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ON $O_m \times GL_n$ HIGHEST WEIGHT VECTORS .

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INTRODUCTION

Let $\mathbb{C}^{m,n}$ be the vector space of $m \times n$ complex matrices and $\mathcal{P}(\mathbb{C}^{m,n})$ be the algebra of complex-valued polynomials on $\mathbb{C}^{m,n}$. Let $GL_m \times GL_n$ act on $\mathcal{P}(\mathbb{C}^{m,n})$ by pre- and post-multiplication as follows:

$$(g_1, g_2)f(x) = f(g_1^{-1}xg_2),$$

where $x \in \mathbb{C}^{m,n}$, $(g_1, g_2) \in GL_m \times GL_n$, $f \in \mathcal{P}(\mathbb{C}^{m,n})$. We choose a system of coordinates on $\mathbb{C}^{m,n}$ as follows:

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}.$$

It is easy to describe the infinitesimal action of the Lie algebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ of $GL_m \times GL_n$ on $\mathcal{P}(\mathbb{C}^{m,n})$. Let

$$\begin{aligned} L_{jk} &= \sum_{s=1}^n x_{js} \frac{\partial}{\partial x_{ks}}, & 1 \leq j, k \leq m, \\ R_{jk} &= \sum_{s=1}^m x_{sj} \frac{\partial}{\partial x_{sk}}, & 1 \leq j, k \leq n. \end{aligned} \tag{1}$$

Then

$$\begin{aligned} \mathfrak{gl}_m &= \text{span}\{L_{jk}, 1 \leq j, k \leq m\}, \\ \mathfrak{gl}_n &= \text{span}\{R_{jk}, 1 \leq j, k \leq n\}. \end{aligned}$$

¹Presenter of the paper at the conference

We have the Cartan decomposition

$$\mathfrak{g}_m = \mathfrak{o}_m \oplus \mathfrak{p},$$

where \mathfrak{o}_m is the Lie algebra of O_m sitting in GL_m , and

$$\begin{aligned} \mathfrak{o}_m &= \text{span}\{L_{jk} - L_{kj}, 1 \leq j < k \leq m\}, \\ \mathfrak{p} &= \text{span}\{L_{jk} + L_{kj}, 1 \leq j \leq k \leq m\}. \end{aligned} \quad (2)$$

Fix a reductive group G and a maximal unipotent subgroup U of G . For any rational representation V of G , let $\mathcal{P}(V)$ be the algebra of polynomial functions on V . The ring of U invariants in $\mathcal{P}(V)$ will be called the ring of G highest weight vectors in $\mathcal{P}(V)$. Our problem can be stated simply as follows:

Problem: Describe the ring $R_{m,n}$ of $O_m \times GL_n$ highest weight vectors in $\mathcal{P}(\mathbb{C}^{m,n})$.

For small m , this is easy.

Proposition 1 (a) $R_{1,n}$ is the polynomial algebra in x_{11} .

(b) $R_{2,1}$ is the polynomial algebra in x_{11} and x_{21} .

(c) If $n \geq 2$, $R_{2,n}$ is freely generated by

$$x_{11}, \quad x_{21}, \quad \text{and} \quad \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}.$$

Let

$$\begin{aligned} \Delta_{jk} &= \sum_{s=1}^m \frac{\partial^2}{\partial x_{sj} \partial x_{sk}}, \quad 1 \leq j \leq k \leq n, \quad \text{and} \\ r_{jk}^2 &= \sum_{s=1}^m x_{sj} x_{sk}, \quad 1 \leq j \leq k \leq n. \end{aligned} \quad (3)$$

Define the space of harmonics

$$\mathcal{H} = \{f \in \mathcal{P}(\mathbb{C}^{m,n}) \mid \Delta_{jk} f = 0, 1 \leq j \leq k \leq n\}, \quad (4)$$

and let

$$\mathcal{I} = \mathcal{P}(\mathbb{C}^{m,n})^{O_m}$$

be the ring of O_m invariants. It is known [6] that

$$\mathcal{P}(\mathbb{C}^{m,n}) = \mathcal{I} \cdot \mathcal{H}. \quad (5)$$

The description of $R_{m,1}$ follow easily from the theory of spherical harmonics. It says that \mathcal{H} is O_m stable and the O_m highest weight vectors in \mathcal{H} (with respect to an appropriate choice of a positive system of roots, which we will elaborate in the next section) are z_1^k for $k = 0, 1, \dots$, where

