Abstract

We show that the Atiyah–Patodi–Singer reduced \( \eta \)-invariant of the twisted Dirac operator on a closed \( 4m - 1 \)-dimensional spin manifold, with the twisted bundle being the Witten bundle appearing in the theory of elliptic genus, is a meromorphic modular form of weight \( 2m \) up to an integral \( q \)-series. We prove this result by combining our construction of certain modular characteristic forms associated to a generalized Witten bundle on spin\(^c\)-manifolds with a deep topological theorem due to Hopkins.

1. Introduction and statement of results

Let \( X \) be a smooth manifold. Let \( T_C X \) be the complexification of the tangent bundle \( TX \). One defines the Witten bundle on \( X \) \[13\] as follows:

\[
\Theta_q(TX) = \bigotimes_{u=1}^{\infty} S_q^u(T_C X - C^{\dim X}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-1/2}}(T_C X - C^{\dim X}),
\]

(1.1)

where \( S_t(\cdot) \) (respectively, \( \Lambda_t(\cdot) \)) denotes the symmetric (respectively, exterior) power and \( q = e^{2\pi \sqrt{-1} \tau} \) with \( \tau \in \mathbb{H} \), the upper half-plane.

Let \( g^{TX} \) be a Riemannian metric on \( TX \) and \( \nabla^{TX} \) be the associated Levi-Civita connection. If we write

\[
\Theta_q(TX) = B_0(TX) + B_1(TX)q^{1/2} + B_2(TX)q + \cdots ,
\]

(1.2)

then each \( B_i(TX) \) carries a Hermitian metric as well as a Hermitian connection \( \nabla^{B_i(TX)} \) canonically induced from \( g^{TX} \) and \( \nabla^{TX} \). In this way, \( \nabla^{TX} \) induces a Hermitian connection \( \nabla^{\Theta_q(TX)} \) on the Witten bundle \( \Theta_q(TX) \).
Now assume that $X$ is closed, spin and of dimension $4m$. Let $S(TX) = S_+(TX) \oplus S_-(TX)$ be the corresponding Hermitian bundle of spinors. For each $i$, let $D_{X,+}^{B_i(TX)} : \Gamma(S_+(TX) \otimes B_i(TX)) \to \Gamma(S_-(TX) \otimes B_i(TX))$ be the corresponding twisted Dirac operator. It is an important and well-known fact (cf. [14]) that the $q$-series

\[
\text{Ind}(D_{X,+}^{\Theta_q(TX)}) = \sum_{i=0}^{\infty} \text{Ind}(D_{X,+}^{B_i(TX)})q^{i/2},
\]

which by the Atiyah–Singer index theorem [1] equals to the elliptic genus (We refer the reader to [9, Section 2.1; 16, Chapter 1] for the notation of the corresponding characteristic forms appearing below.)

\[
\int_X \hat{A}(TX, \nabla^{TX})\text{ch}(\Theta_q(TX), \nabla^{\Theta_q(TX)})
\]

\[
= \sum_{i=0}^{\infty} q^{i/2} \int_X \hat{A}(TX, \nabla^{TX})\text{ch}(B_i(TX), \nabla^{B_i(TX)})
\]

is an integral modular form of weight $2m$ over $\Gamma^0(2)$, where $\Gamma^0(2)$ is the index 2 modular subgroup of $\text{SL}_2(\mathbb{Z})$ defined by

\[
\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid b \equiv 0 \pmod{2} \right\}.
\]

It is natural to look at what would happen if $X$ is a $4m - 1$-dimensional closed spin manifold. In this case, let $E$ be a Hermitian vector bundle over $X$ carrying a Hermitian connection $\nabla^E$. Let $D_X^E : \Gamma(S(TX) \otimes E) \to \Gamma(S(TX) \otimes E)$ be the associated twisted Dirac operator, which is formally self-adjoint.

