) is the complex plane.

Let \( \theta'(0, \tau) = \frac{\partial}{\partial z} \theta(z, \tau)|_{z=0} \). One has the following Jacobi identity (c.f. [5]),

(1.5) \( \theta'(0, \tau) = \pi \theta_1(0, \tau)\theta_2(0, \tau)\theta_3(0, \tau) \).

Let

\[ SL(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \middle| a_1, a_2, a_3, a_4 \in \mathbb{Z}, \ a_1a_4 - a_2a_3 = 1 \right\} \]

be the modular group. Let \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) be the two generators of \( SL(2, \mathbb{Z}) \). Their actions on \( \mathbb{H} \) are given by

\[ S : \tau \mapsto -\frac{1}{\tau}, \ T : \tau \mapsto \tau + 1. \]

The actions on theta-functions by \( S \) and \( T \) are given by the following transformation formulas (c.f. [5]),

(1.6) \( \theta(z, \tau + 1) = e^{2\pi \tau \sqrt{-1}} \theta(z, \tau), \ \theta(z, -1/\tau) = \frac{1}{\sqrt{-1}} \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi \tau \sqrt{-1}z^2} \theta(z, \tau); \)

(1.7) \( \theta_1(z, \tau + 1) = e^{2\pi \tau \sqrt{-1}} \theta_1(z, \tau), \ \theta_1(z, -1/\tau) = \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi \tau \sqrt{-1}z^2} \theta_2(z, \tau); \)

(1.8) \( \theta_2(z, \tau + 1) = \theta_3(z, \tau), \ \theta_2(z, -1/\tau) = \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi \tau \sqrt{-1}z^2} \theta_3(z, \tau); \)

(1.9) \( \theta_3(z, \tau + 1) = \theta_2(z, \tau), \ \theta_3(z, -1/\tau) = \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} e^{\pi \tau \sqrt{-1}z^2} \theta_2(z, \tau). \)

One also has the following formulas about how the theta functions vary along the lattice \( \Gamma = \{a + b\tau|a, b \in \mathbb{Z}\} \) (c.f. [5]),

(1.10) \( \theta(z + a, \tau) = (-1)^a \theta(z, \tau), \ \theta(z + b\tau, \tau) = (-1)^b e^{-2\pi \sqrt{-1}b\tau - \pi \sqrt{-1}b^2 \tau} \theta(z, \tau); \)
(1.11) \[ \theta_1(z + a, \tau) = (-1)^a \theta_1(z, \tau), \quad \theta_1(z + b\tau, \tau) = e^{-2\pi \sqrt{-1}b^2 \tau} \theta_1(z, \tau); \]

(1.12) \[ \theta_2(z + a, \tau) = \theta_2(z, \tau), \quad \theta_2(z + b\tau, \tau) = (-1)^b e^{-2\pi \sqrt{-1}b^2 \tau} \theta_2(z, \tau); \]

(1.13) \[ \theta_3(z + a, \tau) = \theta_3(z, \tau), \quad \theta_3(z + b\tau, \tau) = e^{-2\pi \sqrt{-1}b^2 \tau} \theta_3(z, \tau). \]

1.2. The basic representation for the affine \( E_8 \). In this subsection we briefly review the basic representation for the affine \( E_8 \) following [16] (see also [17]).

Let \( \mathfrak{g} \) be the (complex) Lie algebra of \( E_8 \). Let \( \langle , \rangle \) be the Killing form on \( \mathfrak{g} \). Let \( \tilde{\mathfrak{g}} \) be the affine Lie algebra corresponding to \( \mathfrak{g} \) defined by

\[
\tilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c,
\]

with bracket

\[
[P(t) \otimes x + \lambda c, Q(t) \otimes y + \mu c] = P(t)Q(t) \otimes [x, y] + \langle x, y \rangle \text{ Res}_{t=0} \left( \frac{dP(t)}{dt}Q(t) \right) c.
\]

Let \( \tilde{\mathfrak{g}} \) be the affine Kac-Moody algebra obtained from \( \mathfrak{g} \) by adding a derivation \( \frac{dt}{t^2} \) which operates on \( \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \) in an obvious way and sends \( c \) to 0.