$$z_1 = x_{11} - i x_{21}.$$

Furthermore, \mathcal{I} is generated by r_{11}^2 as an algebra. The ring of O_m highest weight vectors in $\mathcal{P}(\mathbb{C}^{m,1})$ is thus

$$\mathbb{C}[z_1, r_{11}^2].$$

For $n > 1$, the GL_n -structure of \mathcal{I} and in particular its $O_m \times GL_n$ highest weight vectors are well-known from classical invariant theory (see Theorem 1 in Section 3). In [6], Kashiwara and Vergne generalised the theory of spherical harmonics to give the set of $O_m \times GL_n$ highest weight vectors in the space \mathcal{H} . These give partial information on $R_{m,n}$, and to get a complete picture, one has to understand how the highest weight vectors multiply as in (5) above.

We had determined the ring $R_{m,2}$ and also found a set of generators for $R_{m,3}$, along with some relations. Beyond $n = 3$, the situation gets too complicated. This is understandable, because $R_{m,n}$ contains in particular information on the restriction of GL_m representations (with depth at most $\min(m,n)$) to its subgroup O_m , which is known to be a difficult problem.

HARMONIC POLYNOMIALS

We recall some results from Kashiwara and Vergne [6]. We choose O_m so that its Lie algebra \mathfrak{o}_m is described by (2). Now choose a Cartan subalgebra (different from the one in Section 2) for \mathfrak{o}_m as follows:

$$\mathfrak{h}_L = \left\{ H = \begin{bmatrix} A_1 & & & \\ & A_2 & 0 & \\ & & \ddots & \\ & & & 0 \end{bmatrix}, A_j = \begin{bmatrix} 0 & i h_j \\ -i h_j & 0 \end{bmatrix}, h_j \in \mathbb{C}, 1 \leq j \leq \lfloor \frac{m}{2} \rfloor \right\} \quad (6)$$

and also a Cartan subalgebra for \mathfrak{g}_n :

$$\mathfrak{h}_R = \left\{ \begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & 0 & \\ & & & d_n \end{bmatrix}, d_j \in \mathbb{C} \right\}. \quad (7)$$

Define the following linear functionals on \mathfrak{h}_L :

$$e_j(\text{above } H) = h_j, \quad 1 \leq j \leq \lfloor \frac{m}{2} \rfloor. \quad (8)$$

The roots of \mathfrak{o}_m are

$$\Delta = \begin{cases} \{\pm e_j \pm e_k, j \neq k\} \cup \{\pm e_k\} & \text{if } m \text{ is odd,} \\ \{\pm e_j \pm e_k, j \neq k\} & \text{if } m \text{ is even.} \end{cases} \quad (9)$$

We choose a positive system as follows:

$$\Delta^+ = \begin{cases} \{e_j \pm e_k, j < k\} \cup \{e_k\} & \text{if } m \text{ is odd,} \\ \{e_j \pm e_k, j < k\} & \text{if } m \text{ is even.} \end{cases} \quad (10)$$

For GL_n , we choose the Borel subalgebra of upper triangular matrices to fix a positive root system for \mathfrak{g}_n .

We could use the theory of highest weights to parametrize irreducible finite-dimensional representations of SO_m by tuples of integers $(\alpha_1, \alpha_2, \dots, \alpha_{\lfloor \frac{m}{2} \rfloor})$ satisfying

$$\begin{cases} \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{\lfloor \frac{m}{2} \rfloor} \geq 0 & \text{if } m \text{ is odd,} \\ \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{\lfloor \frac{m}{2} \rfloor} & \text{if } m \text{ is even,} \end{cases}$$

and likewise an irreducible finite-dimensional representation of GL_n by an n -tuple of integers $(\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$.

If $m = 2l+1$, O_{2l+1} is isomorphic to a direct product $SO_{2l+1} \times \mathbb{Z}_2$. A representation of O_{2l+1} can then be parametrized by

$$(\alpha_1, \dots, \alpha_l; \varepsilon) \simeq (\alpha_1, \dots, \alpha_l) \otimes \varepsilon,$$

where $(\alpha_1, \dots, \alpha_l) \in \widehat{SO_{2l+1}}$ and ε is a one-dimensional representation of \mathbb{Z}_2 trivial or nontrivial according to $\varepsilon = 1$ or -1 .