Following [2], for any $\text{Re}(s) \gg 0$, set

\[
\eta(D_X^E, s) = \sum_{\lambda \in \text{Spec}(D_X^E) \setminus \{0\}} \frac{\text{Sgn}(\lambda)}{|\lambda|^s}.
\]

By [2], one knows that $\eta(D_X^E, s)$ is a holomorphic function in $s$ with $\text{Re}(s) > \text{dim } X/2$. Moreover, it extends to a meromorphic function over $\mathbb{C}$, which is holomorphic at $s = 0$. The $\eta$ invariant of $D_X^E$, in the sense of Atiyah–Patodi–Singer [2], is defined by

\[
\eta(D_X^E) = \eta(D_X^E, 0),
\]

while the reduced $\eta$ invariant is defined and denoted by

\[
\tilde{\eta}(D_X^E) = \frac{\text{dim}(\ker D_X^E) + \eta(D_X^E)}{2}.
\]

It is the aim of this paper to study the modularity of the $q$-series

\[
\tilde{\eta}(D_X^{\Theta_q(TX)}) = \sum_{i=0}^{\infty} \tilde{\eta}(D_X^{B_i(TX)})q^{i/2},
\]

which is a spectral invariant depending on $g^{TX}$. 
The term of integration over disjoint copies of $X$ manifold.

Indeed, it is a well-known fact in cobordism theory that there is a positive integer $k$ such that $k$ disjoint copies of $X$ bound a spin manifold $Y$. In this case, one has the following analogue of (1.7):

$$
\int_Y \widehat{A}(TY, \nabla^{TY}) \text{ch}(\Theta_q(TY), \nabla^{\Theta_q(TY)}) - \tilde{\eta}(D_{X}^{\Theta_q(TX)}) \in \mathbb{Z}[[q^{1/2}]].
$$

From (1.8), one sees that $\tilde{\eta}(D_{X}^{\Theta_q(TX)})$ is a modular form up to an element in $\mathbb{Z}[[q^{1/2}]]/k$. Thus, the natural classical method gives the conclusion that $\tilde{\eta}(D_{X}^{\Theta_q(TX)})$ is a modular form up to an element in $\mathbb{Q}[[q^{1/2}]]$ instead of $\mathbb{Z}[[q^{1/2}]]$.

On the other hand, if $\tilde{g}$ is another Riemannian metric on $TX$ with $\tilde{\nabla}^{TX}$ being its Levi-Civita connection and $\tilde{D}_{X}^{\Theta_q(TX)}$ being the corresponding twisted Dirac operator, then by the variation formula for the reduced $\eta$ invariant (cf. [2, 3]), one has

$$
\tilde{\eta}(D_{X}^{\Theta_q(TX)}) - \tilde{\eta}(\tilde{D}_{X}^{\Theta_q(TX)}) = \int_X \text{CS} \Phi(\nabla^{TX}, \tilde{\nabla}^{TX}, \tau) \mod \mathbb{Z}[[q^{1/2}]],\n$$

where $\text{CS} \Phi(\nabla^{TX}, \tilde{\nabla}^{TX}, \tau)$ is the Chern–Simons transgression form associated to $\Phi(\nabla^{TX}, \tau) = \{\widehat{A}(TX, \nabla^{TX}) \text{ch}(\Theta_q(TX), \nabla^{\Theta_q(TX)})\}^{4m}$. It is easy to see that $\int_X \text{CS} \Phi(\nabla^{TX}, \tilde{\nabla}^{TX}, \tau)$ is a modular form of weight $2m$ over $\Gamma^0(2)$ (cf. [7]). Thus, the variation of $\tilde{\eta}(D_{X}^{\Theta_q(TX)})$ has mod $\mathbb{Z}$ modularity property. It turns out to be an interesting open problem that whether $\tilde{\eta}(D_{X}^{\Theta_q(TX)})$ is by itself a modular form of weight $2m$ over $\Gamma^0(2)$ up to an element in $\mathbb{Z}[[q^{1/2}]]$.

The purpose of this short note is to give an answer to this question. Our main result can be stated as follows.

**Theorem 1.1** Let $X$ be a $4m - 1$-dimensional closed spin Riemannian manifold. Then the reduced $\eta$-invariant $\tilde{\eta}(D_{X}^{\Theta_q(TX)})$ of the twisted Dirac operator $\tilde{D}_{X}^{\Theta_q(TX)}$ is a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$, up to an element in $\mathbb{Z}[[q^{1/2}]]$.