The basic representation \( V(\Lambda_0) \) is the \( \tilde{\mathfrak{g}} \)-module defined by the property that there is a nonzero vector \( v_0 \) (highest weight vector) in \( V(\Lambda_0) \) such that \( cv_0 = v_0, (\mathbb{C}[t] \otimes \mathfrak{g} \oplus \mathbb{C} \frac{dt}{t})v_0 = 0 \). Setting \( V_i := \{ v \in V(\Lambda_0) | t \frac{dt}{t} v = -iv \} \) gives a \( \mathbb{Z}_4 \)-gradation by finite dimensional subspaces. Since \( [\mathfrak{g}, t \frac{dt}{t}] = 0 \), each \( V_i \) is a representation of \( \mathfrak{g} \). Moreover, \( V_1 \) is the adjoint representation of \( E_8 \).

Fix a basis \( \{ Z_i \}_{i=1}^8 \) for the Cartan subalgebra. The character of the basic representation is given by

(1.14) \[ \text{ch}(z_1, z_2, \ldots, z_8, \tau) := \sum_{i=0}^{\infty} \text{ch} V_i (z_1, z_2, \ldots, z_8) q^i = \varphi(\tau)^{-8} \Theta_{\mathfrak{g}}(z_1, z_2, \ldots, z_8, \tau), \]

where \( \varphi(\tau) = \prod_{n=1}^{\infty} (1 - q^n) \) so that \( \eta(\tau) = q^{1/24} \varphi(\tau) \) is the Dedekind \( \eta \) function; \( \Theta_{\mathfrak{g}}(z_1, z_2, \ldots, z_8, \tau) \) is the theta function defined on the root lattice \( \mathfrak{g} \) by

(1.15) \[ \Theta_{\mathfrak{g}}(z_1, z_2, \ldots, z_8, \tau) = \sum_{\gamma \in \Gamma} q^{\langle \gamma \rangle^2/2} e^{2\pi \sqrt{-1} \tau \langle \gamma, \sum_{i=1}^{8} z_i Z_i \rangle}. \]

It is proved in [10] (cf. [11]) that there is a basis for the \( E_8 \) root lattice such that

(1.16) \[ \Theta_{\mathfrak{g}}(z_1, \ldots, z_8, \tau) = \frac{1}{2} \left( \prod_{l=1}^{8} \theta(z_l, \tau) + \prod_{l=1}^{8} \theta_1(z_l, \tau) + \prod_{l=1}^{8} \theta_2(z_l, \tau) + \prod_{l=1}^{8} \theta_3(z_l, \tau) \right). \]
2. $E_8$ Bundles and Rigidity

In this section we prove two rigidity and vanishing theorems for spin$^c$ manifolds with $E_8$ principal bundles. Theorem 0.1 is deduced from the first one (Theorem 2.1).

Let $X$ be a $2k$ dimensional closed spin$^c$ manifold, which admits a nontrivial $S^1$ action that preserves the spin$^c$ structure. Let $L$ be the complex line bundle associated with the spin$^c$ structure of $X$. It’s the associated line bundle of the $U(1)$-bundle $Q/spin(2k) \to Q/spin^c(2k) \cong X$, where $Q$ is the spin$^c(2k)$ principal bundle over $X$ determined by the spin$^c$ structure. We denote the first equivariant Chern class of $L$ by $c_1(X)_{S^1}$. Let $P$ be an $E_8$ principal bundle over $X$ such that the $S^1$ action on $P$ can be lifted to $P$ as a left action which commutes with the free action of $E_8$ on $P$. Let $W$ be the vector bundle associated to the complex adjoint representation of $E_8$ mentioned above. Then the $S^1$ action on $P$ naturally induces an action on $W$ as described in the introduction.

Let $g^{TX}$ be a Riemannian metric on $X$. Let $\nabla^{TX}$ be the Levi-Civita connection associated to $g^{TX}$. Denote the complexification of $TX$ by $T_C X$. Let $g^{T_C X}$ and $\nabla^{T_C X}$ be the induced Hermitian metric and Hermitian connection on $T_C X$. Let $h^L$ be a Hermitian metric on $L$ and $\nabla^L$ be a Hermitian connection. Let $\overline{L}$ be the complex conjugate of $L$ with the induced Hermitian metric and connection. Assume that the $S^1$ action on $X$ preserves the metrics and connections involved. Let $S_c(TX) = S_{c,+}(TX) \oplus S_{c,-}(TX)$ denote the bundle of spinors associated to the spin$^c$ structure, $(TX, g^{TX})$ and $(L, h^L)$. Then $S_c(TX)$ carries induced Hermitian metric and connection preserving the above $\mathbb{Z}_2$-grading. Let $D_{c,\pm} : \Gamma(S_{c,\pm}(TX)) \to \Gamma(S_{c,\mp}(TX))$ denote the induced spin$^c$ Dirac operators (cf. [21]). If $V$ is an equivariant complex vector bundle over $X$ with equivariant Hermitian metric $h^V$ and Hermitian connection $\nabla^V$, let $D^{V}_{c,\pm} : \Gamma(S_{c,\pm}(TX) \otimes V) \to \Gamma(S_{c,\mp}(TX) \otimes V)$ denote the induced twisted spin$^c$ Dirac operators.