If $m = 2l$, O_{2l} is isomorphic to a semi-direct product $SO_{2l} \rtimes \mathbb{Z}_2$. If $(\alpha_1, \dots, \alpha_l) \in \widehat{SO_{2l}}$ and $\alpha_l \neq 0$, then

$$\text{Ind}_{SO_{2l}}^{O_{2l}}(\alpha_1, \dots, \alpha_l)$$

is irreducible and we denote this by $(\alpha_1, \dots, \alpha_l; 0)$. If $\alpha_l = 0$, then there are two ways to extend $(\alpha_1, \dots, \alpha_l)$ to a representation of O_{2l} , namely

$$(\alpha_1, \dots, \alpha_l; \varepsilon) \simeq (\alpha_1, \dots, \alpha_l) \otimes \varepsilon,$$

where $\varepsilon = \pm 1$. A representation of O_{2l} can then be parametrized by

$$(\alpha_1, \dots, \alpha_l; 0) \quad \text{if } \alpha_l \neq 0,$$

or

$$(\alpha_1, \dots, \alpha_l; \pm 1) \quad \text{if } \alpha_l = 0.$$

Now define

$$z_{jk} = x_{2j-1,k} - ix_{2j,k} \quad (11)$$

for $1 \leq j \leq \lfloor \frac{m}{2} \rfloor$, $1 \leq k \leq n$, and the following $t \times t$ determinants for $1 \leq t \leq \min(\lfloor \frac{m}{2} \rfloor, n)$:

$$\alpha_t = \begin{vmatrix} z_{11} & z_{12} & \dots & z_{1t} \\ \vdots & \vdots & \vdots & \vdots \\ z_{t1} & z_{t2} & \dots & z_{tt} \end{vmatrix}, \quad (12)$$

$$\gamma_t = \begin{vmatrix} r_{11}^2 & r_{12}^2 & \dots & r_{1t}^2 \\ \vdots & \vdots & \vdots & \vdots \\ r_{t1}^2 & r_{t2}^2 & \dots & r_{tt}^2 \end{vmatrix}. \quad (13)$$

It is not difficult to check that with respect to the positive systems for \mathfrak{o}_m and \mathfrak{gl}_n , each α_t is an $O_m \times GL_n$ highest weight vector of weight

$$\underbrace{(1, \dots, 1, 0, \dots, 0)}_{t \text{ copies}}; (-1)^t \otimes \underbrace{(1, \dots, 1, 0, \dots, 0)}_{t \text{ copies}}, \quad (14)$$

and each γ_t is an $O_m \times GL_n$ highest weight vector of weight

$$(0, \dots, 0; 1) \otimes \underbrace{(2, \dots, 2, 0, \dots, 0)}_{t \text{ copies}}. \quad (15)$$

The following results are well known, and can be found in [6] and [8]:

Theorem 1 (a) *The space of O_m -invariant polynomials in $\mathcal{P}(\mathbb{C}^{m,n})$ is generated by r_{jk}^2 , $1 \leq j \leq k \leq n$. The subring of $O_m \times GL_n$ highest weight vectors in \mathcal{I} is generated freely by*

$$\gamma_1, \dots, \gamma_{\min(\lfloor \frac{m}{2} \rfloor, n)}.$$

(b) *Assume that $m \geq 2n$. The ring of $O_m \times GL_n$ highest weight vectors in \mathcal{H} is generated freely by*

$$\alpha_1, \dots, \alpha_n.$$

(c) $\mathcal{P}(\mathbb{C}^{m,n}) = \mathcal{H} \cdot \mathcal{I}$.

Remarks. Analogous results hold for the cases where $m < 2n$ and they can be found in [6]. It turns out that there are more generators in \mathcal{H} for $m < 2n$. In this paper, we will only discuss the cases when $m \geq 2n$ to facilitate exposition. In fact, the cases where $m < 2n$ could be treated in a similar fashion, but it is more messy, and so we will only present the results for $m \geq 2n$.

STRUCTURE OF $R_{m,2}$ AND APPLICATIONS

We have already described $R_{1,2}$ and $R_{2,2}$. We treat the case when $n = 2$ and $m \geq 4$, and simply state the result for $R_{3,2}$. First, we have a simple result about tensor products of GL_2 representations. Let

$$\mathfrak{gl}_2 = \text{span}\{R_{11}, R_{22}, R_{12}, R_{21}\},$$

with the usual commutation relations

$$[R_{ij}, R_{st}] = \delta_{js}R_{it} - \delta_{it}R_{sj}.$$

We will select R_{12} to be the positive root vector in \mathfrak{gl}_2 . As usual, if V is a representation of GL_2 , we say that $v \in V$ is a GL_2 highest weight vector if $R_{12}v = 0$ and v is an eigenvector for R_{11} and R_{22} . The following lemma is easy.