Here meromorphic modular form is a weaker notion than modular form without requiring holomorphicity, but only meromorphicity on the upper half-plane.
To prove Theorem 1.1, instead of using the cobordism result as above, we make use of a result due to Hopkins (cf. [10, Section 8]) which asserts that for any complex vector bundle $V$ over $X$, there is a non-negative integer $s$ such that $X \times \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ ($s$-copies of $\mathbb{C}P^1$) bounds a spin manifold $Y$ and $V \boxtimes H^s$ on $X \times \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ extends to $Y$, where $H$ denotes the Hopf hyperplane bundle on $\mathbb{C}P^1$. We then apply the modular characteristic forms, which is associated to a generalized Witten bundle we have constructed in [9], on the bounding manifold, as well as the Atiyah–Patodi–Singer index theorem [2] to get the modularity of the reduced $\eta$-invariant in question.

It remains a challenge to find a purely analytic proof of Theorem 1.1 without using the deep topological results as the above-mentioned Hopkins’ theorem.

Theorem 1.1 immediately implies that the quantity in (1.8) is a meromorphic modular form up to an element in $k\mathbb{Z}[[q^{1/2}]]$, where $k$ is the positive integer such that $k$ disjoint copies of $X$ bounds $Y$ as explained before (1.8). Observe that in (1.8), each $q$-coefficient mod $k$ is a mod $k$ index studied by Freed and Melrose [8]. It is a topological invariant and the main result in [8] provides a topological interpretation of it. Therefore, as an application of Theorem 1.1, we have the following corollary.

**Corollary 1.1** Let $Y$ be a $4m$-dimensional spin $\mathbb{Z}/k$-manifold in the sense of Sullivan (cf. [8]). Then the mod $k$ index associated to the Witten bundle $\Theta_q(TY)$ can be represented by a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$.

On the other hand, in view of [5, (25)], which corresponds to the case of $k = 1$ in (1.8) for the category of stable almost complex manifolds, Theorem 1.1 might become a starting point of a kind of tertiary index theory, in the sense of [5, Theorem 4.2], for spin manifolds. Recently, Bunke informed us that Theorem 1.1 can be given an alternative proof by using the theory of the universal $\eta$ invariant [4, Lemma 3.1], and a spin version of the $f$-invariant has also been constructed in [4, Definition 13.2].

For completeness, we would like to point out what happens in dimension $4m + 1$. Actually, when $X$ is an $8n + 5$-dimensional closed spin manifold, since for each $i$, $\eta(D^0_X(TX)) = 0$ and $\dim(\ker D^0_X(TX))$ is even (cf. [2]), we have $\tilde{\eta}(D^0_X(TX)) = 0 \mod \mathbb{Z}[[q^{1/2}]]$. In dimension $8n + 1$, since $\eta(D^0_X(TX)) = 0$ for each $i$ (cf. [2]), we have $\tilde{\eta}(D^0_X(TX)) = \frac{\dim(\ker D^0_X(TX))}{2}$. Therefore, in view of the Atiyah–Singer mod 2 index theorem, $\tilde{\eta}(D^0_X(TX))$ can be identified with Ochanine’s beta invariant $\beta_q(X)$, the modularity of which has been shown in [12].

This paper is organized as follows. In Section 2, we briefly recall our construction (in [9]) of the modular form associated to a generalized Witten bundle involving a complex line bundle. In Section 3, we combine our modular form and the Hopkins boundary theorem to prove Theorem 1.1. In Section 4, we propose a possible refinement of Theorem 1.1 in $8n + 3$ dimension.

## 2. Complex line bundles and modular forms

In this section, we briefly review our construction (in [9]) of a modular form, which is associated to a generalized Witten bundle involving a complex line bundle.

Let $M$ be a $4l$-dimensional Riemannian manifold. Let $\nabla^TM$ be the associated Levi-Civita connection.

Let $\xi$ be a complex line bundle over $M$. Equivalently, one can view $\xi$ as a rank 2 real oriented vector bundle over $M$. Let $\xi$ carry a Euclidean metric and also a Euclidean connection $\nabla^\xi$, let $c = e(\xi, \nabla^\xi)$ be the Euler form associated to $\nabla^\xi$ (cf. [16, Section 3.4]). Let $\xi_C$ be the complexification of $\xi$.

