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K-Theory for the Integer Heisenberg Groups

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Abstract. We give a closed formula for topological K-theory of the homogeneous space N/Γ, where Γ is the standard integer lattice in the simply connected Heisenberg Lie group N of dimension $2n+1$, $n \in \mathbb{Z}^+$. The main tools in our calculations are obtained by computing diagonal forms for certain incidence matrices that arise naturally in combinatorics.

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1. Introduction

Let n be a natural number and let Γ denote the standard integer lattice of the $(2n+1)$ dimensional simply connected Heisenberg Lie group N. Both of these groups together with other discrete subgroups of N occur in a wide variety of situations in representation theory, geometry and mathematical physics. In this paper we will compute a closed form of the topological K-groups associated to Γ, that is, we compute the topological K-groups for the classifying space N/Γ associated to Γ. Our main tool in our calculations are certain incidence matrices that arise naturally in combinatorics, and in the course of computing the K-groups for N/Γ, we are able to find diagonal forms for these incidence matrices that are of independent interest.

The tools for computing the ranks of the K-groups have been available for some time [10]. In particular, it is well known that for any finite CW complex X, the Chern characters

$$\mathrm{ch}\colon K^0(X) \otimes \mathbb{Q} \to H^{\mathrm{even}}(X, \mathbb{Q})$$

and

$$\mathrm{ch}\colon K^1(X) \otimes \mathbb{Q} \to H^{\mathrm{odd}}(X, \mathbb{Q})$$

are bijective, so that the ranks of the K-groups of N/Γ will satisfy

$$\operatorname{rank} K^i(N/\Gamma) = \sum_{j=0} \beta_{2j+i}, \quad i = 0, 1, \tag{1.1}$$

where β_k is the kth Betti number of N/Γ. The last two authors [10] recently used Equation (1.1) together with formulas for the Betti numbers available from the work of Dupré [3] and Howe [5] to compute

$$\operatorname{rank} K^i(N/\Gamma) = \binom{2n+1}{n}, \quad i = 0, 1,$$

for any lattice D (including Γ) in N. However, in [10] no general formula for the torsion subgroups of the K-groups was given. In this paper, by using the combinatorial methods outlined above, we are able to obtain precise formulas for these torsion groups of $K^*(N/\Gamma)$. This serves as a natural complement to another recent paper of the last two authors [11] in which the cohomology groups for N/Γ with coefficients in the integers were computed. In particular we will prove the following theorem.

THEOREM 3.9. *Let $n \in \mathbb{Z}^+$ and let Γ be the standard integer lattice in the $(2n+1)$-dimensional simply connected Heisenberg Lie group N. The topological K-groups for N/Γ are given by*

$$K^0(N/\Gamma) \cong \bigoplus_{r=0}^{n} H^{2r}(N/\Gamma, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-1-2k}}, & n \text{ odd}, \\ \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{n/2} \mathbb{Z}_k^{2\binom{2n}{n-2k}}, & n \text{ even}, \end{cases}$$

$$K^1(N/\Gamma) \cong \bigoplus_{r=0}^{n} H^{2r+1}(N/\Gamma, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_k^{2\binom{2n}{n-2k}}, & n \text{ odd}, \\ \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-2)/2} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-1-2k}}, & n \text{ even}. \end{cases}$$

These formulas can also be written as

$$K^i(N/\Gamma) \cong \begin{cases} \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{[(n-1)/2]} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-1-2k}}, & n - i \equiv 1 \mod 2, \\ \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{[n/2]} \mathbb{Z}_k^{2\binom{2n}{n-2k}}, & n - i \equiv 0 \mod 2. \end{cases}$$

From Theorem 3.9 we can immediately obtain the following closed formula for $K^*(N/\Gamma) \cong K^0(N/\Gamma) \oplus K^1(N/\Gamma)$ that is valid for all $n \in \mathbb{Z}^+$.

COROLLARY 3.10. *Let $n \in \mathbb{Z}^+$ and let Γ denote the integer Heisenberg group of rank $2n+1$ sitting inside the Heisenberg Lie group N. As an Abelian group,*

$$K^*(N/\Gamma) \cong \bigoplus_{k=0}^{2n+1} H^k(N/\Gamma, \mathbb{Z}) \cong \bigoplus_{k=0}^{[(n+1)/2]} \mathbb{Z}_k^{\binom{2n+2}{n+1-2k}}.$$

We remark that the above formulae, in addition to being of interest to topologists and group theorists, are also of interest in C^*-algebra theory. In particular, by using the methods of the third author and I. Raeburn [13], we are able to use Theorem 3.9 to compute the K-groups of *any* twisted group C^*-algebra $C^*(\Gamma, \sigma)$ corresponding to any multiplier σ defined on the integer Heisenberg group Γ, which we do in Corollary 3.11.