Lemma 1 *Let $V_{(\lambda_1, \lambda_2)}$ be an irreducible finite-dimensional representation of GL_2 with highest weight (λ_1, λ_2) . Then*

$$V_{(\lambda_1, \lambda_2)} \otimes V_{(1,0)} \simeq \begin{cases} V_{(\lambda_1+1, \lambda_2)} \oplus V_{(\lambda_1, \lambda_2+1)} & \text{if } \lambda_1 \neq \lambda_2, \\ V_{(\lambda_1+1, \lambda_2)} & \text{if } \lambda_1 = \lambda_2, \end{cases}$$

and if $\lambda_1 \neq \lambda_2$, the highest weight vector of $V_{(\lambda_1, \lambda_2+1)}$ is given (up to a scalar) by

$$v = R_{21}v_{(\lambda_1, \lambda_2)} \otimes v_{(1,0)} - (\lambda_1 - \lambda_2)v_{(\lambda_1, \lambda_2)} \otimes R_{21}v_{(1,0)},$$

where $v_{(\lambda_1, \lambda_2)}$ and $v_{(1,0)}$ are the highest weight vectors in $V_{(\lambda_1, \lambda_2)}$ and $V_{(1,0)}$, respectively.

From now on, we restrict our discussion to vectors in $\mathcal{P}(\mathbb{C}^{m,n})$. Consider the map

$$\pi : \mathcal{I} \otimes \mathcal{H} \longrightarrow \mathcal{P}(\mathbb{C}^{m,n})$$

by polynomial multiplication. Theorem 1(c) says that the map is onto. To find the $O_m \times GL_n$ representations appearing in $\mathcal{P}(\mathbb{C}^{m,n})$, consider $1 \otimes \sigma_1$ appearing in \mathcal{I} and $\rho \otimes \sigma_2$ appearing in \mathcal{H} . Since

$$(1 \otimes \sigma_1) \otimes (\rho \otimes \sigma_2) \simeq \rho \otimes (\sigma_1 \otimes \sigma_2)$$

as $O_m \times GL_n$ representations, it suffices to understand the projection of $\rho \otimes \tilde{\sigma}$ under π , where $\tilde{\sigma}$ is a GL_n subrepresentation of $\sigma_1 \otimes \sigma_2$.

Let $\langle f \rangle$ denote the GL_2 module in $\mathcal{P}(\mathbb{C}^{m,2})$ generated by the polynomial f . From the discussion above, we are to find the GL_2 highest weight vectors in the tensor product

$$\langle \alpha_1^a \alpha_2^b \rangle \otimes \langle \gamma_1^c \gamma_2^d \rangle.$$

Since γ_2 and α_2 each generates a one-dimensional representation of GL_2 , it suffices to consider

$$\langle \alpha_1^a \rangle \otimes \langle \gamma_1^c \rangle.$$

Observe from Lemma 1 that if v_1 and v_2 are two GL_2 highest weight vectors of weights (λ_1', λ_2') and $(\lambda_1'', \lambda_2'')$, then the highest weight vectors of

$$\langle \alpha_1 \rangle \otimes \langle v_1 v_2 \rangle$$

are $\alpha_1 v_1 v_2$ and

$$\begin{aligned} & (R_{21}(v_1 v_2))\alpha_1 - (\lambda_1' + \lambda_1'' - \lambda_2' - \lambda_2'')v_1 v_2 (R_{21}\alpha_1) \\ &= [(R_{21}v_1)\alpha_1 - (\lambda_1' - \lambda_2')v_1 (R_{21}\alpha_1)]v_2 \\ &+ [(R_{21}v_2)\alpha_1 - (\lambda_1'' - \lambda_2'')v_2 (R_{21}\alpha_1)]v_1. \end{aligned} \quad (16)$$

Now using Lemma 1 on $\langle \alpha_1 \rangle \otimes \langle \gamma_1 \rangle$ we get

$$\beta_1 = \begin{vmatrix} z_{11} & z_{12} \\ r_{11}^2 & r_{12}^2 \end{vmatrix}$$

of $O_m \times GL_2$ weight $(1, 0, \dots, 0; -1) \otimes (2, 1)$. Another application of Lemma 1 on $\langle \alpha_1 \rangle \otimes \langle \beta_1 \rangle$ gives

$$\beta_2 = \begin{vmatrix} 0 & z_{11} & z_{12} \\ z_{11} & r_{11}^2 & r_{12}^2 \\ z_{12} & r_{21}^2 & r_{22}^2 \end{vmatrix}$$

of $O_m \times GL_2$ weight $(2, 0, \dots, 0; 1) \otimes (2, 2)$. Since β_2 transforms under GL_2 by the determinant representation,

$$\langle \alpha_1 \rangle \otimes \langle \beta_2 \rangle = \langle \alpha_1 \beta_2 \rangle.$$

We also have the relation

$$\alpha_1^2 \gamma_2 + \beta_1^2 + \gamma_1 \beta_2 = 0$$

by a direct computation.