**Theorem 2.1.** Assume the action only has isolated fixed points and the restriction of the equivariant characteristic class

$$\frac{1}{30} c_2(W)_{S^1} + 3 c_1(X)_{S^1}^2 - p_1(TX)_{S^1}$$

to $X^{S^1}$ is equal to $n \cdot \pi^* u^2$ for some integer $n$.

(i) If $n < 0$, then

$$\text{Ind}(g, D_{c,+}^{(1+\overline{L}) \otimes W}) + \text{Ind}(g, D_{c,+}^{(1+\overline{L}) \otimes (T_C X - (L^2 + \overline{L}^2) + (L + \overline{L})))}) \equiv 0.$$ 

In particular,

$$\text{Ind}D_{c,+}^{(1+\overline{L}) \otimes W} + \text{Ind}D_{c,+}^{(1+\overline{L}) \otimes (T_C X - (L^2 + \overline{L}^2) + (L + \overline{L})))} = 0.$$

(ii) If $n = 0$, then

$$\text{Ind}(g, D_{c,+}^{(1+\overline{L}) \otimes W}) + \text{Ind}(g, D_{c,+}^{(1+\overline{L}) \otimes (T_C X - (L^2 + \overline{L}^2) + (L + \overline{L})))})$$
is independent of \( g \). Moreover, when \( k \) is odd, one has

\[
\text{Ind}(g, D_{c,+}^{(1+T) \otimes W}) + \text{Ind}(g, D_{c,+}^{(1+\tilde{T}) \otimes (\mathcal{T}_c X - (L^2 + \tilde{T}^2) + (L + \tilde{T}))}) \equiv 0.
\]

(iii) If \( n = 2 \) and \( k \) is odd, then

\[
\text{Ind}(g, D_{c,+}^{(1+T) \otimes W}) + \text{Ind}(g, D_{c,+}^{(1+\tilde{T}) \otimes (\mathcal{T}_c X - (L^2 + \tilde{T}^2) + (L + \tilde{T}))}) \equiv 0.
\]

**Proof.** Let \( g = e^{2\pi \sqrt{-1} t} \in S^1 \) be the generator of the action group. Let \( X^{S^1} = \{ p \} \) be the set of fixed points. Let \( TX|_p = E_1 \oplus \cdots \oplus E_k \) be the decomposition of the tangent bundle into the \( S^1 \)-invariant 2-planes. Assume that \( g \) acts on \( E_j \) by \( e^{2\pi \sqrt{-1} \alpha_j t} \), \( \alpha_j \in \mathbb{Z} \). Assume \( g \) acts on \( L|_p \) by \( e^{2\pi \sqrt{-1} ct} \), \( c \in \mathbb{Z} \). Clearly,

\[
p_1(TM|_p)_{S^1} = (2\pi \sqrt{-1})^2 \sum_{j=1}^{k} \alpha_j^2 t^2, \quad c_1(L|_p)_{S^1} = 2\pi \sqrt{-1} ct.
\]

Denote \( L \oplus L \) by \( L_C \). If \( E \) is a complex vector bundle over \( X \), set \( \tilde{E} = E - C^{\text{tr}(E)} \in K(X) \).

Let \( \Theta(X, L, \tau) \) be the virtual complex vector bundle over \( X \) defined by

\[
\Theta(X, L, \tau) := \left( \bigotimes_{m=1}^{\infty} S_{q^m}(\tilde{T}_C X) \right) \otimes \left( \bigotimes_{u=1}^{\infty} \Lambda_q^u(\tilde{L}_C) \right)
\]

\[
\otimes \left( \bigotimes_{v=1}^{\infty} \Lambda_{q^{-v-1/2}}(\tilde{L}_C) \right) \otimes \left( \bigotimes_{w=1}^{\infty} \Lambda_{q^{-w-1/2}}(\tilde{L}_C) \right),
\]

Let \( W_i \) (\( i = 0, 1, \cdots \)) be the associated bundles \( P \times_{\mu_i} V_i \), where \( V_i \)'s are the representations of \( E_8 \) as in §1.2. Then \( W = W_1 \).