Theorem 3.9 and Corollary 3.10 suggest the interesting possibility that for N/Γ as above, we might be able to directly define a ring isomorphism

$$\mathrm{ch}\colon K^*(N/\Gamma) \xrightarrow{\cong} H^*(N/\Gamma, \mathbb{Z}),$$

without having to tensor over $\mathbb{Q}$. We leave this conjecture as an open problem.

As in our computation of the cohomology groups of N/Γ, the method used in the proof of Theorem 3.9 will be the Gysin exact sequence [8], applied to our situation by viewing N/Γ as a principal $\mathbb{T}$ bundle over $\mathbb{T}^{2n}$. In principle, the K-theoretic version of this sequence has been available for over 30 years to allow the computation of the K-groups of N/Γ; however, the main technical problem that needs to be overcome is the calculation of the cokernels in the connecting maps of the sequence. As in [11], we find explicit forms for these maps using combinatorial methods. In particular, we are able to write out the connecting maps as direct sums of certain combinatorial incidence matrices related to the work of Wilson [17]. We have called these matrices *proper inclusion incidence matrices*, which we define as follows:

DEFINITION 1.1. Let $\nu \in \mathbb{Z}^+ \cup \{0\}$ and let X be a ν-set. The proper inclusion incidence matrix $P(\nu)$ is defined to be the $2^\nu \times 2^\nu$ $\{0,1\}$-matrix whose rows and columns are indexed by all subsets I and J of X, and where

$$P(\nu)_{IJ} = P_{IJ} = \begin{cases} 1, & \text{if } J \subsetneq I, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

Recall that a $m\times n$ integer diagonal matrix D is called a *diagonal form* for the $m\times n$ matrix Q if we can reduce Q to D by elementary row and column operations. This is equivalent to finding matrices E and F in $\mathrm{GL}(m,\mathbb{Z})$ and $\mathrm{GL}(n,\mathbb{Z})$, respectively, such that $D = EQF$. By applying results of Wilson [17], we are able to compute two explicit diagonal forms for $P(\nu)$, one of which we mention here.

THEOREM 2.7. *Let $\nu \in \mathbb{Z}^+$, and let X be a ν-set. The $2^\nu \times 2^\nu$ proper inclusion incidence matrix $P(\nu)$ has as a diagonal form the $2^\nu \times 2^\nu$ diagonal matrix $D(\nu)$ with the following entries*:

1 *with multiplicity* $2^{\nu-1}$ *and* $k(k+1)$ *with multiplicity* $\binom{\nu}{(\nu-1)/2-k}$ *for* $0 \leqslant k \leqslant (\nu-1)/2$ *when* ν *is odd,*
1 *with multiplicity* $2^{\nu-1}$ *and* $k(k+1)$ *with multiplicity* $2\binom{\nu-1}{(\nu-2)/2-k}$ *for* $0 \leqslant k \leqslant (\nu-2)/2$ *when* ν *is even.*

We study the proper inclusion incidence matrix $P(\nu)$ in Section 2, where we establish its relationship to the incidence matrices studied by Wilson, thus enabling us to verify the above and other diagonal forms for $P(\nu)$. In our third section, we use these diagonal forms to read off the kernels and cokernels of the connecting maps in the Gysin sequence for $K^*(N/\Gamma)$, and thus establish our main theorem.

2. Diagonal Forms for the Proper Inclusion Incidence Matrix $P(\nu)$

In this section we study the proper inclusion incidence matrix $P(\nu)$ defined in the introduction and clarify its relationship to certain incidence matrices that have appeared widely in the combinatorial literature, most recently in work of Wilson [17].

For fixed $\nu \in \mathbb{Z}^+ \cup \{0\}$, let X be a ν-set, which we shall write as $\emptyset$ for $\nu = 0$ and as $\{1, 2, \ldots, \nu\}$ for $\nu \geqslant 1$. Recall (Definition 1.1) that the proper inclusion incidence matrix $P(\nu)$ is defined to be the $2^\nu \times 2^\nu$ $\{0, 1\}$-matrix whose rows and columns are indexed by all subsets I and J of X, and where

$$P(\nu)_{IJ} = P_{IJ} = \begin{cases} 1, & \text{if } J \subsetneq I, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

So for example, by suitably ordering the set of subsets of X for $\nu = 0$, 1, and 2, we get

$$P(0) = (0), \qquad P(1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Our main aim in this section is to establish a similarity between $P(\nu)$ and another incidence matrix, $L(\nu)$, for which diagonal forms can easily be computed using results from [17].