Theorem 2 (Roger Howe)

(a) The ring $R_{1,2}$ is the polynomial algebra $\mathbb{C}[x_{11}]$ while $R_{2,2}$ is freely generated by

$$\alpha_1, \quad \bar{\alpha}_1, \quad \text{and} \quad \begin{vmatrix} z_{11} & \bar{z}_{11} \\ z_{12} & \bar{z}_{12} \end{vmatrix}.$$

(b) The ring $R_{3,2}$ is freely generated by α_1, γ_1 and δ where

$$\delta = \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}.$$

(c) If $m \geq 4$, $R_{m,2}$ is generated by $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ and γ_2 with one relation:

$$\alpha_1^2 \gamma_2 + \beta_1^2 + \gamma_1 \beta_2 = 0.$$

Proof. We have already shown (a). We will prove (c). Because

$$\langle \alpha_1^a \rangle \otimes \langle \gamma_1^c \rangle \mapsto \langle \alpha_1 \rangle \otimes (\langle \alpha_1^{a-1} \rangle \otimes \langle \gamma_1^c \rangle),$$

we can use equation (16) and a straightforward induction to conclude that $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ and γ_2 generate $R_{m,2}$. It is not difficult to compute that the Krull dimension of $R_{m,2}$ is 5. It is also simple to see that $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ and γ_2 are algebraically independent. The ideal of relations among $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ and γ_2 will therefore have to be a principal ideal. Since the polynomial

$$\alpha_1^2 \gamma_2 + \beta_1^2 + \gamma_1 \beta_2$$

is irreducible, it generates this ideal. \square

A consequence of the above result is a branching law for O_m representations appearing in a GL_m representation with depth at most two.

Corollary 1 We have the following formula for the restriction of a representation $(a, b, 0, \dots, 0)$ of GL_m to O_m :

$$(a, b, 0, \dots, 0) \Big|_{O_m} = \sum_{\delta \in S} (x_\delta, y_\delta, 0, \dots, 0; (-1)^\epsilon)$$

where

$$\begin{aligned} S = \{ \delta = (a_1, a_2, b_1, b_2, c_1, c_2) \in \mathbb{Z}_+^6 \mid & x_\delta = a_1 + a_2 + b_1 + 2b_2, \quad y_\delta = a_2, \\ & a_1 + a_2 + 2b_1 + 2b_2 + 2c_1 + 2c_2 = a, \quad \epsilon = a_1 + b_1, \\ & a_2 + b_1 + 2b_2 + 2c_2 = b, \quad b_1 = 0, 1 \} \end{aligned}$$

GENERATORS FOR $R_{m,3}$

We proceed to study the case $n = 3$ and $m \geq 6$. As in the previous section we use Theorem 1 to reduce our study to tensor products of the form

$$\langle \alpha_1^a \alpha_2^b \alpha_3^c \rangle \otimes \langle \gamma_1^d \gamma_2^e \gamma_3^f \rangle.$$

Since $\langle \alpha_3 \rangle$ and $\langle \gamma_3 \rangle$ each generates a one-dimensional representation of GL_3 , it suffices to study

$$\langle \alpha_1^a \alpha_2^b \rangle \otimes \langle \gamma_1^c \gamma_2^d \rangle.$$

Because

$$\langle \alpha_1^a \alpha_2^b \rangle \otimes \langle \gamma_1^c \gamma_2^d \rangle \hookrightarrow \langle \alpha_1 \rangle \otimes (\langle \alpha_1^{a-1} \alpha_2^b \rangle \otimes \langle \gamma_1^c \gamma_2^d \rangle)$$

if $a \neq 0$, and

$$\langle \alpha_1^a \alpha_2^b \rangle \otimes \langle \gamma_1^c \gamma_2^d \rangle \hookrightarrow \langle \alpha_2 \rangle \otimes (\langle \alpha_1^a \alpha_2^{b-1} \rangle \otimes \langle \gamma_1^c \gamma_2^d \rangle)$$

if $b \neq 0$, we shall first consider decompositions of the GL_3 modules

$$\langle \alpha_1 \rangle \otimes V \quad \text{and} \quad \langle \alpha_2 \rangle \otimes V$$

for an arbitrary irreducible GL_3 module V .

