

[L-L] T.Y. LAM and A. LEROY, Vandermonde and Wronskian Matrices over Division Rings, *J. of Algebra* 119 (1988), 308-336.

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GENERATORS AND RELATIONS OF INVARIANTS OF 2×2 MATRICES

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1 Introduction

Let G be a classical group of matrices of order $n \times n$ over F , where F is \mathbb{R} or \mathbb{C} . Let G act on m -tuples of $n \times n$ matrices by simultaneous conjugation. In [3, 9, 11] systems of generators of the polynomial invariants were given. These sets, however, contain infinitely many generators. When $n = 2$, it is possible to determine minimal systems of generators. This was done for $GL(2, F)$, $U(2)$ and $O(2, F)$ in [11] and for $SO(2, F)$ in [1]. In Section 2, we will summarize the basic results from [3, 9, 11] about systems of generators of invariants of $n \times n$ matrices. In Section 3, we will use these results to obtain minimal systems of generators of the remaining classical groups of 2×2 matrices. In Section 4, we will find algebraically independent sets of invariants of $GL(2, F)$, $O(2, F)$ and $SO(2, F)$, and we will discuss some syzygies between the invariants of these groups. Another related question is finding a linear basis for the ring of generic matrices. This has been done by Procesi [10] for $GL(2, F)$ and $SL(2, F)$. For $O(2, F)$ we refer to [2].

Table 1

	\mathbb{R}	\mathbb{C}	\mathbb{H}
bilinear and symmetric	$O(p, q)$	$O(n, \mathbb{C})$	—
bilinear and skewsymmetric	$Sp(k, \mathbb{R})$	$Sp(k, \mathbb{C})$	—
sesquilinear and Hermitian	—	$U(p, q)$	$Sp(p, q)$
sesquilinear and skew-Hermitian	—	$U(p, q)$	$Sp(k, \mathbb{H})$

2 Background results

Let us first define what we mean by a classical group. Let D be a division algebra over F with involution $a \mapsto a^\#$. Let V be a vector space over D , and let $(,)$ be a nondegenerate sesquilinear form on V that is Hermitian or skew-Hermitian relative to $^\#$. (If the involution is the identity we get bilinear symmetric and skewsymmetric forms.) Let G be the isometry group of $(,)$. We will call such groups classical isometry groups.

If the involution is either the identity or complex or quaternionic conjugation, we can describe the classical isometry groups by Table 1. (We are assuming that $n = p + q$ or $n = 2k$ when appropriate.) A “—” means that there are no such forms. Helgason [7] writes $SO^*(2k)$ for the group we have called $Sp(k, \mathbb{H})$.

The remaining classical groups are the general and special linear groups $GL(n, F)$ and $SL(n, F)$ and the special isometry groups, i.e., the intersections of $SL(n, F)$ with the classical isometry groups.

Let G be a classical group and let $M(n, D)$ be the set of $n \times n$ matrices with entries from D . We let

$$W = W(n, m, D) = \{(M_1, \dots, M_m) \mid M_i \in M(n, D)\}$$

be the set of m -tuples of matrices from $M(n, D)$. We let G act on W by simultaneous conjugation and denote the ring of polynomial invariants by $P[W]^G$.

We will call f_1, \dots, f_n a system of generators if all the polynomial invariants can be expressed as polynomials of the f_i .

Let G be a classical isometry group that is not quaternionic, i.e., one of the groups in the first two columns of Table 1. We use the following notation.

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad J_k = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}.$$

The form on V is given by $(x, y) = (x^\#)^t K y$ where K is given in Table 2. For $A \in M(n, D)$ we define

$$A^* = K^{-1}(A^\#)^t K. \tag{1}$$

Table 2

	$O(p, q)$	$O(n, \mathbb{C})$	$Sp(k, \mathbb{R})$	$Sp(k, \mathbb{C})$	$U(p, q)$
K	$I_{p,q}$	I_n	J_k	J_k	$I_{p,q}$

It follows that $(Ax, y) = (x, A^*y)$, and the isometry group G of the form is then defined by the condition $(g^\#)^t K g = K$ or $g^*g = I_n$.

If A is a skewsymmetric $2k \times 2k$ matrix over F , we denote the Pfaffian of A by $\text{pf } A$. It satisfies $\det A = \text{pf}^2 A$ and $\text{pf}(gAg^t) = \det g \text{pf } A$. For an arbitrary $2k \times 2k$ matrix M , we define $\widetilde{\text{pf}}(M) = \text{pf}(M - M^t)$ to be the Pfaffian of the skewsymmetric projection of M . This is clearly an $SO(2k, F)$ invariant. By abuse of notation we will refer to $\widetilde{\text{pf}}$ as the Pfaffian, too. (In [1], we used the notation $\text{pf } A$ instead of $\widetilde{\text{pf}} A$). For 2×2 matrices, we have

$$\widetilde{\text{pf}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = b - c.$$

When $k > 1$ the Pfaffian will no longer be linear. Instead we will consider the polarized Pfaffian, which we will denote by pl . Thus $\text{pl}(A_1, \dots, A_k)$ is the coefficient of $t_1 \cdots t_k$ in the expansion of $\widetilde{\text{pf}}(t_1 A_1 + \cdots + t_k A_k)$. It is a symmetric, multilinear function of k matrices and satisfies $\text{pl}(A, \dots, A) = k! \widetilde{\text{pf}}(A)$.