Consider the twisted operator

\[
D_{c,+}^{(1+\tilde{T}) \otimes \Theta(X, L, \tau) \otimes (\varphi^8(\tau) \sum_{i=0}^{\infty} W_i q^i)}.
\]

Expanding \( q \)-series, we have

\[
\Theta(X, L, \tau) \otimes (\varphi^8(\tau) \sum_{i=0}^{\infty} W_i q^i)
\]

\[
= (1 + (T \mathcal{C} X - 2k)q + O(q^2)) \otimes (1 + \tilde{L}_C q + O(q^2))
\]

\[
\otimes (1 - \tilde{L}_C q^{1/2} - 2\tilde{L}_C q + O(q^{3/2})) \otimes (1 + \tilde{L}_C q^{1/2} - 2\tilde{L}_C q + O(q^{3/2}))
\]

\[
\otimes (1 - 8q + O(q^2)) \otimes (1 + W q + O(q^2))
\]

\[
= 1 + (W - 8 + T \mathcal{C} X - 2k - 3\tilde{L}_C - \tilde{L}_C \otimes \tilde{L}_C)q + O(q^2).
\]

It’s not hard to see that \( \tilde{L}_C \otimes \tilde{L}_C = L^2 + \tilde{T}^2 - 4(L + \tilde{T}) + 6 \). So

\[
D_{c,+}^{(1+\tilde{T}) \otimes \Theta(M, L, \tau) \otimes (\varphi^8(\tau) \sum_{i=0}^{\infty} W_i q^i)}
\]

\[
= D_{c,+}^{(1+\tilde{T})} + D_{c,+}^{(1+T)(W + T \mathcal{C} X - (L^2 + \tilde{T}^2) + (L + \tilde{T}) - 8 - 2k) q + O(q^2)}.
\]
By the Atiyah-Bott-Segal-Singer Letschetz fixed point formula, for the twisted operator $D^{(1+L)\otimes \Theta(X,L,\tau)\otimes (\varphi^8(\tau)\sum_{i=0}^{\infty} W_i q^i)}$, the equivariant index

$$I(t, \tau) = 2 \sum_p \left\{ \frac{1}{(2\pi \sqrt{-1})^k} \prod_{j=1}^k \theta'(0, \tau) \theta_1(ct, \tau) \theta_2(ct, \tau) \theta_3(ct, \tau) \theta_j(0, \tau) \theta(\alpha_j t, \tau) \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau) \right\} \cdot \varphi^8(\tau) \cdot \left( \sum_{i=0}^{\infty} \text{ch}(W_i|_p) q^i \right).$$

(2.5)

On the fixed point $p$, fixing an element $s \in P|_p$, one can define a map $f_s : S^1 \rightarrow E_8$ by $g \cdot s = s \cdot f_s(g)$. It’s not hard to check that $f_s$ is a group homomorphism. Moreover, for $h \in E_8$, we have

$$g \cdot (s \cdot h) = (g \cdot s) \cdot h = s \cdot f_s(g) \cdot h = (s \cdot h) \cdot (h^{-1} f_s(g) h).$$

As all the maximal tori in $E_8$ are conjugate, then one may choose $s \in P|_p$ such that $f_s$ maps $S^1$ into the maximal torus $t$ that corresponds to the Cartan subalgebra such that the theta function $\Theta(z_1, \cdots, z_8, \tau)$ appears as in (1.16). For any unitary representation $\rho : E_8 \rightarrow U(N)$, let $\mathfrak{T}$ be a maximal torus of $U(N)$ that contains $\rho(t)$. Let

$$\widehat{\mathfrak{T}} \xrightarrow{\widehat{\rho}} \widehat{\mathfrak{f}} \xrightarrow{\widehat{\varphi}} \widehat{S^1}$$

be the induced maps on the character groups. Assume $\widehat{f}_s(z_i) = \beta_i$. Let $\{x_i\}$ be basis for $\widehat{\mathfrak{T}}$. By definition,

$$(\text{ch}\rho)(z_1, z_2, \cdots, z_8) = \sum_{i=1}^{N} e^{\widehat{\beta}(x_i)},$$