DEFINITION 2.1. Let $\nu \in \mathbb{Z}^+ \cup \{0\}$, and let X be a ν-set. The matrix $L(\nu)$ is defined to be the $2^\nu \times 2^\nu$ $\{0, 1\}$-matrix whose rows and columns are indexed by subsets I and J of X, and where

$$L(\nu)_{IJ} = L_{IJ} = \begin{cases} 1, & \text{if } J \subset I \text{ and } |I| = |J| + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

As in the case of $P(\nu)$ we can calculate

$$L(0)=(0),\qquad L(1)=\begin{pmatrix}0&0\\1&0\end{pmatrix}\quad\text{and}\quad L(2)=\begin{pmatrix}0&0&0&0\\1&0&0&0\\1&0&0&0\\0&1&1&0\end{pmatrix}.$$

We now wish to give $P(\nu)$ and $L(\nu)$ block decompositions where the block entries are variants of incidence matrices studied by Wilson in [17]. We first recall Wilson's notation.

DEFINITION 2.2. Let $\nu\in\mathbb{Z}^+\cup\{0\}$, let X be a ν-set, and let $0\leqslant t\leqslant k\leqslant\nu$. The Wilson incidence matrix $W_{tk}(\nu)$ is defined to be the $\binom{\nu}{t}\times\binom{\nu}{k}$ $\{0,1\}$-matrix whose rows are indexed by the t-sets T of X, whose columns are indexed by the k-sets K of X, and where

$$W_{tk}(\nu)_{TK}=W_{TK}=\begin{cases}1, & \text{if } T\subset K,\\ 0, & \text{otherwise.}\end{cases}\tag{2.3}$$

By partitioning the subsets of our ν-set X into sets of order $0,1,\dots,\nu$, we can give $L(\nu)$ and $P(\nu)$ block decompositions involving the $W_{tk}(\nu)$. Indeed, we easily check that

$$L(\nu)=(L_{ij}(\nu)),\qquad 0\leqslant i,j\leqslant\nu,$$

where each $L_{ij}(\nu)$ is a $\binom{\nu}{i}\times\binom{\nu}{j}$ matrix defined by

$$L_{ij}(\nu)=\begin{cases}(W_{j,j+1}(\nu))^t, & \text{if } i=j+1 \quad\text{and}\quad 0\leqslant j\leqslant\nu-1,\\ 0\left(\binom{\nu}{i}\times\binom{\nu}{j}\right), & \text{otherwise,}\end{cases}\tag{2.4}$$

and

$$P(\nu)=(P_{ij}(\nu)),\qquad 0\leqslant i,j\leqslant\nu,$$

where P_{ij} is the $\binom{\nu}{i}\times\binom{\nu}{j}$ matrix given by

$$P_{ij}(\nu)=\begin{cases}(W_{j,i}(\nu))^t, & 0\leqslant j<i\leqslant\nu,\\ 0\left(\binom{\nu}{i}\times\binom{\nu}{j}\right), & 0\leqslant i\leqslant j\leqslant\nu.\end{cases}\tag{2.5}$$

The decompositions (2.4) and (2.5) are obtained from the definitions of $L(\nu)$, $P(\nu)$ and $W_{tk}(\nu)$ by indexing the rows and columns of $L_{ij}(\nu)$ and $P_{ij}(\nu)$ by the i-sets of X and the j-sets of X, respectively.

PROPOSITION 2.3. *Let* $\nu\in\mathbb{Z}^+\cup\{0\}$. *We have* $P(\nu)=\exp(L(\nu))-I_{2^\nu}$.