When $p + q = 2k$ is even, we also need to change the Pfaffian to the (p, q) Pfaffian to get an $SO(p, q)$ invariant. It is defined by

$$\widetilde{\text{pf}}_{p,q}(A) = \text{pf}(I_{p,q}A - A^t I_{p,q}) = \widetilde{\text{pf}}(I_{p,q}A).$$

The relevant results in [3, 9, 11] can now be stated as follows.

Theorem 1

1. Let G be a classical isometry group that is not quaternionic. The ring of invariants $P[W]^G$ is generated by invariants of the form

$$\text{tr } P(A, A^*),$$

where P is a noncommutative polynomial and $A^* = K^{-1}(A^\#)^t K$ where K is given by Table 2.

2. Let $G = SO(p, q)$. Define $A^* = I_{p,q}^{-1} A^t I_{p,q}$. When $p + q$ is odd, the invariants $P[W(p + q, m, \mathbb{R})]^G$ are the same as for $O(p, q)$. When $p + q = 2k$ is even, the invariants $P[W(p + q, m, \mathbb{R})]^G$ are generated by traces and polarized (p, q) Pfaffians of the A_i and A_i^* , i.e.,

$$\text{tr } P(A, A^*) \quad \text{and} \quad \text{pl}_{p,q}(P_1(A, A^*), \dots, P_k(A, A^*)),$$

where P, P_1, \dots, P_k are noncommutative polynomials.

3. Let $G = Sp(k, \mathbb{H})$. Define $A^* = J_k^{-1} \bar{A}^t J_k$. The invariants $P[W(2k, m, \mathbb{C})]^G$ are generated by traces and polarized Pfaffians of A_i, A_i^t, A_i^* and $(A_i^*)^t$, i.e.,

$$\text{tr } P(A, A^t, A^*, (A^*)^t) \quad \text{and} \\ \text{pl}(P_1(A, A^t, A^*, (A^*)^t), \dots, P_k(A, A^t, A^*, (A^*)^t)),$$

where P, P_1, \dots, P_k are noncommutative polynomials.

For $n = 2$, we will now determine minimal systems of generators. The following theorem sums up the relevant results from [1, 11].

Theorem 2

1. Let $G = GL(2, F)$. The polynomials

$$\text{tr } A_i, \text{tr } A_i A_j \ (i \leq j), \text{tr } A_i A_j A_k \ (i < j < k)$$

form a minimal system of generators of $P[W]^G$.

2. Let $G = U(2)$. Define $A^* = \bar{A}^t$. The polynomials

$$\text{tr } A_i, \text{tr } A_i^* = \overline{\text{tr } A_i}, \\ \text{tr } A_i A_j \ (i \leq j), \text{tr } A_j^* A_i^* = \overline{\text{tr } A_i A_j} \ (i \leq j), \\ \text{tr } A_i A_j^* \ (i \leq j), \text{tr } A_i A_j^* \ (i < j), \\ \text{tr } A_i A_j A_l^* \ (i < j), \text{tr } A_l A_j^* A_i^* = \overline{\text{tr } A_i A_j A_l^*} \ (i < j), \\ \text{tr } A_i A_j A_k \ (i < j < k), \text{tr } A_k^* A_j^* A_i^* = \overline{\text{tr } A_i A_j A_k} \ (i < j < k)$$

form a minimal system of generators of $P[W]^G$.

3. Let $G = O(2, F)$. The polynomials

$$\text{tr } A_i, \text{tr } A_i A_j \ (i \leq j), \text{tr } A_i A_j^t \ (i \leq j), \text{tr } A_i A_l A_l^t \ (i \neq l), \\ \text{tr } A_i^t A_j A_k \ (i < j < k), \text{tr } A_i A_j^t A_k \ (i < j < k), \text{tr } A_i A_j A_k^t \ (i < j < k)$$

form a minimal system of generators of $P[W]^G$.

4. Let $G = SO(2, F)$. The polynomials

$$\text{tr } A_i, \text{tr } A_i A_j \ (i \leq j), \widetilde{\text{pf}} A_i, \widetilde{\text{pf}} A_i A_j \ (i < j)$$

form a minimal system of generators of $P[W]^G$.

We also observe that $Sp(1, F) = SL(2, F)$. Since the invariants of $SL(n, F)$ under conjugation are the same as for $GL(n, F)$, it only remains to consider $O(1, 1), SO(1, 1), U(1, 1)$ and $Sp(1, \mathbb{H})$.

3 $O(1, 1), SO(1, 1), U(1, 1)$ and $Sp(1, \mathbb{H})$

Let G be a real connected Lie group that acts rationally on a real vector space V . We can then find a complex connected group $G_{\mathbb{C}}$ such that the Lie algebra of $G_{\mathbb{C}}$ is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ where \mathfrak{g} is the Lie algebra of G and with $G \subset G_{\mathbb{C}}$. Then both G and $G_{\mathbb{C}}$ act on $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus iV$, and for $A = X + iY \in V_{\mathbb{C}}$ with $X, Y \in V$ we define $A^* = X - iY$. The following Lemma was proved in [3].

Lemma 3 Consider $f(A, B) \in P(V_{\mathbb{C}} \oplus V_{\mathbb{C}})$. Then $f(A, B)$ is $G_{\mathbb{C}}$ invariant if and only if $f(A, A^*)$ is G invariant.