Let us introduce some notations. Recall our selection of the set of diagonal matrices

$$\mathfrak{h}_R = \left\{ \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \middle| d_j \in \mathbb{C} \right\}$$

as the Cartan subalgebra in \mathfrak{gl}_3 . Define the functionals

$$\varepsilon_j \left(\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \right) = d_j, \quad j = 1, 2, 3.$$

The non-zero roots of \mathfrak{gl}_3 are

$$\Delta = \{ \pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq 3 \},$$

where the root vectors of $\varepsilon_i - \varepsilon_j$ and $-(\varepsilon_i - \varepsilon_j)$ are the matrix units E_{ij} and E_{ji} respectively. These root vectors act on $\mathcal{P}(\mathbb{C}^{m,3})$ as R_{ij} described by (1.1). Select a positive system as follows

$$\Xi = \{ e_1 = \varepsilon_1 - \varepsilon_2, e_2 = \varepsilon_2 - \varepsilon_3 \}.$$

If h is a GL_3 highest weight vector in $\mathcal{P}(\mathbb{C}^{m,3})$, we shall denote by $\langle h \rangle$ the GL_3 module generated by h . If V_1 and V_2 are two GL_3 sub-modules in $\mathcal{P}(\mathbb{C}^{m,3})$, we have the multiplication map

$$\begin{aligned} \Phi : V_1 \otimes V_2 &\longrightarrow \mathcal{P}(\mathbb{C}^{m,3}), \\ \Phi(v_1 \otimes v_2) &= v_1 v_2, \quad v_1 \in V_1, v_2 \in V_2. \end{aligned}$$

If V_{λ_1} and V_{λ_2} are GL_3 modules of highest weight λ_1 and λ_2 , respectively, standard representation theory says that

$$V_{\lambda_1} \otimes V_{\lambda_2} = \sum_{\lambda \in P} m_\lambda V_{\lambda_1 + \lambda_2 - \lambda} = \sum_{\lambda \in P} W_{\lambda_1 + \lambda_2 - \lambda},$$

where P is the set of sums of positive roots (including the zero sum), m_λ is the multiplicity of the representation $V_{\lambda_1 + \lambda_2 - \lambda}$ in the tensor product, and $W_{\lambda_1 + \lambda_2 - \lambda}$ is the $V_{\lambda_1 + \lambda_2 - \lambda}$ -isotypic component. For $\lambda \in P$, let

$$\pi_\lambda : V_{\lambda_1} \otimes V_{\lambda_2} \longrightarrow W_{\lambda_1 + \lambda_2 - \lambda}$$

be the projection onto the $V_{\lambda_1 + \lambda_2 - \lambda}$ -isotypic component, and define another map

$$\begin{aligned} F_\lambda : V_{\lambda_1} \otimes V_{\lambda_2} &\longrightarrow \Phi(W_{\lambda_1 + \lambda_2 - \lambda}), \\ F_\lambda &= \Phi \circ \pi_\lambda. \end{aligned}$$

Thus, if $V_1 = \langle v_1 \rangle$ and $V_2 = \langle v_2 \rangle$, then

$$F_0(v_1 \otimes v_2) = v_1 v_2,$$

and so

$$F_0(V_1 \otimes V_2) = \langle v_1 v_2 \rangle.$$

The following lemma is straightforward.

Lemma 2 Let $V_1 = \langle v_1 \rangle$ and $V_2 = \langle v_2 \rangle$ be irreducible GL_3 modules generated by highest weight vectors v_1 and v_2 of weight $(\lambda_1, \lambda_2, \lambda_3)$ and (μ_1, μ_2, μ_3) respectively. We have the following formulae.

- (a) $F_{e_1}(v_1 \otimes v_2) = (\lambda_1 - \lambda_2)v_1(R_{21}v_2) - (\mu_1 - \mu_2)(R_{21}v_1)v_2,$
- (b) $F_{e_2}(v_1 \otimes v_2) = (\lambda_2 - \lambda_3)v_1(R_{32}v_2) - (\mu_2 - \mu_3)(R_{32}v_1)v_2,$
- (c) $F_{e_1+e_2}(\alpha_1 \otimes v_1) = \alpha_1(R_{21}R_{32}v_1) - (\lambda_2 - \lambda_3)\alpha_1(R_{31}v_1) - (\lambda_1 - \lambda_3 + 1)(R_{21}\alpha_1)(R_{32}v_1) + (\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3 + 1)(R_{31}\alpha_1)v_1,$
- (d) $F_{e_1+e_2}(\alpha_2 \otimes v_1) = \alpha_2(R_{21}R_{32}v_1) - (\lambda_1 - \lambda_2 + 1)\alpha_2(R_{31}v_1) - (\lambda_1 - \lambda_3 + 1)(R_{32}\alpha_2)(R_{21}v_1) + (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3 + 1)(R_{31}\alpha_2)v_1.$