If $E$ is a complex vector bundle over $M$, set $\overline{E} = E - \dim E \in K(M)$. 

Following [9, (2.5)], set

\[
\Theta_q(TM, \xi) = \bigotimes_{u=1}^{\infty} S^u(TC_{\tilde{M}}) \otimes \bigotimes_{v=1}^{\infty} \Lambda_{-q^{v-1/2}}(TC_{\tilde{M}} - 2\xi_C) \otimes \bigotimes_{r=1}^{\infty} \Lambda_{q^{r-1/2}}(\xi_C) \otimes \bigotimes_{t=1}^{\infty} \Lambda_{q^t}(\xi_C),
\]

which is an element in \( K(M)[[q^{1/2}]] \). As before, \( \nabla^{TM} \) and \( \nabla^{\xi} \) induce a Hermitian connection \( \nabla^{\Theta_q(TM, \xi)} \) on \( \Theta_q(TM, \xi) \).

Let \( P(TM, \xi, \tau) \in \Omega^4(M) \) be the characteristic form defined by

\[
P(TM, \xi, \tau) := \left\{ \hat{A}(TM, \nabla^{TM}) \cosh \left( \frac{\xi}{2} \right) \text{ch}(\Theta_q(TM, \xi), \nabla^{\Theta_q(TM, \xi)}) \right\}^{(4l)}.
\]

It is shown in [9] that \( P(TM, \xi, \tau) \) can be expressed by using the formal Chern roots of \((TC_M, \nabla^{TC_M})\) and \( c \) through the Jacobi theta functions, which are defined as follows (cf. [6; 9, Section 2.3]):

\[
\theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi \sqrt{-1}v} q^j)(1 - e^{-2\pi \sqrt{-1}v} q^j)],
\]

\[
\theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi \sqrt{-1}v} q^j)(1 + e^{-2\pi \sqrt{-1}v} q^j)],
\]

\[
\theta_2(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 - e^{2\pi \sqrt{-1}v} q^{j-1/2})(1 - e^{-2\pi \sqrt{-1}v} q^{j-1/2})],
\]

\[
\theta_3(v, \tau) = \prod_{j=1}^{\infty} [(1 - q^j)(1 + e^{2\pi \sqrt{-1}v} q^{j-1/2})(1 + e^{-2\pi \sqrt{-1}v} q^{j-1/2})].
\]

The theta functions are all holomorphic functions for \((v, \tau) \in \mathbb{C} \times \mathbb{H}\), where \( \mathbb{C} \) is the complex plane and \( \mathbb{H} \) is the upper half-plane. Let \( \{ \pm 2\pi \sqrt{-1}x_i \} \) be the formal Chern roots for \((TC_M, \nabla^{TC_M})\) and \( c = 2\pi \sqrt{-1}u \), we have

\[
P(TM, \xi, \tau) = \left\{ \left( \prod_{i=1}^{2l} x_i \frac{\theta'(0, \tau) \theta_2(x_i, \tau)}{\theta_1(x_i, \tau) \theta_2(0, \tau)} \right) \frac{\theta_1(u, \tau) \theta_2^2(0, \tau) \theta_3(u, \tau)}{\theta_1(0, \tau) \theta_2^2(u, \tau) \theta_3(0, \tau)} \right\}^{(4l)}.
\]

By using the transformation laws of theta functions (cf. [6; 9, Section 2.3]), one sees as in [9, Proposition 2.6] that \( P(TM, \xi, \tau) \) is a modular form of weight \( 2l \) over \( \Gamma^0(2) \).

3. Proof of the main theorem

In this section, we will prove our main result Theorem 1.1.
The topological tool we will use is the following boundary theorem of Hopkins (cf. [10, Section 8]).

**Theorem 3.1 (Hopkins)** Let $X$ be a compact, odd-dimensional spin manifold and $V \to X$ be a complex vector bundle over $X$. Then there is an integer $s$ such that the vector bundle $V \boxtimes (\mathbb{R}_j^{1}H_j) \to X \times (\mathbb{CP}^1)^s$ is a boundary, where $H_j$ denotes the Hopf hyperplane bundle on the $j$th copy of $\mathbb{CP}^1$. In other words, there is a spin manifold $Y$ with a complex vector bundle $W$ on $Y$ such that $W|_{\partial Y} = V \boxtimes (\mathbb{R}_j^{1}H_j)$.