Conversely, suppose that we start with a complex connected Lie group $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ and an involution $*$ of $M(n, \mathbb{C})$ such that $(iI_n)^* = -iI_n$. We then obtain a real form G of $G_{\mathbb{C}}$ by setting $G = G_{\mathbb{C}} \cap \{gg^* = I_n\}$, and we get the following.

Corollary 4 Let $*$ be an involution of $M(n, \mathbb{C})$ such that $(iI_n)^* = -iI_n$. Let $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ be a connected complex Lie group. Set $G = G_{\mathbb{C}} \cap \{gg^* = I_n\}$. Let $W = W(n, m, \mathbb{C})$ and $f : W \oplus W \rightarrow \mathbb{C}$ be a polynomial function.

1. $f(A, B)$ is $G_{\mathbb{C}}$ invariant if and only if $f(A, A^*)$ is G invariant.
2. The set of polynomials $f_i(A, A^*)$ with $i \in I$ forms a system of generators of the $G_{\mathbb{C}}$ invariants of $P[W \oplus W]$ if and only if the set of polynomials $f_i(A, A^*)$ with $i \in I$ forms a system of generators of the G invariants of $P[W]$.

We can now state the following.

Theorem 5

1. Let $G = U(1, 1)$. Define $A^* = I_{1,1}^{-1} \bar{A}^t I_{1,1}$. The polynomials

$$\text{tr } A_i, \text{tr } A_i^* = \overline{\text{tr } A_i}, \\ \text{tr } A_i A_j \ (i \leq j), \text{tr } A_j^* A_i^* = \overline{\text{tr } A_i A_j} \ (i \leq j), \\ \text{tr } A_i A_j^* \ (i \leq j), \text{tr } A_i A_j^* \ (i < j), \\ \text{tr } A_i A_j A_l^* \ (i < j), \text{tr } A_l A_j^* A_i^* = \overline{\text{tr } A_i A_j A_l^*} \ (i < j), \\ \text{tr } A_i A_j A_k \ (i < j < k), \text{tr } A_k^* A_j^* A_i^* = \overline{\text{tr } A_i A_j A_k} \ (i < j < k)$$

form a minimal system of generators of $P[W]^G$.

2. Let $G = O(1, 1)$. Define $A^* = I_{1,1}^{-1} A^t I_{1,1}$. The polynomials

$$\text{tr } A_i, \text{tr } A_i A_j \ (i \leq j), \text{tr } A_i A_j^* \ (i \leq j), \text{tr } A_i A_l A_l^* \ (i \neq l), \\ \text{tr } A_i^* A_j A_k \ (i < j < k), \text{tr } A_i A_j^* A_k \ (i < j < k), \text{tr } A_i A_j A_k^* \ (i < j < k)$$

form a minimal system of generators of $P[W]^G$.

3. Let $G = SO(1, 1)$. The polynomials

$$\text{tr } A_i, \text{tr } A_i A_j \ (i \leq j), \widetilde{\text{pf}}_{1,1} A_i, \widetilde{\text{pf}}_{1,1} A_i A_j \ (i < j)$$

form a minimal system of generators of $P[W]^G$.

4. Let $G = Sp(1, \mathbb{H})$. Define $A^* = J_1^{-1} \bar{A}^t J_1$. The polynomials

$$\text{tr } A_i, \text{tr } A_i^*, \text{tr } A_i A_j \ (i \leq j), \text{tr } A_i^* A_j^* \ (i \leq j), \text{tr } A_i A_i^*, \\ \widetilde{\text{pf}} A_i, \widetilde{\text{pf}} A_i^*, \widetilde{\text{pf}} A_i A_j \ (i < j), \widetilde{\text{pf}} A_i^* A_j^* \ (i < j), \widetilde{\text{pf}} A_i A_i^*$$

form a minimal system of generators of $P[W]^G$.

Proof. Using Corollary 4 and the corresponding results for $GL(2, F)$, $O(2, F)$ and $SO(2, F)$, we get a system of generators. The reduction to a minimal system of generators, is similar to the arguments in the case of $GL(2, F)$, $O(2, F)$ and $SO(2, F)$ [1, 11]. We only need to make slight changes, even though we use different $*$ operators and the $\widetilde{\text{pf}}_{1,1}$ function. In particular, we can replace the formula [1]

$$\widetilde{\text{pf}} A^2 = (\widetilde{\text{pf}} A \text{tr } A)/2 \quad \text{by} \quad \widetilde{\text{pf}}_{1,1} A^2 = (\widetilde{\text{pf}}_{1,1} A \text{tr } A)/2.$$

□

4 Algebraically independent sets of invariants and syzygies

In this section we will first determine algebraically independent sets of invariants of $GL(2, F)$, $O(2, F)$ and $SO(2, F)$.

Theorem 6

1. Let $G = GL(2, F)$. The polynomials

$$\text{tr } A_i, \text{tr } A_i^2, \text{tr } A_1 A_i \ (1 < i), \text{tr } A_2 A_j \ (2 < j)$$

form a maximal set of algebraically independent elements of $P[W]^G$.

2. Let $G = O(2, F)$. The polynomials

$$\text{tr } A_i, \text{tr } A_i^2, \text{tr } A_i A_i^t, \text{tr } A_1 A_i \ (1 < i)$$

form a maximal set of algebraically independent elements of $P[W]^G$.