Proof. By formulas given in [17, Equation (3.1)], we know that for $0 \leqslant j \leqslant t \leqslant k \leqslant \nu$, we have

$$W_{jt}(\nu)W_{tk}(\nu) = \binom{k-j}{t-j} W_{jk}(\nu). \tag{2.6}$$

By using formulas (2.4) and (2.6), we easily verify that for $1 \leqslant k \leqslant \nu$, the powers $L(\nu)^k$ have a block decomposition

$$L(\nu)^k = (L^k_{ij}(\nu)), \qquad 0 \leqslant i \leqslant j \leqslant \nu,$$

where

$$L^k_{ij}(\nu) = \begin{cases} k!W_{j,j+k}(\nu)^t, & \text{if } i - j = k, \\ 0\left(\binom{\nu}{i} \times \binom{\nu}{j}\right), & \text{otherwise.} \end{cases} \tag{2.7}$$

Thus $L(\nu)^k/k!$ has as its only nonzero blocks the matrices

$$\{\, W_{j,j+k}(\nu)^t \mid 0 \leqslant j \leqslant \nu - k \,\},$$

which occur along its kth block subdiagonal, and $L(\nu)^{\nu+1} = 0$, that is, $L(\nu)$ is nilpotent of degree $\nu + 1$. Therefore the matrix

$$\exp(L(\nu)) - I = \sum_{k=0}^{\infty} L(\nu)^k/k! - I = \sum_{k=0}^{\nu} L(\nu)^k/k! - I$$

has at its block decomposition M_{ij}, $0 \leqslant i, j \leqslant \nu$, where

$$M^{(\nu)}_{ij} = \begin{cases} W_{ji}(\nu)^t, & 0 \leqslant j < i \leqslant \nu, \\ 0\left(\binom{\nu}{i} \times \binom{\nu}{j}\right), & 0 \leqslant i \leqslant j \leqslant \nu. \end{cases} \tag{2.8}$$

Comparing (2.8) with (2.5) we get $P(\nu) = \exp(L(\nu)) - I$ as desired. □

We now wish to prove that the proper inclusion incidence matrix $P(\nu)$ is similar to $L(\nu)$ via a unimodular matrix. Since diagonal forms for $L(\nu)$ can easily be obtained by using formulas available in [17], this will enable us to obtain the desired diagonal forms for $P(\nu)$. We now partition the rows and columns of $P(\nu)$ and $L(\nu)$, that is, the subsets of $X = \{1, \ldots, \nu\}$ according to whether or not they contain the element ν. In this way we obtain the following useful block decompositions of $L(\nu)$ and $P(\nu)$.

LEMMA 2.4. *Let* $\nu \geqslant 1$. *We have the following block decomposition of* $L(\nu)$ *and* $P(\nu)$ *into four* $2^{\nu-1} \times 2^{\nu-1}$ *subblocks:*

$$L(\nu) = \begin{pmatrix} L(\nu-1) & 0 \\ I & L(\nu-1) \end{pmatrix}, \tag{2.9}$$

$$P(\nu) = \begin{pmatrix} P(\nu-1) & 0 \\ P(\nu-1)+I & P(\nu-1) \end{pmatrix}. \tag{2.10}$$

Proof. Let the first $2^{\nu-1}$ rows and columns for $L(\nu)$ and $P(\nu)$ consist of all subsets of $\{1,\ldots,\nu-1\}$, and let the second 2^{n-1} rows and columns consist of all subsets of $\{1,\ldots,\nu\}$ that contain ν. Giving $L(\nu)$ and $P(\nu)$ the associated block decompositions and using the defining Equations (2.2) and (2.1) for $L(\nu)$ and $P(\nu)$, respectively, we obtain the decompositions (2.9) and (2.10). □

Before proving the similarity of $L(\nu)$ and $P(\nu)$, we need one final lemma.

LEMMA 2.5. *Let $\nu \in \mathbb{Z} \cup \{0\}$ and let $L(\nu)$ be the $2^\nu \times 2^\nu$ $\{0,1\}$-matrix with block decomposition given by (2.4). There is then a $2^\nu \times 2^\nu$ integer matrix $X(\nu)$ that satisfies*

$$[X(\nu), L(\nu)] = X(\nu)L(\nu) - L(\nu)X(\nu) = L(\nu).$$

Proof. For $\nu = 0$, take $X(\nu) = 0$. For $\nu > 0$, again partition the subsets of $\{1,\ldots,n\}$ into sets of order $0 \leqslant i \leqslant \nu$. This gives $X(\nu)$ a block decomposition

$$X(\nu) = (X_{ij}(\nu)), \qquad 0 \leqslant i, j \leqslant \nu,$$

where $X_{ij}(\nu)$ is the integer $\binom{\nu}{i} \times \binom{\nu}{j}$ matrix defined by

$$X_{ij}(\nu) = \begin{cases} (i+1)I_{\binom{\nu}{i}\times\binom{\nu}{j}}, & i = j, \quad 0 \leqslant i \leqslant n, \\ 0\left(\binom{\nu}{i} \times \binom{\nu}{j}\right), & i \neq j, \quad 0 \leqslant i, j \leqslant \nu. \end{cases} \tag{2.11}$$