In what follows, we will combine this Hopkins boundary theorem with the modular characteristic form constructed in Section 2 to give a proof of Theorem 1.1.

**Proof of Theorem 1.1** Without loss of generality, for the $4m-1$-dimensional closed spin manifold $X$, in view of the Hopkins boundary theorem, we take an even integer $s$ so that the complex line bundle

$$p^*(\mathbb{R}_j^{1}H_j) \to X \times (\mathbb{CP}^1)^s$$

bounds, where $p : X \times (\mathbb{CP}^1)^s \to (\mathbb{CP}^1)^s$ is the natural projection. This means that there is a spin manifold $Y$ and a complex line bundle $\zeta$ over $Y$ such that $\partial Y = X \times (\mathbb{CP}^1)^s$ and $\zeta|_{X \times (\mathbb{CP}^1)^s} = p^*(\mathbb{R}_j^{1}H_j)$.

Let $g^{TX}$ be any Riemannian metric on $X$. Equip $(\mathbb{CP}^1)^s$ with arbitrary Riemannian metrics and the $H_j$’s with arbitrary Euclidean metrics and Euclidean connections.

Let $g^{TY}$ be a metric on $TY$ such that it is of product structure near $X \times (\mathbb{CP}^1)^s$ and restricts to the product metric on $X \times (\mathbb{CP}^1)^s$. Let $\nabla^{TY}$ be the Levi-Civita connection associated to $g^{TY}$.

Let $g^\xi$ be an Euclidean metric on $\xi$ (viewed as an oriented real plane bundle) such that $g^\xi$ is of product structure near $X \times (\mathbb{CP}^1)^s$ and restricts to the Euclidean metric on $p^* (\mathbb{R}_j^{1}H_j)$ on $X \times (\mathbb{CP}^1)^s$. Let $\nabla^\xi$ be an Euclidean connection of $g^\xi$ which is of product structure near $X \times (\mathbb{CP}^1)^s$ and restricts to the canonically induced Euclidean connection on $p^*(\mathbb{R}_j^{1}H_j)$ on $X \times (\mathbb{CP}^1)^s$.

Let $c = e(\xi)$ and $z_j = c_1(H_j)/\pi \sqrt{-1}$, $1 \leq j \leq s$.

By applying the Atiyah–Patodi–Singer index theorem [2] to the twisted Dirac operator $D_Y^{\Theta_q(TY, \xi^2) \otimes \xi}$, in noting that

$$(\Theta_q(TY, \xi^2) \otimes \xi)|_{X \times (\mathbb{CP}^1)^s} = \Theta_q(T(X \times (\mathbb{CP}^1)^s), (p^*(\mathbb{R}_j^{1}H_j))^2) \otimes p^*(\mathbb{R}_j^{1}H_j),$$

one finds that there exist integers $a_i$’s such that

$$\tilde{\eta}(D_X^{\Theta_q(T(X \times (\mathbb{CP}^1)^s), (p^*(\mathbb{R}_j^{1}H_j))^2) \otimes p^*(\mathbb{R}_j^{1}H_j)})$$

$$= \int_Y \hat{A}(TY, \nabla^{TY}) \text{ch}(\Theta_q(TY, \xi^2) \otimes \xi, \nabla^{\Theta_q(TY, \xi^2) \otimes \xi}) - \sum_{i=0}^{\infty} a_i q^{i/2}$$

$$= \int_Y \hat{A}(TY, \nabla^{TY}) e^c \text{ch}(\Theta_q(TY, \xi^2), \nabla^{\Theta_q(TY, \xi^2)}) - \sum_{i=0}^{\infty} a_i q^{i/2}$$

$$= \int_Y \hat{A}(TY, \nabla^{TY}) \cosh(c) \text{ch}(\Theta_q(TY, \xi^2), \nabla^{\Theta_q(TY, \xi^2)}) - \sum_{i=0}^{\infty} a_i q^{i/2}, \quad (3.1)$$

where the last equality follows from the fact that $s$ is an even integer.
Let \( r : X \times (\mathbb{C}P^1)^s \to X \) be the natural projection. For bundles \( E \to X \) and \( F \to (\mathbb{C}P^1)^s \), by separation of variables, we have

\[
\eta(D_{X \times (\mathbb{C}P^1)^s}^{(r^*E) \otimes (p^*F)}) = \eta(D_X^E) \cdot \text{Ind}(D_{(\mathbb{C}P^1)^s,+}^F).
\]