3. Let $G = SO(2, F)$. The polynomials

$$\text{tr } A_i, \text{tr } A_i^2, \text{tr } A_1 A_i \ (1 < i), \widetilde{\text{pf}} A_i$$

form a maximal set of algebraically independent elements of $P[W]^G$.

Remark. In the $GL(2, F)$ case, a different set of algebraically independent invariants (and a different proof) is given in [13].

Proof. We will first consider the $GL(2, F)$ case. For a generic m -tuple with $m \geq 2$, the stabilizer will consist of the scalar matrices. Hence the principal orbits will have dimension three, and the dimension of the orbit space is $4m - 3$. Since the dimension of the orbit space is equal to the Krull dimension of the ring of invariants, we know that the maximal number of algebraically independent invariants is $4m - 3$. The set of invariants listed above contains $4m - 3$ invariants, so we only have to prove that they are algebraically independent. We will do this by induction. Consider the Jacobian of the $4m - 3$ functions with respect to the $4m - 3$ variables

$$a_{11}^{(1)}, a_{11}^{(2)}, a_{12}^{(2)}, a_{22}^{(2)}, a_{11}^{(3)}, a_{12}^{(3)}, a_{21}^{(3)}, a_{22}^{(3)}, \dots, a_{11}^{(m)}, a_{12}^{(m)}, a_{21}^{(m)}, a_{22}^{(m)}.$$

If we let the last 4 rows correspond to the functions $\text{tr } A_m, \text{tr } A_m^2, \text{tr } A_1 A_m, \text{tr } A_2 A_m$, the Jacobian is of the form

$$\begin{pmatrix} M & 0 \\ N & P \end{pmatrix}$$

where M is invertible by induction and

$$P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 2a_{11}^{(m)} & 2a_{12}^{(m)} & 2a_{21}^{(m)} & 2a_{22}^{(m)} \\ a_{11}^{(1)} & a_{21}^{(1)} & a_{12}^{(1)} & a_{22}^{(1)} \\ a_{11}^{(2)} & a_{21}^{(2)} & a_{12}^{(2)} & a_{22}^{(2)} \end{pmatrix}.$$

It follows that the Jacobian is invertible, so the functions are algebraically independent.

In the $O(2, F)$ and $SO(2, F)$ cases, the principal isotropy groups are again the scalar matrices. Hence the principal orbits are of dimension 1, so the dimension of the orbit space is $4m - 1$. Thus we only have to prove that the invariants listed in the theorem are algebraically independent. This is proved in the same way as in the $GL(2, F)$ case by considering the Jacobian. □

5 Final remarks

In this section we will make some comparisons between Theorem 2 and Theorem 6. The algebraically independent invariants listed in Theorem 6 form a subset of the invariants listed in the minimal systems of generators in Theorem 2. Consider the invariants that are listed in Theorem 2 but not listed in Theorem 6. We will call them *additional invariants*. We will now try to determine the syzygies we obtain when we add the additional invariants to the algebraically independent ones.

Let us first make some general remarks about trace relations. Procesi [9] has proved that in the $GL(2, F)$ case, all relations among traces are consequences of the Cayley-Hamilton Theorem. The polarized version of the Cayley-Hamilton Theorem says that

$$XY + YX - Y \operatorname{tr} X - X \operatorname{tr} Y - \operatorname{tr} XY + \operatorname{tr} X \operatorname{tr} Y = 0. \quad (2)$$

If we multiply by Z and take the trace, we get what Procesi calls the fundamental trace relation

$$\begin{aligned} &\operatorname{tr} XYZ + \operatorname{tr} YXZ \\ &- \operatorname{tr} YZ \operatorname{tr} X - \operatorname{tr} XZ \operatorname{tr} Y - \operatorname{tr} XY \operatorname{tr} Z + \operatorname{tr} X \operatorname{tr} Y \operatorname{tr} Z = 0. \end{aligned} \quad (3)$$

By substituting monomials of matrices for X, Y and Z , we can deduce all the relations between the $GL(2, F)$ invariants.

In the $O(2, F)$ case, Procesi [9] proved that all relations are consequences of two relations. The first is the fundamental trace relation (4) and the other is given by

$$4 \operatorname{tr}(X^- Z^- Y) - 2 \operatorname{tr}(X^- Z^-) \operatorname{tr} Y = 0, \quad (4)$$

where

$$M^- = (M - M^t)/2$$

is the skewsymmetric projection. It is easy to see that (5) is equivalent to (2).

If we take the trace in (3), everything cancels. (That's why we first multiplied (3) with Z when deriving (4).) But if we take the Pfaffian, we get the relation

$$\widetilde{\operatorname{pf}} XY + \widetilde{\operatorname{pf}} YX - \widetilde{\operatorname{pf}} X \operatorname{tr} Y - \operatorname{tr} X \widetilde{\operatorname{pf}} Y = 0. \quad (5)$$

This can be thought of as a fundamental Pfaffian relation for relations among $SO(2, F)$ invariants.