Then we calculate that $X(\nu)L(\nu)$ has a block decomposition

$$(X(\nu)L(\nu))_{ij} = \begin{cases} (j+1)W_{j,j+1}(\nu)^t, & i = j+1, \quad 0 \leqslant j \leqslant \nu - 1, \\ 0\left(\binom{\nu}{i} \times \binom{\nu}{j}\right), & i \neq j, \quad 0 \leqslant i, j \leqslant \nu. \end{cases} \tag{2.12}$$

Similarly, $L(\nu)X(\nu)$ has a block decomposition

$$(L(\nu)X(\nu))_{ij} = \begin{cases} jW_{j,j+1}(\nu)^t, & i = j+1, \quad 0 \leqslant j \leqslant \nu - 1, \\ 0\left(\binom{\nu}{i} \times \binom{\nu}{j}\right), & i \neq j+1, \quad 0 \leqslant i, j \leqslant \nu. \end{cases} \tag{2.13}$$

Thus clearly $X(\nu)L(\nu) - L(\nu)X(\nu) = L(\nu)$ as desired. □

We are now in position to prove the similarity over $M_n(\mathbb{Z})$ of $P(\nu)$ and $L(\nu)$.

PROPOSITION 2.6. *Let $\nu \in \mathbb{Z}^+ \cup \{0\}$. There is a matrix $U(\nu) \in \mathrm{SL}(2^\nu, \mathbb{Z})$ such that $P(\nu) = U(\nu)^{-1}L(\nu)U(\nu)$.*

Proof. We prove the Lemma by induction on ν. The statement is clearly true for $\nu = 0$, since in this case $L(0) = P(0) = (0)$. Assuming that the statement is true for $\nu = k$, we prove it to be true for $\nu = k+1$. Find $U(k) \in \mathrm{SL}(2^k, \mathbb{Z})$ such that $U(k)^{-1}L(k)U(k) = P(k)$ and $X(k)$ as in Lemma 2.5 such that $[X(k), L(k)] = L(k)$. Taking $\nu = k+1$ and giving $L(k+1)$ and $P(k+1)$ the block decompositions from Lemma 2.4 we have

$$L(k+1) = \begin{pmatrix} L(k) & 0 \\ I & L(k) \end{pmatrix} \quad \text{and} \quad P(k+1) = \begin{pmatrix} P(k) & 0 \\ P(k)+I & P(k) \end{pmatrix}.$$

We now define

$$U(k+1) = \begin{pmatrix} U(k) & 0 \\ -X(k)U(k) & U(k) \end{pmatrix}. \tag{2.14}$$

We easily see that $U(k+1) \in \mathrm{SL}(2^{k+1}, \mathbb{Z})$ with

$$U(k+1)^{-1} = \begin{pmatrix} U(k)^{-1} & 0 \\ U(k)^{-1}X(k) & U(k)^{-1} \end{pmatrix}, \tag{2.15}$$

so that

$$\begin{aligned}
&U(k+1)^{-1}L(k+1)U(k+1) \\
&\quad = \begin{pmatrix} U(k)^{-1} & 0 \\ U(k)^{-1}X(k) & U(k)^{-1} \end{pmatrix} \begin{pmatrix} L(k) & 0 \\ I & L(k) \end{pmatrix} \begin{pmatrix} U(k) & 0 \\ -X(k)U(k) & U(k) \end{pmatrix} \\
&\quad = \begin{pmatrix} U(k)^{-1}L(k)U(k) & 0 \\ U(k)^{-1}X(k)L(k)U(k) + I - U(k)^{-1}L(k)X(k)U(k) & U(k)^{-1}L(k)U(k) \end{pmatrix} \\
&\quad = \begin{pmatrix} P(k) & 0 \\ U(k)^{-1}(X(k)L(k) - L(k)X(k))U(k) + I & P(k) \end{pmatrix} \\
&\quad = \begin{pmatrix} P(k) & 0 \\ U(k)^{-1}L(k)U(k) + I & P(k) \end{pmatrix} = \begin{pmatrix} P(k) & 0 \\ P(k)+I & P(k) \end{pmatrix} = P(k+1).
\end{aligned}$$

Thus the result is true for $\nu = k+1$, and by induction, it is true for all $\nu \in \mathbb{Z}^+ \cup \{0\}$. □

Equation (2.7) shows that $L(\nu)^k$ has a subdiagonal block decomposition easily described in terms of the Wilson incidence matrices $W_{j,j+k}(\nu)^t$, $0 \leqslant j \leqslant \nu - k$. Wilson has established diagonal forms for the $W_{j,j+k}(\nu)$ in [17, Theorem 2]. Thus we obtain diagonal forms for $L(\nu)^k$ and $P(\nu)^k$ by summing the diagonal forms for $W_{j,j+k}(\nu)^t$, $0 \leqslant j \leqslant \nu - k$ and applying Proposition 2.6. This leads to the following Theorem. (We give another diagonal form for $P(\nu)$ in Theorem 2.10.)