So we have

\[
\tilde{\eta}(D_{X \times (\mathbb{C}P^1)^s}^{(r^*E) \otimes (p^*F)}) = \tilde{\eta}(D_X^E) \cdot \text{Ind}(D_{(\mathbb{C}P^1)^s,+}^F) + \dim(\ker D_X^E) \dim(\ker(D_{(\mathbb{C}P^1)^s,-}^F)).
\]

From the above formula, we can see that there are integers \( b_i \) such that

\[
\tilde{\eta}(D_{X \times (\mathbb{C}P^1)^s}^{(r^*E) \otimes (p^*F)}) = \sum_{i=0}^{\infty} b_i q^{i/2}
\]

\[
= \tilde{\eta}(D_{X \times (\mathbb{C}P^1)^s}^{(r^*E) \otimes (p^*F)}) - \sum_{i=0}^{\infty} b_i q^{i/2}
\]

\[
= \tilde{\eta}(D_X^{\Theta_q(T(X\times (\mathbb{C}P^1)^s) \otimes (p^*H_j)^2)}) - \sum_{i=0}^{\infty} b_i q^{i/2}
\]

\[
= \tilde{\eta}(D_X^{\Theta_q(T(X))}) \cdot \text{Ind}(D_{(\mathbb{C}P^1)^s,+}^{(r^*E) \otimes (p^*H_j)^2}) - \sum_{i=0}^{\infty} b_i q^{i/2}
\]

\[
= \tilde{\eta}(D_X^{\Theta_q(T(X))) \cdot \int_{(\mathbb{C}P^1)^s} \hat{A}(T(CP^1)^s, \nabla T(CP^1)^s) e^{\ell_1(H_j) + \cdots + \ell_1(H_s)} ch(\Theta_q(T(CP^1)^s, (\mathbb{Z}^s_j = H_j)^2))
\]

\[
= \tilde{\eta}(D_X^{\Theta_q(T(X))) \cdot \int_{(\mathbb{C}P^1)^s} \left( \prod_{j=1}^{s} \frac{\theta_1(\sum_{j=1}^{s} z_j, \tau)}{\theta_1(0, \tau)} \frac{\theta_2(\sum_{j=1}^{s} z_j, \tau)}{\theta_2(0, \tau)} \right) \frac{\theta_1(\sum_{j=1}^{s} z_j, \tau)}{\theta_1(0, \tau)} \frac{\theta_2(\sum_{j=1}^{s} z_j, \tau)}{\theta_2(0, \tau)} \frac{\theta_3(\sum_{j=1}^{s} z_j, \tau)}{\theta_3(0, \tau)}
\]

\[
= \tilde{\eta}(D_X^{\Theta_q(T(X))) \cdot \int_{(\mathbb{C}P^1)^s} \left( \frac{\theta_1(\sum_{j=1}^{s} z_j, \tau)}{\theta_1(0, \tau)} \frac{\theta_2(\sum_{j=1}^{s} z_j, \tau)}{\theta_2(0, \tau)} \frac{\theta_3(\sum_{j=1}^{s} z_j, \tau)}{\theta_3(0, \tau)}
\]

where the last equality holds due to the fact that \( x/\theta(x, \tau) \) and \( \theta_2(x, \tau) \) are both even functions about \( x \) and \( \int_{C_{\mathbb{P}^1}} \frac{x^n}{2} = 0 \) if \( n > 1 \).