In the $GL(2, F)$ case, the minimal system of generators has $(m^3 + 11m)/6$ elements, while the algebraically independent set of invariants has $4m - 3$ elements for $m \geq 2$ and 1 element when $m = 1$. For $m = 1$ or $m = 2$, the sets are the same, but for $m = 3$, the minimal system of generators contains $\operatorname{tr} ABC$ in addition to the 9 algebraically independent invariants. It turns out that $\operatorname{tr} ABC$ satisfies the following quadratic relation with coefficients expressible in terms of the 9 invariants that are traces of one or two factors [8, 11]. (Here $\{ \}$ denotes the sum of the terms obtained by cyclic permutation of A, B and C .)

$$4(\operatorname{tr} ABC)^2 - 4a_1 \operatorname{tr} ABC + a_2 = 0, \quad (6)$$

where

$$\begin{aligned} a_1 &= \{\operatorname{tr} A \operatorname{tr} BC\} - \operatorname{tr} A \operatorname{tr} B \operatorname{tr} C, \\ a_2 &= 2\{\operatorname{tr} AB(\operatorname{tr} AB - \operatorname{tr} A \operatorname{tr} B)(\operatorname{tr}^2 C - \operatorname{tr} C^2)\} + 4 \operatorname{tr} AB \operatorname{tr} BC \operatorname{tr} CA \\ &\quad + 2 \operatorname{tr} A^2 \operatorname{tr} B^2 \operatorname{tr} C^2 + \operatorname{tr}^2 A \operatorname{tr}^2 B \operatorname{tr}^2 C - \{\operatorname{tr} A^2 \operatorname{tr} B^2 \operatorname{tr}^2 C\}. \end{aligned}$$

Notice that this is a relation of degree 6 involving 3 matrices. We expect that it could also have been deduced from (4) by replacing XYZ by $(ABC)^2$, e.g., substituting $X = AB, Y = CA, Z = BC$.

For $m = 4$, we have 5 additional invariants. One of them is of degree two, namely $\operatorname{tr} A_3 A_4$. We want to find a relation between the 14 invariants $\operatorname{tr} A_i$ and $\operatorname{tr} A_i A_j$. Such a relation was found using the computer system Macaulay [4] and the Gröbner basis methods described in [12]. It is given as equation (12) in the Appendix.

We will try to indicate why it is natural to expect such a relation. We know from [9] that it can be obtained from (4) by replacing XYZ by some monomial. The key idea is that we need to use (7) to get rid of expression of the form $\operatorname{tr} XYZ$, so each matrix must appear twice. We therefore expect that replacing XYZ by $(ABCD)^2$ in (4) in some way will give a relation of degree 8 involving 4 matrices. We could also argue as follows: Multiply (3) by ZW and take the trace. This gives a trace relation of degree 4 involving 4 matrices. We can then replace $XYZW$ by $(ABCD)^2$ in some way, and we would expect to get a relation of degree 8 involving 4 matrices.

In the $O(2, F)$ case, the two sets of invariants are the same when $m = 1$. When $m = 2$, the minimal system of generators contains the additional invariants $\operatorname{tr} AB^t, \operatorname{tr} ABB^t$ and $\operatorname{tr} BAA^t$. The last two are related to the algebraically independent invariants by relations of the form (7). We now observe [1] that

$$\widetilde{\operatorname{pf}} A \widetilde{\operatorname{pf}} B = \operatorname{tr} AB^t - \operatorname{tr} AB. \quad (7)$$

From

$$(\widetilde{\operatorname{pf}} A \widetilde{\operatorname{pf}} B)^2 = \widetilde{\operatorname{pf}}^2 A \widetilde{\operatorname{pf}}^2 B,$$

we get an immediate proof of the following syzygy from [11]

$$(\operatorname{tr} AB^t - \operatorname{tr} AB)^2 = (\operatorname{tr} AA^t - \operatorname{tr} A^2)(\operatorname{tr} BB^t - \operatorname{tr} B^2), \quad (8)$$

which gives the desired equation involving $\operatorname{tr} AB^t$.

When $m = 3$, the algebraically independent set of invariants only contains 11 invariants, so there must be one relation between the 12 invariants $\operatorname{tr} A_i, \operatorname{tr} A_i A_j (i \leq j)$ and $A_i A_i^t$. We have not been able to find this, but we know that it must have degree at least 12.

For $SO(2, F)$ we get the same sets of invariants when $m = 1$. For $m = 2$ the minimal system of generators contains 8 invariants. Using Macaulay [4],

we found that $\widetilde{\text{pf}} AB$ is related to the 7 algebraically independent invariants by the following relation of degree 4

$$\begin{aligned} & -\text{tr}^2 B \text{tr} A^2 - \text{tr}^2 A \text{tr} B^2 + 2 \text{tr} A^2 \text{tr} B^2 + 2 \text{tr} A \text{tr} B \text{tr} AB \\ & - 2 \text{tr}^2 AB - \text{tr}^2 B \widetilde{\text{pf}}^2 A + \text{tr} B^2 \widetilde{\text{pf}}^2 A - 2 \text{tr} AB \widetilde{\text{pf}} A \widetilde{\text{pf}} B - \text{tr}^2 A \widetilde{\text{pf}}^2 B (9) \\ & + \text{tr} A^2 \widetilde{\text{pf}}^2 B + 2 \text{tr} B \widetilde{\text{pf}} A \widetilde{\text{pf}} AB + 2 \text{tr} A \widetilde{\text{pf}} B \widetilde{\text{pf}} AB - 2 \widetilde{\text{pf}}^2 AB = 0. \end{aligned}$$

Notice that (6) is of degree 2 and involves 2 matrices, while (10) is of degree 4 and involves 2 matrices. We thus expect that (10) could have been deduced from (6) by replacing XY by $(AB)^2$ in some way.