Proposition 2 Let $V_1 = \langle v_1 \rangle$ and $V_2 = \langle v_2 \rangle$ be irreducible GL_3 modules generated by highest weight vectors v_1 and v_2 of weight $(\lambda_1, \lambda_2, \lambda_3)$ and (μ_1, μ_2, μ_3) respectively. We have the following formulae.

- (a) $F_{e_1}(\alpha_1 \otimes (v_1 v_2)) = v_1 F_{e_1}(\alpha_1 \otimes v_2) + v_2 F_{e_1}(\alpha_1 \otimes v_1),$
- (a) $F_{e_2}(\alpha_2 \otimes (v_1 v_2)) = v_1 F_{e_2}(\alpha_2 \otimes v_2) + v_2 F_{e_2}(\alpha_2 \otimes v_1),$
- (c) $F_{e_1+e_2}(\alpha_1 \otimes (v_1 v_2)) = \frac{[F_{e_2}(F_{e_1}(\alpha_1 \otimes v_2) \otimes v_1) + (\lambda_2 + \mu_2 - \lambda_3 - \mu_3 + 1)v_1 F_{e_1+e_2}(\alpha_1 \otimes v_2)]}{(\mu_2 - \mu_3 + 1)} + \frac{[F_{e_2}(F_{e_1}(\alpha_1 \otimes v_1) \otimes v_2) + (\lambda_2 + \mu_2 - \lambda_3 - \mu_3 + 1)v_2 F_{e_1+e_2}(\alpha_1 \otimes v_1)]}{(\lambda_2 - \lambda_3 + 1)},$
- (d) $F_{e_1+e_2}(\alpha_2 \otimes (v_1 v_2)) = \frac{[F_{e_1}(F_{e_2}(\alpha_2 \otimes v_1) \otimes v_2) + (\lambda_1 + \mu_1 - \lambda_2 - \mu_2 + 1)v_2 F_{e_1+e_2}(\alpha_2 \otimes v_1)]}{(\mu_1 - \mu_2 + 1)} + \frac{[F_{e_1}(F_{e_2}(\alpha_2 \otimes v_2) \otimes v_1) + (\lambda_1 + \mu_1 - \lambda_2 - \mu_2 + 1)v_1 F_{e_1+e_2}(\alpha_2 \otimes v_2)]}{(\lambda_1 - \lambda_2 + 1)}.$

Proof. The first two formulae follow directly from Lemma 2. It is not difficult to verify (c) and (d). The difficult part is to believe such a formula holds and then to find it. For full details, please refer to [1]. \square

Define

$$\beta_1 = \begin{vmatrix} z_{11} & z_{12} \\ r_{11}^2 & r_{12}^2 \end{vmatrix}$$

$$\beta_2 = \begin{vmatrix} 0 & z_{11} & z_{12} \\ z_{11} & r_{11}^2 & r_{12}^2 \\ z_{12} & r_{21}^2 & r_{22}^2 \end{vmatrix} \quad \beta_3 = \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ r_{11}^2 & r_{12}^2 & r_{13}^2 \end{vmatrix} \quad \beta_4 = \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ r_{11}^2 & r_{12}^2 & r_{13}^2 \\ r_{21}^2 & r_{22}^2 & r_{23}^2 \end{vmatrix}$$

$$\beta_5 = \begin{vmatrix} z_{11} & 0 & z_{12} & z_{13} \\ z_{21} & 0 & z_{22} & z_{23} \\ 0 & r_{11}^2 & r_{12}^2 & r_{13}^2 \\ 0 & r_{21}^2 & r_{22}^2 & r_{23}^2 \end{vmatrix} \quad \beta_6 = \begin{vmatrix} 0 & z_{11} & z_{12} & z_{13} \\ z_{11} & r_{11}^2 & r_{12}^2 & r_{13}^2 \\ z_{12} & r_{21}^2 & r_{22}^2 & r_{23}^2 \\ z_{13} & r_{31}^2 & r_{32}^2 & r_{33}^2 \end{vmatrix} \quad \beta_7 = \begin{vmatrix} 0 & z_{11} & z_{12} & z_{13} \\ 0 & z_{21} & z_{22} & z_{23} \\ z_{11} & r_{11}^2 & r_{12}^2 & r_{13}^2 \\ z_{12} & r_{21}^2 & r_{22}^2 & r_{23}^2 \end{vmatrix}$$