THEOREM 2.7. *Let $\nu \in \mathbb{Z}^+$, and let X be a ν-set. The $2^\nu \times 2^\nu$ proper inclusion incidence matrix $P(\nu)$ has as a diagonal form the $2^\nu \times 2^\nu$ diagonal matrix $D(\nu)$ with the following entries*:

1 *with multiplicity* $2^{\nu-1}$ *and* $k(k+1)$ *with multiplicity* $\binom{\nu}{(\nu-1)/2-k}$ *for* $0 \leqslant k \leqslant (\nu-1)/2$ *when* ν *is odd,*
1 *with multiplicity* $2^{\nu-1}$ *and* $k(k+1)$ *with multiplicity* $2\binom{\nu-1}{(\nu-2)/2-k}$ *for* $0 \leqslant k \leqslant (\nu-2)/2$ *when* ν *is even.*

Before beginning the proof of Theorem 2.7 we establish some definitions and notation and recall some basic algebraic facts. If M_1 and M_2 are two $n \times m$ integer matrices, we will say that M_1 is equivalent to M_2, written $M_1 \sim M_2$, if there are matrices $E \in \mathrm{GL}(n, \mathbb{Z})$ and $F \in \mathrm{GL}(m, \mathbb{Z})$ such that $EM_1F = M_2$. To say that $M_1 \sim M_2$ is exactly the same as saying that M_1 can be transformed into M_2 by a sequence of elementary row and column operations. For any $m \times n$ integer matrix M we can always find a matrix D whose only non-zero entries are along the diagonal with $M \sim D$. In fact, we can find a unique D with

$$D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ & & d_k & \\ 0 & & & 0 \end{pmatrix},$$

where $k \leqslant \min(m, n)$ and $1 \leqslant d_1|d_2|\dots|d_k$. This is exactly the *Smith normal form* for M [6, Theorem 3.9]. Clearly if $M_1 \sim M_2$ and D is a diagonal form for M_1, then D is also a diagonal form for M_2. The benefit of finding a diagonal form for M is that if we think of M as a map $M\colon \mathbb{Z}^n \to \mathbb{Z}^m$, then we can then easily read off that

$$\operatorname{coker} M = \bigoplus_{i=1}^{k} \mathbb{Z}_{d_k} \oplus \mathbb{Z}^{m-k}.$$

If $\{ d_i \mid 1 \leqslant i \leqslant k \}$ is a sequence of integers, we will denote the $k \times k$ diagonal matrix

$$\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_k \end{pmatrix}$$

by $\bigoplus_{i=1}^{k}[d_i]$. For a fixed integer k and $k \in \mathbb{Z}^+$, we let $[d]^k$ denote the $k \times k$ matrix

$$\begin{pmatrix} d & & 0 \\ & \ddots & \\ 0 & & d \end{pmatrix}.$$

Finally, if D_1 is a $m \times m$ matrix and D_2 is a $n \times n$ matrix, $D_1 \oplus D_2$ will denote the $(m+n) \times (m+n)$ matrix

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.$$

We now give the proof of Theorem 2.7.

Proof of Theorem 2.7. For $\nu = 1$,

$$P(1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

so the formula in the theorem holds for this case. We now assume that $\nu \geqslant 2$. By Lemma 2.4, we know that

$$P(\nu) = \begin{pmatrix} P(\nu-1) & 0 \\ P(\nu-1)+I & P(\nu-1) \end{pmatrix},$$

and by performing elementary row and column operations, we see that the right-hand matrix is equivalent to

$$\begin{aligned} P(\nu) &\sim \begin{pmatrix} P(\nu-1) & 0 \\ I & P(\nu-1) \end{pmatrix} \\ &\sim \begin{pmatrix} I & P(\nu-1) \\ P(\nu-1) & 0 \end{pmatrix} \sim \begin{pmatrix} I & P(\nu-1) \\ 0 & P(\nu-1)^2 \end{pmatrix} \sim \begin{pmatrix} I & 0 \\ 0 & P(\nu-1)^2 \end{pmatrix}. \end{aligned}$$