When $m = 3$, the minimal system of generators has 15 invariants, but only 11 of them are algebraically independent. We have 3 relations of the form (10) in addition to a relation of degree 6 and involving 3 matrices which we found using Macaulay [4]. It is given as equation (11) in the Appendix. We believe that this relation could be obtained from replacing XY by $(ABC)^2$ in (6).

6 Appendix

A syzygy of degree 6 involving the traces and Pfaffians of $3 \times 2 \times 2$ matrices.

$$\begin{aligned} & -\widetilde{\text{pf}}^2 C \text{tr}^2 B \text{tr} A^2 + 2 \widetilde{\text{pf}} B \widetilde{\text{pf}} C \text{tr} B \text{tr} C \text{tr} A^2 - \widetilde{\text{pf}}^2 B \text{tr}^2 C \text{tr} A^2 \\ & - \text{tr}^2 A \widetilde{\text{pf}}^2 C \text{tr} B^2 + 2 \text{tr} A \widetilde{\text{pf}} A \widetilde{\text{pf}} C \text{tr} C \text{tr} B^2 - \widetilde{\text{pf}}^2 A \text{tr}^2 C \text{tr} B^2 \\ & + 2 \widetilde{\text{pf}}^2 C \text{tr} A^2 \text{tr} B^2 - 2 \text{tr}^2 C \text{tr} A^2 \text{tr} B^2 - \text{tr}^2 A \widetilde{\text{pf}}^2 B \text{tr} C^2 \\ & + 2 \text{tr} A \widetilde{\text{pf}} A \widetilde{\text{pf}} B \text{tr} B \text{tr} C^2 - \widetilde{\text{pf}}^2 A \text{tr}^2 B \text{tr} C^2 + 2 \widetilde{\text{pf}}^2 B \text{tr} A^2 \text{tr} C^2 \\ & - 2 \text{tr}^2 B \text{tr} A^2 \text{tr} C^2 - 2 \text{tr}^2 A \text{tr} B^2 \text{tr} C^2 + 2 \widetilde{\text{pf}}^2 A \text{tr} B^2 \text{tr} C^2 + 4 \text{tr} A^2 \text{tr} B^2 \text{tr} C^2 \\ & + 2 \text{tr} A \widetilde{\text{pf}}^2 C \text{tr} B \text{tr} AB - 2 \text{tr} A \widetilde{\text{pf}} B \widetilde{\text{pf}} C \text{tr} C \text{tr} AB \\ & - 2 \widetilde{\text{pf}} A \widetilde{\text{pf}} C \text{tr} B \text{tr} C \text{tr} AB + 2 \widetilde{\text{pf}} A \widetilde{\text{pf}} B \text{tr}^2 C \text{tr} - 4 \widetilde{\text{pf}} A \widetilde{\text{pf}} B \text{tr} C^2 \text{tr} ABAB \\ & + 4 \text{tr} A \text{tr} B \text{tr} C^2 \text{tr} AB - 2 \widetilde{\text{pf}}^2 C \text{tr}^2 AB + 2 \text{tr}^2 C \text{tr}^2 AB - 4 \text{tr} C^2 \text{tr}^2 AB \\ & - 2 \text{tr} A \widetilde{\text{pf}} B \widetilde{\text{pf}} C \text{tr} B \text{tr} AC + 2 \widetilde{\text{pf}} A \widetilde{\text{pf}} C \text{tr}^2 B \text{tr} AC + 2 \text{tr} A \widetilde{\text{pf}}^2 B \text{tr} C \text{tr} A Q(10) \\ & - 2 \widetilde{\text{pf}} A \widetilde{\text{pf}} B \text{tr} B \text{tr} C \text{tr} AC - 4 \widetilde{\text{pf}} A \widetilde{\text{pf}} C \text{tr} B^2 \text{tr} AC + 4 \text{tr} A \text{tr} C \text{tr} B^2 \text{tr} AC \\ & + 4 \widetilde{\text{pf}} B \widetilde{\text{pf}} C \text{tr} AB \text{tr} AC - 4 \text{tr} B \text{tr} C \text{tr} AB \text{tr} AC - 2 \widetilde{\text{pf}}^2 B \text{tr}^2 AC \\ & + 2 \text{tr}^2 B \text{tr}^2 AC - 4 \text{tr} B^2 \text{tr}^2 AC + 2 \text{tr}^2 A \widetilde{\text{pf}} B \widetilde{\text{pf}} C \text{tr} BC \\ & - 2 \text{tr} A \widetilde{\text{pf}} A \widetilde{\text{pf}} C \text{tr} B \text{tr} BC - 2 \text{tr} A \widetilde{\text{pf}} A \widetilde{\text{pf}} B \text{tr} C \text{tr} BC + 2 \widetilde{\text{pf}}^2 A \text{tr} B \text{tr} C \text{tr} BC \\ & - 4 \widetilde{\text{pf}} B \widetilde{\text{pf}} C \text{tr} A^2 \text{tr} BC + 4 \text{tr} B \text{tr} C \text{tr} A^2 \text{tr} BC + 4 \widetilde{\text{pf}} A \widetilde{\text{pf}} C \text{tr} AB \text{tr} BC \\ & - 4 \text{tr} A \text{tr} C \text{tr} AB \text{tr} BC + 4 \widetilde{\text{pf}} A \widetilde{\text{pf}} B \text{tr} AC \text{tr} BC - 4 \text{tr} A \text{tr} B \text{tr} AC \text{tr} BC \\ & + 8 \text{tr} AB \text{tr} AC \text{tr} BC + 2 \text{tr}^2 A \text{tr}^2 BC - 2 \widetilde{\text{pf}}^2 A \text{tr}^2 BC - 4 \text{tr} A^2 \text{tr}^2 BC = 0. \end{aligned}$$