and therefore

$$(\text{ch}\rho)(\beta_1 t, \beta_2 t, \cdots, \beta_8 t)$$

$$= \widehat{f}_s((\text{ch}\rho)(z_1, z_2, \cdots, z_8))$$

$$= \sum_{i=1}^{N} e(\widehat{f}_s \circ \widehat{\rho})(x_i)$$

$$= \text{ch}((P \times_{\rho} \mathbb{C}^N)|_p)_{S^1}.$$  

So for each $i$, we have $\text{ch}(W_i|_p)_{S^1} = (\text{ch}V_i)(\beta_1 t, \beta_2 t, \cdots, \beta_8 t)$. Then by (1.14) and (1.16), we have

$$\varphi^8(\tau) \cdot \left( \sum_{i=0}^{\infty} \text{ch}(W_i|_p) q^i \right)$$

$$= \frac{1}{2} \left( \prod_{l=1}^{8} \theta(\beta_l t, \tau) + \prod_{l=1}^{8} \theta_1(\beta_l t, \tau) + \prod_{l=1}^{8} \theta_2(\beta_l t, \tau) + \prod_{l=1}^{8} \theta_3(\beta_l t, \tau) \right).$$  

(2.6)
Comparing both sides of (2.6), we can see by direct computation that

\[(2.7)\quad 30 \cdot (2\pi \sqrt{-1})^2 \sum_{l=1}^{8} \beta_l^2 t^2 = c_2(W|_p)_{S^1}.\]

By (2.5) and (2.6), we have

\[(2.8)\quad I(t, \tau) = \sum_p \left\{ \frac{1}{(2\pi \sqrt{-1})^k} \prod_{j=1}^{k} \theta'(0, \tau) \theta_1(\alpha_j t, \tau) \theta_2(0, \tau) \theta_3(0, \tau) \cdot \left( \prod_{l=1}^{8} \theta(\beta_l t, \tau) + \prod_{l=1}^{8} \theta_1(\beta_l t, \tau) + \prod_{l=1}^{8} \theta_2(\beta_l t, \tau) + \prod_{l=1}^{8} \theta_3(\beta_l t, \tau) \right) \right\}.\]

From the transformation laws of theta functions (1.10)-(1.13), for \(a, b \in 2\mathbb{Z}\), it’s not hard to see that

\[I(t + a\tau + b, \tau) = e^{-\pi \sqrt{-1} n (b^2 \tau + 2br)} I(t, \tau).\]

Since when restricted to fixed points, \(\frac{1}{30} c_2(W)_{S^1} + 3c_1(L)^2_{S^1} - p_1(T_X)_{S^1}\) is equal to \(n \cdot \pi^* u^2\), then for each fixed point, from (2.1) and (2.7) we have

\[\sum_{l=1}^{8} \beta_l^2 + 3c_2 - \sum_{j=1}^{k} \alpha_j^2 = n\]

and therefore

\[(2.9)\quad I(t + a\tau + b, \tau) = e^{-\pi \sqrt{-1} n (b^2 \tau + 2br)} I(t, \tau).\]

It’s easy to deduce from (1.6) that

\[\theta'(0, \tau + 1) = e^{\frac{\pi \sqrt{-1}}{\tau}} \theta'(0, \tau), \quad \theta'(0, -1/\tau) = \frac{1}{\sqrt{-1}} \left( \frac{\tau}{\sqrt{-1}} \right)^{1/2} \tau \theta'(0, \tau).\]

Using the above two formulas and the transformation laws of theta functions (1.6)-(1.9), we have

\[(2.10)\quad I(t, \tau + 1) = I(t, \tau)\]

and

\[(2.11)\quad I\left( \frac{t}{\tau}, -\frac{1}{\tau} \right) = \tau^{k+4} e^{\frac{\pi \sqrt{-1}}{\tau} n u^2} I(t, \tau).\]

(2.9)-(2.11) tell us that \(I(t, \tau)\) obeys the transformation laws that a Jacobi form (see [9]) should satisfy.

Next we shall prove that \(I(t, \tau)\) is holomorphic for \((t, \tau) \in \mathbb{C} \times \mathbb{H}\). First, we have the following lemma:

**Lemma 2.1.** \(I(t, \tau)\) is holomorphic for \((t, \tau) \in \mathbb{R} \times \mathbb{H}\).
The proof of this lemma is almost verbatimly same as the proof of Lemma 1.3 in [22]. We shall prove that \( I(t, \tau) \) is actually holomorphic on \( \mathbb{C} \times \mathbb{H} \). The possible polar divisor of \( I(t, \tau) \) can be written in the form \( t = \frac{m(c\tau + d)}{l} \) for integers \( m, l, c, d \) with \( (c, d) = 1 \). Assume \( \frac{m(c\tau + d)}{l} \) is a pole for \( I(t, \tau) \). Find integers \( a, b \) such that \( ad - bc = 1 \). Consider the function \( I\left(\frac{t}{c\tau + d} - a, \frac{d\tau - b}{c\tau + d} \right) \).