Since \( s \) is an even integer, from the knowledge about the modular form \( P(TM, \xi, \tau) \) constructed in Section 2, we know that

\[
f_s(\tau) := \int_{(\mathbb{C}P^1)^s} \frac{\theta_1(\sum_{j=1}^{s} z_j, \tau)}{\theta_1(0, \tau)} \frac{\theta_2(\sum_{j=1}^{s} z_j, \tau)}{\theta_2(0, \tau)} \frac{\theta_3(\sum_{j=1}^{s} z_j, \tau)}{\theta_3(0, \tau)}
\]

is an integral modular form of weight \( s \) over \( \Gamma^0(2) \). Moreover, since

\[
\int_{(\mathbb{C}P^1)^s} \hat{A}(T(CP^1)^s, \nabla T(CP^1)^s) e^{\ell_1(H_j) + \cdots + \ell_1(H_s)} = 1,
\]

we see that \( f_s(\tau) \) has constant term 1. Therefore, \( f_s^{-1}(\tau) \in \mathbb{Z}[1/2] \).
From (3.1) and (3.2), we have
\[
\bar{\eta}(D_{X}^{\Theta_q(TX)}) = f_{s}^{-1} (\tau) \cdot \int_{Y} \hat{A}(TY, \nabla^{TY}) \cosh(c) \ 	ext{ch}(\Theta_q(TY, \zeta^2), \nabla^{\Theta_q(TY, \zeta^2)}) \\
- f_{s}^{-1} (\tau) \cdot \left( \sum_{i=0}^{\infty} (a_i + b_i) q^{i/2} \right).
\]  
(3.3)

Still by the modularity of $P(TM, \xi, \tau)$ constructed in Section 2, we know that
\[
\int_{Y} \hat{A}(TY, \nabla^{TY}) \cosh(c) \ 	ext{ch}(\Theta_q(TY, \zeta^2), \nabla^{\Theta_q(TY, \zeta^2)})
\]
is a modular form of weight $2m + s$ over $\Gamma^0(2)$. So
\[
 f_{s}^{-1} (\tau) \cdot \int_{Y} \hat{A}(TY, \nabla^{TY}) \cosh(c) \ 	ext{ch}(\Theta_q(TY, \zeta^2), \nabla^{\Theta_q(TY, \zeta^2)})
\]
is a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$. Therefore, from (3.3), we see that
\[
\bar{\eta}(D_{X}^{\Theta_q(TX)}) = f_{s}^{-1} (\tau) \cdot \int_{Y} \hat{A}(TY, \nabla^{TY}) \cosh(c) \ 	ext{ch}(\Theta_q(TY, \zeta^2), \nabla^{\Theta_q(TY, \zeta^2)}) \mod Z[[q^{1/2}]],
\]
a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$. The proof of Theorem 1.1 is complete. □

**Remark 3.1** The modular form $f_s(\tau)$ in the above proof can be explicitly expressed by theta functions and their derivatives. For example, we have
\[
f_2(\tau) = -\frac{1}{\pi^2} \left( \frac{\theta_1''(0, \tau)}{\theta_1(0, \tau)} - \frac{2 \theta_2''(0, \tau)}{\theta_2(0, \tau)} + \frac{\theta_3''(0, \tau)}{\theta_3(0, \tau)} \right)
\]  
(3.4)

and
\[
f_4(\tau) = \frac{1}{\pi^4} \left( \frac{\theta_1^{(4)}(0, \tau)}{\theta_1(0, \tau)} - \frac{2 \theta_2^{(4)}(0, \tau)}{\theta_2(0, \tau)} + \frac{\theta_3^{(4)}(0, \tau)}{\theta_3(0, \tau)} + 18 \left( \frac{\theta_2''(0, \tau)}{\theta_2(0, \tau)} \right)^2 \right)
\]
\[
- 12 \frac{\theta_1''(0, \tau) \theta_2''(0, \tau)}{\theta_1(0, \tau) \theta_2(0, \tau)} - 12 \frac{\theta_2''(0, \tau) \theta_3''(0, \tau)}{\theta_2(0, \tau) \theta_3(0, \tau)} + 6 \frac{\theta_1''(0, \tau) \theta_3''(0, \tau)}{\theta_1(0, \tau) \theta_3(0, \tau)} \right)
\]  
(3.5)