A syzygy of degree 8 involving the traces of $4 \times 2 \times 2$ matrices.

$$\begin{aligned} & -\text{tr} D^2 \text{tr}^2 A \text{tr}^2 BC + 2 \text{tr} D \text{tr} A \text{tr} AD \text{tr}^2 BC - 2 \text{tr}^2 AD \text{tr}^2 BC \\ & - 2 \text{tr} D \text{tr} A \text{tr} AC \text{tr} BC \text{tr} BD + 4 \text{tr} AC \text{tr} AD \text{tr} BC \text{tr} BD \end{aligned}$$

$$\begin{aligned} & - 2 \text{tr}^2 AC \text{tr}^2 BD + 2 \text{tr}^2 A \text{tr} BC \text{tr} BD \text{tr} CD + 2 \text{tr} D^2 \text{tr} A \text{tr} AC \text{tr} BC \text{tr} B \\ & - 2 \text{tr} D \text{tr} AC \text{tr} AD \text{tr} BC \text{tr} B + 2 \text{tr} D \text{tr}^2 AC \text{tr} BD \text{tr} B \\ & - 2 \text{tr} A \text{tr} AD \text{tr} BC \text{tr} CD \text{tr} B - 2 \text{tr} A \text{tr} AC \text{tr} BD \text{tr} CD \text{tr} B \\ & - \text{tr} D^2 \text{tr}^2 AC \text{tr}^2 B + 2 \text{tr} AC \text{tr} AD \text{tr} CD \text{tr}^2 B - 2 \text{tr} A \text{tr} AD \text{tr} BC \text{tr} BD \text{tr} C \\ & + 2 \text{tr} A \text{tr} AC \text{tr}^2 BD \text{tr} C + 2 \text{tr}^2 AD \text{tr} BC \text{tr} B \text{tr} C \\ & - 2 \text{tr} AC \text{tr} AD \text{tr} BD \text{tr} B \text{tr} C + (2 \text{tr} D^2 - \text{tr}^2 D) \text{tr}^2 BC \text{tr} A^2 \\ & - 4 \text{tr} BC \text{tr} BD \text{tr} CD \text{tr} A^2 + 2 \text{tr} D \text{tr} BC \text{tr} CD \text{tr} B \text{tr} A^2 \\ & - \text{tr}^2 CD \text{tr}^2 B \text{tr} A^2 + 2 \text{tr} D \text{tr} BC \text{tr} BD \text{tr} C \text{tr} A^2 - 2 \text{tr} D^2 \text{tr} BC \text{tr} B \text{tr} C \text{tr} A^2 \\ & + 2 \text{tr} BD \text{tr} CD \text{tr} B \text{tr} C \text{tr} A^2 - \text{tr}^2 BD \text{tr}^2 C \text{tr} A^2 \\ & + (2 \text{tr} D^2 - \text{tr}^2 D) \text{tr}^2 AC \text{tr} B^2 + 2 \text{tr} D \text{tr} A \text{tr} AC \text{tr} CD \text{tr} B^2 \\ & - 4 \text{tr} AC \text{tr} AD \text{tr} CD \text{tr} B^2 - \text{tr}^2 A \text{tr}^2 CD \text{tr} B^2 - 2 \text{tr} D^2 \text{tr} A \text{tr} AC \text{tr} C \text{tr} B^2 \\ & + 2 \text{tr} D \text{tr} AC \text{tr} AD \text{tr} C \text{tr} B^2 + 2 \text{tr} A \text{tr} AD \text{tr} CD \text{tr} C \text{tr} B^2 \\ & - \text{tr}^2 AD \text{tr}^2 C \text{tr} B^2 + 2 \text{tr}^2 CD \text{tr} A^2 \text{tr} B^2 - 2 \text{tr} D \text{tr} CD \text{tr} C \text{tr} A^2 \text{tr} B^2 \\ & + \text{tr} D^2 \text{tr}^2 C \text{tr} A^2 \text{tr} B^2 - \text{tr}^2 A \text{tr}^2 BD \text{tr} C^2 + 2 \text{tr} A \text{tr} AD \text{tr} BD \text{tr} B \text{tr} C^2 (11) \\ & - \text{tr}^2 AD \text{tr}^2 B \text{tr} C^2 + 2 \text{tr}^2 BD \text{tr} A^2 \text{tr} C^2 - 2 \text{tr} D \text{tr} BD \text{tr} B \text{tr} A^2 \text{tr} C^2 \\ & + \text{tr} D^2 \text{tr}^2 B \text{tr} A^2 \text{tr} C^2 + \text{tr} D^2 \text{tr}^2 A \text{tr} B^2 \text{tr} C^2 - 2 \text{tr} D \text{tr} A \text{tr} AD \text{tr} B^2 \text{tr} C^2 \\ & + 2 \text{tr}^2 AD \text{tr} B^2 \text{tr} C^2 - (2 \text{tr} D^2 - \text{tr}^2 D) \text{tr} A^2 \text{tr} B^2 \text{tr} C^2 \\ & - 2(2 \text{tr} D^2 - \text{tr}^2 D) \text{tr} AC \text{tr} BC \text{tr} AB - 2 \text{tr} D \text{tr} A \text{tr} BC \text{tr} CD \text{tr} AB \\ & + 4 \text{tr} AD \text{tr} BC \text{tr} CD \text{tr} AB + 4 \text{tr} AC \text{tr} BD \text{tr} CD \text{tr} AB \\ & - 2 \text{tr} D \text{tr} AC \text{tr} CD \text{tr} B \text{tr} AB + 2 \text{tr} A \text{tr}^2 CD \text{tr} B \text{tr} AB \\ & + 2 \text{tr} D^2 \text{tr} A \text{tr} BC \text{tr} C \text{tr} AB \\ & - 2 \text{tr} D \text{tr} AD \text{tr} BC \text{tr} C \text{tr} AB - 2 \text{tr} D \text{tr} AC \text{tr} BD \text{tr} C \text{tr} AB \\ & - 2 \text{tr} A \text{tr} BD \text{tr} CD \text{tr} C \text{tr} AB + 2 \text{tr} D^2 \text{tr} AC \text{tr} B \text{tr} C \text{tr} AB \\ & - 2 \text{tr} AD \text{tr} CD \text{tr} B \text{tr} C \text{tr} AB + 2 \text{tr} AD \text{tr} BD \text{tr}^2 C \text{tr} AB \\ & + 2 \text{tr} D \text{tr} A \text{tr} BD \text{tr} C^2 \text{tr} AB - 4 \text{tr} AD \text{tr} BD \text{tr} C^2 \text{tr} AB \\ & - 2 \text{tr} D^2 \text{tr} A \text{tr} B \text{tr} C^2 \text{tr} AB + 2 \text{tr} D \text{tr} AD \text{tr} B \text{tr} C^2 \text{tr} AB - 2 \text{tr}^2 CD \text{tr}^2 AB \\ & + 2 \text{tr} D \text{tr} CD \text{tr} C \text{tr}^2 AB - \text{tr} D^2 \text{tr}^2 C \text{tr}^2 AB + (2 \text{tr} D^2 - \text{tr}^2 D) \text{tr} C^2 \text{tr}^2 AB = 0. \end{aligned}$$