By (2.10) and (2.11), it’s easy to see that

\[
(2.12) \quad I\left(\frac{t}{c\tau + d} - a, \frac{d\tau - b}{c\tau + d} \right) = f(t, \tau) \cdot I(t, \tau),
\]

where \( f(t, \tau) \) is an entire function of \( t \) for every \( \tau \in \mathbb{H} \). If \( \tau' = \frac{a\tau + b}{c\tau + d} \), then \( \tau = \frac{d\tau' - b}{c\tau' + a} \) and \( \frac{m(c\tau' - b)}{l} \) is a pole for the function \( I\left(\frac{t}{c\tau' + d} - a, \frac{d\tau' - b}{c\tau' + d} \right) \).

However by (2.12), we have

\[
I\left(\frac{m(c\tau' - b)}{l}, \frac{d\tau' - b}{-c\tau' + a} \right) = I\left(\frac{m}{l}, \frac{d\tau' - b}{-c\tau' + a} \right) = f\left(\frac{m}{l}, \tau' \right) \cdot I\left(\frac{m}{l}, \tau' \right).
\]

As \( \frac{m}{l} \) is real, by Lemma 2.1, we get a contradiction. Therefore \( I(t, \tau) \) is holomorphic for \( (t, \tau) \in \mathbb{C} \times \mathbb{H} \).

Combining the transformation formulas (2.9)-(2.11) and the holomorphicity of \( I(t, \tau) \) on \( \mathbb{C} \times \mathbb{H} \), we see that \( I(t, \tau) \) is a weak Jacobi form of index \( \frac{n^2}{2} \) and weight \( k + 4 \) over \( (2\mathbb{Z})^2 \rtimes SL(2, \mathbb{Z}) \). Here by weak Jacobi form, we don’t require the regularity condition at the cusp but only require that at the cusp \( q \) appears with nonnegative powers only. We refer to [9] for the precise definition of the Jacobi forms.

If \( n = 0 \), by (2.9), we see that \( I(t, \tau) \) is holomorphic on the torus \( \mathbb{C}/2\mathbb{Z} + 2\mathbb{Z}\tau \)

and therefore must be independent of \( t \). So, by (2.4), we see that

\[
\text{Ind}(g, D_{c,+}^{(1+L)}),
\]

\[
\text{Ind}(g, D_{c,+}^{(1+L)} \otimes (W + T_{c,X} - (L^2 + L^2) + (L + L) - 8 - 2k))
\]

are both independent of \( g \). So

\[
\text{Ind}(g, D_{c,+}^{(1+L)} \otimes W) + \text{Ind}(g, D_{c,+}^{(1+L)} \otimes (T_{c,X} - (L^2 + L^2) + (L + L)))
\]

must be independent of \( g \). The index density of the operator

\[
D_{c,+}^{(1+L)} \otimes W + D_{c,+}^{(1+L)} \otimes (T_{c,X} - (L^2 + L^2) + (L + L))
\]
involves the characteristic forms
\[ \hat{\Delta}(TM), e^{-c_1(L)/2}(1 + e^{-c_1(L)}), \text{ch}(W), \text{ch}(T_C M), \text{ch}(L + \mathcal{L}), \text{ch}(L^2 + L^2), \]
which are all of degree 4 (noting that \( W \) is the complexification of the real adjoint representation of compact \( E_8 \)). Therefore by the Atiyah-Singer index theorem, \( \text{Ind}(1+L)^{c +} \otimes \text{ch}(W) + \text{ch}(T_C X -(L^2 + L^2) + (L+\mathcal{L})) \) (i.e. when \( g = id \)) must be 0 when the dimension of the manifold is not divisible by 4.

So when \( k \) is odd,
\[ \text{Ind}(1+L)^{c +} \otimes \text{ch}(W) + \text{ch}(T_C X -(L^2 + L^2) + (L+\mathcal{L})) \equiv 0. \]
This finishes the proof of part (ii).

If \( n \neq 0 \), i.e in the case of nonzero anomaly, we need the following two lemmas.