$$\beta_8 = \begin{vmatrix} z_{11} & z_{21} & 0 & 0 & 0 \\ 0 & 0 & z_{11} & z_{12} & z_{13} \\ 0 & 0 & r_{11}^2 & r_{12}^2 & r_{13}^2 \\ z_{12} & z_{22} & r_{21}^2 & r_{22}^2 & r_{23}^2 \\ z_{13} & z_{23} & r_{31}^2 & r_{32}^2 & r_{33}^2 \end{vmatrix} \quad \beta_9 = \begin{vmatrix} 0 & 0 & z_{11} & z_{21} & z_{13} \\ 0 & 0 & z_{21} & z_{22} & z_{23} \\ z_{11} & z_{22} & r_{11}^2 & r_{12}^2 & r_{13}^2 \\ z_{12} & z_{22} & r_{21}^2 & r_{22}^2 & r_{23}^2 \\ z_{13} & z_{23} & r_{31}^2 & r_{32}^2 & r_{33}^2 \end{vmatrix}$$

The $O_m \times GL_3$ weights of the above polynomials are given in the following table.

	O_m weight	GL_3 weight
α_1	$(1, 0, \dots, 0; -1)$	$(1, 0, 0)$
α_2	$(1, 1, 0, \dots, 0; 1)$	$(1, 1, 0)$
α_3	$(1, 1, 1, 0, \dots, 0; -1)$	$(1, 1, 1)$
γ_1	$(0, \dots, 0; 1)$	$(2, 0, 0)$
γ_2	$(0, \dots, 0; 1)$	$(2, 2, 0)$
γ_3	$(0, \dots, 0; 1)$	$(2, 2, 2)$
β_1	$(1, 0, \dots, 0; -1)$	$(2, 1, 0)$
β_2	$(2, 0, \dots, 0; 1)$	$(2, 2, 0)$
β_3	$(1, 1, 0, \dots, 0; 1)$	$(2, 1, 1)$
β_4	$(1, 0, \dots, 0; -1)$	$(2, 2, 1)$
β_5	$(1, 1, 0, \dots, 0; 1)$	$(3, 2, 1)$
β_6	$(2, 0, \dots, 0; 1)$	$(2, 2, 2)$
β_7	$(2, 1, 0, \dots, 0; -1)$	$(2, 2, 1)$
β_8	$(2, 1, 0, \dots, 0; -1)$	$(3, 2, 2)$
β_9	$(2, 2, 0, \dots, 0; 1)$	$(2, 2, 2)$

Theorem 3 If $m \geq 6$, the ring of $O_m \times GL_3$ highest weight vectors in $\mathcal{P}(\mathbb{C}^{m,3})$ is generated by

$$\alpha_i, \quad i = 1, 2, 3, \quad \beta_j, \quad j = 1, \dots, 9, \quad \gamma_k, \quad k = 1, 2, 3.$$

Remarks. We also have an (incomplete) set of relations:

- (1) $\beta_1^2 + \beta_2\gamma_1 + \alpha_1^2\gamma_2 = 0$,
- (2) $\beta_4^2 + \beta_6\gamma_2 - \beta_2\gamma_3 = 0$,
- (3) $\beta_5^2 + \alpha_2^2\gamma_1\gamma_3 + \beta_3^2\gamma_2 - \beta_9\gamma_1\gamma_2 = 0$,
- (4) $\beta_7^2 + \beta_2\beta_9 - \alpha_2^2\beta_6 = 0$,
- (5) $\beta_8^2 + \alpha_1^2\beta_9\gamma_3 + \beta_6\beta_9\gamma_1 - \beta_3^2\beta_6 = 0$,
- (6) $\beta_1\beta_3 - \beta_7\gamma_1 - \alpha_1\beta_5 = 0$.

These relations are some of those that were found using computer algebra software [2] and [9] and the methods of [7].

Proof. By Proposition 2, it suffices to find a set of highest weights S that are closed under F_{e_1} and F_{e_2} , i.e., $x, y \in S$ implies that $F_{e_i}(x \otimes y)$ is generated by elements in S . This can be done, after some tedious computations (see [1]). \square

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