References

- [1] H. Aslaksen, *SO(2) invariants of a set of 2×2 matrices*, Math. Scand. **65** (1989), 59–66.
- [2] H. Aslaksen and E.-C. Tan, *2×2 matrices with involution*, preprint, National Univ. of Singapore, 1993.
- [3] H. Aslaksen, E.-C. Tan and C. Zhu, *Invariant theory of special orthogonal groups*, to appear in Pacific J. Math.
- [4] D. Bayer and M. Stillman, *Macaulay: A system for computation in algebraic geometry and commutative algebra*. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from zariski.harvard.edu via anonymous ftp.

- [5] E. Formanek, *Invariants and the ring of generic matrices*, J. Algebra **89** (1984), 178–223.
- [6] ———, *The polynomial identities and invariants of $n \times n$ matrices*, CBMS Regional Conf. Ser. in Math. **78** (1991).
- [7] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.
- [8] L. Le Bruyn, *Trace rings of generic 2×2 matrices*, Mem. Amer. Math. Soc. **363**, Amer. Math. Soc., Providence, RI, 1987.
- [9] C. Procesi, *The invariant theory of $n \times n$ matrices*, Adv. Math. **19** (1976), 306–381.
- [10] C. Procesi, *Computing with 2×2 matrices*, J. of Algebra **87** (1984), 342–359.
- [11] K. S. Sibirskii, *Algebraic invariants for a set of matrices*, Siberian Math. J. **9** (1968), 115–124.
- [12] M. Stillman, *Methods for computing in algebraic geometry and commutative algebra*, Acta Appl. Math., **21** (1990), 77–103.
- [13] Y. Teranishi, *The Hilbert series of rings of matrix concomitants*, Nagoya Math. J. **111** (1988), 143–156.

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STRONGLY RIGID AND I-RIGID RINGS

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Abstract

In this paper we consider questions connected with the problem of rigidity of rings for the class of finite-rank torsion-free rings (A ring is called rigid, if it has only trivial endomorphisms). We study strongly rigid and I-rigid rings [3] (A ring R is called strongly rigid, if any ring quasi-isomorphic to R is rigid). The main results of this paper are the characterization of strongly rigid rings and the establishment of relations between strongly rigid and I-rigid rings. It is shown that the verification of strong rigidity of the ring reduces to testing the rigidity of some field of algebraic numbers over its subfield. This gives rise to examples of non-trivial strongly rigid rings.

1 Introduction

In recent publications a certain interest has appeared in the problem of rigidity of rings, ($[1],[2]$, etc.) and various types of rigidity ($[3],[4]$). (It should be remembered that a ring is called rigid, if it has only the trivial endomorphisms: that is, identity and zero). The problem of investigating

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