**Lemma 2.2** (Theorem 1.2 in [9]). Let \( I \) be a weak Jacobi form of index \( m \) and weight \( h \). Then for fixed \( \tau \), if not identically 0, \( I \) has exactly \( 2m \) zeros in any fundamental domain for the action of the lattice on \( \mathbb{C} \).

**Lemma 2.3** (Theorem 2.2 in [9]). Let \( I \) be a weak Jacobi form of index \( m \) and weight \( h \). If \( m = 1 \) and \( h \) is odd, then \( I \) is identically 0.

We would like to point that Lemma 2.2 and Lemma 2.3 are stated in [9] for Jacobi forms. However, as in the proofs of them no regularity condition at the cusp are used, we state them here for weak Jacobi forms. See [9] for details.

If \( n < 0 \), then by Lemma 2.2, \( I(t, \tau) \equiv 0 \), therefore
\[ \text{Ind}(D^{(1+\mathcal{L}) \otimes W} + \text{Ind}(D^{(1+\mathcal{L}) \otimes (T_C X -(L^2 + L^2) + (L+\mathcal{L}))}) \equiv 0. \]
So part (i) follows.

If \( n = 2 \), as the the weight of \( I(t, \tau) \) is \( k + 4 \), so part (iii) similarly follows clearly from Lemma 2.3.

\[ \square \]

Theorem 0.1 can be easily deduced from Theorem 2.1 as follows.

**Proof of Theorem 0.1:** When \( X \) is a spin manifold, \( L \) is trivial and \( D_{c,+} = D \). By the Atiyah-Hirzebruch vanishing theorem ([2]), we have \( \text{Ind}(g, D) \equiv 0 \). Moreover by the Witten rigidity theorem ([29, 4, 22], the operator \( D^{T_C X} \) is rigid. i.e. \( \text{Ind}(g, D^{T_C X}) \equiv \text{Ind}D^{T_C X} \). Also note that \( \text{Ind}D^{T_C X} \) equals to 0 when \( k \) is odd. Then the three parts in Theorem 0.1 easily follow from the corresponding three parts in Theorem 2.1. \( \square \)

For Spin\( ^c \) manifolds, we have rigidity and vanishing theorem for another type of twisted operators.
**Theorem 2.2.** Assume the action only has isolated fixed points and the restriction of the equivariant characteristic class

\[ \frac{1}{30} c_2(W)_{S^1} + c_1(X)_{S^1}^2 - p_1(TX)_{S^1} \]

to \( X_{S^1} \) is equal to \( n \cdot \pi^* u^2 \) for some integer \( n \).

(i) If \( n < 0 \), then

\[ \text{Ind}(g, D^{(1-L)\otimes W}_{c,+}) + \text{Ind}(g, D^{(1-L)\otimes (TCX-(L+\overline{L}))}_{c,+}) \equiv 0. \]

In particular,

\[ \text{Ind} D^{(1-L)\otimes W}_{c,+} + \text{Ind} D^{(1-L)\otimes (TCX-(L+\overline{L}))}_{c,+} = 0. \]

(ii) If \( n = 0 \), then

\[ \text{Ind}(g, D^{(1-L)\otimes W}_{c,+}) + \text{Ind}(g, D^{(1-L)\otimes (TCX-(L+\overline{L}))}_{c,+}) \]

is independent of \( g \). Moreover, when \( k \) is even, one has

\[ \text{Ind}(g, D^{(1-L)\otimes W}_{c,+}) + \text{Ind}(g, D^{(1-L)\otimes (TCX-(L+\overline{L}))}_{c,+}) \equiv 0. \]

(iii) If \( n = 2 \) and \( k \) is even, then

\[ \text{Ind}(g, D^{(1-L)\otimes W}_{c,+}) + \text{Ind}(g, D^{(1-L)\otimes (TCX-(L+\overline{L}))}_{c,+}) \equiv 0. \]

**Proof.** We will use the same notations as in the proof of Theorem 2.1.

Let \( \Theta^*(X, L,\tau) \) be the virtual complex vector bundles over \( X \) defined by

\[ \Theta^*(X, L,\tau) := \left( \bigotimes_{m=1}^{\infty} S_{q^m}(\widetilde{T\text{C}X}) \right) \otimes \left( \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\widetilde{\text{L}C}) \right). \]

Consider the twisted operator

\[ D^{(1-L)\otimes \Theta^*(X,L,\tau)\otimes (\varphi^8(\tau)\sum_{i=0}^{\infty} W_i q^i)}. \]

(2.13)