Therefore, $P(\nu)$ has the same diagonal form as $I_{2^{\nu-1}\times 2^{\nu-1}} \oplus P(\nu-1)^2$. By Equation (2.7), we have

$$\begin{aligned} P(\nu-1)^2 &\sim L(\nu-1)^2 \\ &= \begin{pmatrix} 0 & & & & & 0 \\ 0 & 0 & & & & \\ 2W_{0,2}(\nu-1)^t & 0 & \ddots & & & \\ & 2W_{1,3}(\nu-1)^t & \ddots & 0 & & \\ & & \ddots & 0 & 0 & \\ 0 & & & 2W_{\nu-3,\nu-1}(\nu-1)^t & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.16}$$

(Recall that the block decomposition for $L(\nu-1)^2$ is obtained by partitioning the subsets of $\{1, \ldots, \nu-1\}$ into sets of order $0, \ldots, \nu-1$.)

By moving the top $\binom{\nu-1}{0} + \binom{\nu-1}{1}$ rows of the matrix in (2.16) to the bottom of the matrix, we see that this matrix is equivalent to the matrix

$$\begin{array}{c} \\ \binom{\nu-1}{2} \\ \binom{\nu-1}{3} \\ \vdots \\ \binom{\nu-1}{\nu-1} \\ \binom{\nu-1}{0} \\ \binom{\nu-1}{1} \end{array} \begin{array}{c} \begin{array}{cccccc} \binom{\nu-1}{0} & \binom{\nu-1}{1} & \cdots & \binom{\nu-1}{\nu-3} & \binom{\nu-1}{\nu-2} & \binom{\nu-1}{\nu-1} \end{array} \\ \begin{pmatrix} 2W_{0,2}(\nu-1)^t & & & & & 0 \\ & 2W_{1,3}(\nu-1)^t & & & & \\ & & \ddots & & & \\ & & & 2W_{\nu-3,\nu-1}(\nu-1)^t & & \\ & & & & 0 & \\ 0 & & & & & 0 \end{pmatrix} \end{array}. \tag{2.17}$$

Thus in order to find a diagonal form for $P(\nu-1)$, it will suffice to find diagonal forms for each of the block matrices along the diagonal above, and then sum these diagonal forms together. We note that diagonal forms for the matrices $2W_{j,j+2}(\nu-1)^t$, $0 \leqslant j \leqslant \nu-3$, can be computed by applying [17, Theorem 2]. In particular, for $j+(j+2) \leqslant \nu-1$, we have $\binom{\nu-1}{j} \leqslant \binom{\nu-1}{j+2}$ and the $\binom{\nu-1}{j+2} \times \binom{\nu-1}{j}$ matrix $2W_{j,j+2}(\nu-1)^t$ is equivalent to the matrix

$$\begin{array}{c} \\ \binom{\nu-1}{j} \\ \binom{\nu-1}{j+2}-\binom{\nu-1}{j} \end{array} \begin{array}{c} \binom{\nu-1}{j} \\ \left(\begin{array}{c} \bigoplus_{k=1}^{j+1} [k(k+1)]^{\binom{\nu-1}{j+1-k}-\binom{\nu-1}{j-k}} \\ 0 \end{array} \right) \end{array}. \tag{2.18}$$

For $j+(j+2) > \nu-1$ we have $\binom{\nu-1}{j} > \binom{\nu-1}{j+2}$, and we can easily check by taking complements in the $\nu-1$ set that $2W_{j,j+2}(\nu-1)^t = 2W_{\nu-3-j,\nu-1-j}(\nu-1)$. Applying [17, Theorem 2] again, we get that $2W_{j,j+2}(\nu-1)^t$ is equivalent to the matrix

$$\begin{array}{c} \\ \binom{\nu-1}{\nu-3-j} \end{array} \begin{array}{c} \begin{array}{cc} \binom{\nu-1}{\nu-3-j} & \binom{\nu-1}{\nu-1-j}-\binom{\nu-1}{\nu-3-j} \end{array} \\ \left(\begin{array}{cc} \bigoplus_{k=1}^{\nu-2-j} [k(k+1)]^{\binom{\nu-1}{\nu-2-j-k}-\binom{\nu-1}{\nu-3-j-k}} & 0 \end{array} \right) \end{array}. \tag{2.19}$$

We now sum up these diagonal forms, keeping in mind that interchanging the order of summation is the same as permuting our basis elements and therefore preserves similarity. Recall that we have the cases $2j+2 \leqslant \nu-1$ and $2j+2 > \nu-1$ to consider. Thus we consider the case of ν odd and even separately.