**Remark 3.2** Let $X$ be a compact, odd-dimensional spin manifold. Define
\[
H(X) := \{ h \in Z : \text{the line bundle } p^*([\bigoplus_{j=1}^{h} H_j]) \to X \times (CP^1)^h \text{ bounds} \},
\]
where $p : X \times (CP^1)^h \to (CP^1)^h$ is the natural projection and $H_j$ denotes the Hopf hyperplane bundle on the $j$th copy of $CP^1$. Define the Hopkins’ index of $X$, $h(X) := \min H(X)$. Obviously, when $X$ is a boundary by itself, $h(X) = 0$. It is clear that $H(X) = \{ s \in Z : s \geq h(X) \}$. 

In the proof of Theorem 1.1, we may take any even number $s \in H(X)$ and denote the corresponding $Y$ and $\zeta$ by $Y_s$ and $\zeta_s$. Then the proof of Theorem 1.1 tells us that, up to an element in $\mathbb{Z}[[q^{1/2}]]$,

$$\tilde{\eta}(D_X^{\Theta_q(TX)}) = f^{-1}_s(\tau) \cdot \int_{Y_s} \hat{A}(TY_s, \nabla^{TY_s}) \cosh(e(\zeta_s)) \operatorname{ch}(\Theta_q(TY_s, \zeta_s), \nabla^{\Theta_q(TY_s, \zeta_s)}).$$

Clearly, if $h(X) = 0$, one gets (1.7). Therefore, for every even number $s \geq 2[(h(X) + 1)/2]$, one can construct a meromorphic modular form of weight $2m$ over $\Gamma^0(2)$ of above form, that is equal to $\tilde{\eta}(D_X^{\Theta_q(TX)})$ up to an element in $\mathbb{Z}[[q^{1/2}]]$. The poles of these meromorphic modular forms are just the zeros of the modular forms $f_s(\tau)$. We hope that further study of the modular forms $f_s(\tau)$ will bring better understanding of modularity of $\tilde{\eta}(D_X^{\Theta_q(TX)})$.

**Remark 3.3** We refer the reader to [4] for an alternative approach to the modularity of $\tilde{\eta}(D_X^{\Theta_q(TX)})$, which is shown to be not only a meromorphic modular form, but also a modular form using the theory of universal $\eta$-invariant.

### 4. The cases of dimension $8n + 3$

In this section, we discuss the case of dimension $8n + 3$. In this dimension, it is known that $\tilde{\eta}(D_X^{\Theta_q(TX)})$ is mod $2\mathbb{Z}[[q^{1/2}]]$ smooth. That is, in the right-hand side of (1.9), the term mod $\mathbb{Z}[[q^{1/2}]]$ can be replaced by mod $2\mathbb{Z}[[q^{1/2}]]$. Therefore, it is natural to propose the following conjecture whose statement refines Theorem 1.1 in this case.

**Conjecture 4.1** Let $X$ be an $8n + 3$-dimensional closed spin Riemannian manifold. Then the reduced $\eta$-invariant $\tilde{\eta}(D_X^{\Theta_q(TX)})$ of the twisted Dirac operator $D_X^{\Theta_q(TX)}$ is a meromorphic modular form of weight $4n + 2$ over $\Gamma^0(2)$, up to an element in $2\mathbb{Z}[[q^{1/2}]]$.

Recall that a mod $2k$ refinement of the Freed–Melrose mod $k$ index for real vector bundles over $8n + 4$-dimensional manifolds has been defined in [15, Section 3]. In view of this, one can propose a refinement of Corollary 1.1, in the case of dim $Y = 8n + 4$, as follows.

**Conjecture 4.2** Let $Y$ be an $8n + 4$-dimensional spin $\mathbb{Z}/k$-manifold in the sense of Sullivan (cf. [8]). Then the mod $2k$ index associated to the Witten bundle $\Theta_q(TY)$ can be represented by a meromorphic modular form of weight $4n + 2$ over $\Gamma^0(2)$.

By the method of this paper, in order to prove Conjectures 4.1 and 4.2, one perhaps needs a kind of Hopkins boundary theorem for real vector bundles. Or, one may try to develop a direct analytic approach, which, even for Theorem 1.1, is a challenging problem as we indicated in Section 1.

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