Expanding \( q \)-series, we have

\[ \Theta^*(X, L,\tau) \otimes (\varphi^8(\tau) \sum_{i=0}^{\infty} W_i q^i) \]

(2.14)

\[ = (1 + (T\text{C}X - 2k)q + O(q^2)) \otimes (1 - \widetilde{\text{L}C}q + O(q^2)) \]

\[ \otimes (1 - 8q + O(q^2)) \otimes (1 + Wq + O(q^2)) \]

\[ = 1 + (W + T\text{C}X - (L + \overline{L}) - 2k - 6)q + O(q^2). \]

So

\[ D^{(1-L)\otimes \Theta^*(X,L,\tau)\otimes (\varphi^8(\tau)\sum_{i=0}^{\infty} W_i q^i)} \]

(2.15)

\[ = D^{(1-L)} + D^{(1-L)\otimes (W + T\text{C}X - (L + \overline{L}) - 2k - 6)q + O(q^2)}. \]
By the Atiyah-Bott-Segal-Singer Letzech fixed point formula, for this twisted operator $D^{(1-L)\otimes \Theta^*(X,L,\tau)\otimes (\varphi^8(\tau) \sum_{i=0}^{\infty} W_i q^i)}_{c,+}$, the equivariant index

\[(2.16)\]

\[J(t, \tau) = 2 \sum_p \left\{ \frac{1}{(2\pi \sqrt{-1})^k} \prod_{j=1}^{k} \frac{\theta'(0, \tau)}{\theta(\alpha_j t, \tau)} \frac{\theta(ct, \tau)}{\theta(0, \tau)\theta_2(0, \tau)\theta_3(0, \tau)} \right. \]

\[\cdot \varphi^8(\tau) \cdot \left( \sum_{i=0}^{\infty} \text{ch}(W_i|_p) q^{i} \right) \]

\[= \sum_p \left\{ \frac{1}{(2\pi \sqrt{-1})^k} \prod_{j=1}^{k} \frac{\theta'(0, \tau)}{\theta(\alpha_j t, \tau)} \frac{\theta(ct, \tau)}{\theta(0, \tau)\theta_2(0, \tau)\theta_3(0, \tau)} \right. \]

\[\cdot \left( \prod_{l=1}^{8} \theta(\beta_l t, \tau) + \prod_{l=1}^{8} \theta_1(\beta_l t, \tau) + \prod_{l=1}^{8} \theta_2(\beta_l t, \tau) + \prod_{l=1}^{8} \theta_3(\beta_l t, \tau) \right) \right\}.

As when restricted to fixed points, $1/24 c_2(W)_{S^1} + c_1(L)_{S^1}^2 \equiv p_1(TX)_{S^1}$ is equal to $n \cdot \pi^* u^2$, then for each fixed point, we have

\[\sum_{l=1}^{8} \beta_l^2 + c^2 = \sum_{j=1}^{k} \alpha_j^2 = n.

Therefore, similar to (2.9), one can show that for $a, b \in 2\mathbb{Z}$

\[(2.17)\]

\[J(t + a\tau + b, \tau) = e^{-\pi \sqrt{-1} n (b^2 \tau + 2br)} J(t, \tau).

One can also show that

\[(2.18)\]

\[J(t, \tau + 1) = J(t, \tau)

and

\[(2.19)\]

\[J\left( \frac{t}{\tau} - \frac{1}{\tau} \right) = \tau^{k+3} e^{\frac{\pi \sqrt{-1} n t^2}{\tau}} J(t, \tau).

So similar to $I(t, \tau)$ in the proof of Theorem 2.1, combing Lemma 2.1 and the above transformation laws, we can prove that $J(t, \tau)$ is a weak Jacobi form of index $\frac{n}{2}$ and weight $k + 3$ over $(2\mathbb{Z})^2 \rtimes SL(2, \mathbb{Z})$.

Then one can prove the three parts of Theorem 2.2 almost the same as those in Theorem 2.1. The only difference one needs to notice is that by the Atiyah-Singer index theorem, $\text{Ind} D^{(1-L)\otimes W}_{c,+} + \text{Ind} D^{(1-L)\otimes (TX - (L+L))}_{c,+}$ must be 0 when the dimension of the manifold is divisible by 4 as the index density of the operator

\[D^{(1-L)\otimes W}_{c,+} + D^{(1-L)\otimes (TX - (L+L))}_{c,+}

is a differential form of degree $4l + 2$. \hfill \square
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