For ν odd, $\nu-1$ is even, and $2j+2 \leqslant \nu-1$ exactly for $j \leqslant (\nu-3)/2$. Therefore

$$\begin{aligned} P(\nu-1)^2 \sim\ & \bigoplus_{j=0}^{(\nu-3)/2} \bigoplus_{k=1}^{j+1} [k(k+1)]^{\binom{\nu-1}{j+1-k}-\binom{\nu-1}{j-k}} \\ & \oplus \bigoplus_{j=(\nu-1)/2}^{\nu-3} \bigoplus_{k=1}^{\nu-2-j} [k(k+1)]^{\binom{\nu-1}{\nu-2-j-k}-\binom{\nu-1}{\nu-3-j-k}} \\ & \oplus \bigoplus_{j=(\nu-1)/2}^{\nu-3} [0]^{\binom{\nu-1}{\nu-1-j}-\binom{\nu-1}{\nu-3-j}} \oplus [0]^{\binom{\nu-1}{\nu-2}} \oplus [0]^{\binom{\nu-1}{\nu-1}} \\ \sim\ & \bigoplus_{j=0}^{(\nu-3)/2} \bigoplus_{k=1}^{j+1} [k(k+1)]^{\binom{\nu-1}{j+1-k}-\binom{\nu-1}{j-k}} \\ & \oplus \bigoplus_{j=0}^{(\nu-5)/2} \bigoplus_{k=1}^{j+1} [k(k+1)]^{\binom{\nu-1}{j+1-k}-\binom{\nu-1}{j-k}} \oplus [0]^{\binom{\nu}{(\nu-1)/2}} \end{aligned}$$

$$\sim \bigoplus_{j=0}^{(\nu-5)/2} \bigoplus_{k=1}^{j+1} [k(k+1)]^{2(\binom{\nu-1}{j+1-k}-\binom{\nu-1}{j-k})}$$

$$\oplus \bigoplus_{k=1}^{(\nu-1)/2} [k(k+1)]^{\binom{\nu-1}{(\nu-1)/2-k}-\binom{\nu-1}{(\nu-3)/2-k}} \oplus [0]^{\binom{\nu}{(\nu-1)/2}}$$

(by interchanging the order of summation in the first summand)

$$\sim \bigoplus_{k=1}^{(\nu-3)/2} [k(k+1)]^{\sum_{j=k-1}^{(\nu-5)/2} 2\left(\binom{\nu-1}{j+1-k}-\binom{\nu-1}{j-k}\right)}$$

$$\oplus \bigoplus_{k=1}^{(\nu-1)/2} [k(k+1)]^{\binom{\nu-1}{(\nu-1)/2-k}-\binom{\nu-1}{(\nu-3)/2-k}} \oplus [0]^{\binom{\nu}{(\nu-1)/2}}$$

$$\sim \bigoplus_{k=1}^{(\nu-3)/2} [k(k+1)]^{2\binom{\nu-1}{(\nu-1)/2-k}} \oplus \bigoplus_{k=1}^{(\nu-3)/2} [k(k+1)]^{\binom{\nu-1}{(\nu-1)/2-k}-\binom{\nu-1}{(\nu-3)/2-k}}$$

$$\oplus [(\tfrac{\nu-1}{2})(\tfrac{\nu+1}{2})]^1 \oplus [0]^{\binom{\nu}{(\nu-1)/2}} \sim \bigoplus_{k=0}^{(\nu-1)/2} [k(k+1)]^{\binom{\nu}{(\nu-1)/2-k}}.$$

Therefore for ν odd, we have

$$P(\nu) \sim I_{2^{\nu-1}} \oplus P(\nu-1)^2 \sim [1]^{2^{\nu-1}} \oplus \bigoplus_{k=0}^{(\nu-1)/2} [k(k+1)]^{\binom{\nu}{(\nu+1)/2-k}},$$

giving the desired diagonal form.

For ν even, we have $2j+2 \leqslant \nu-1$ exactly for $j \leqslant (\nu-4)/2$. Therefore (recall that for ν even, $2\binom{\nu-1}{(\nu-2)/2} = \binom{\nu}{\nu/2}$)

$$P(\nu-1)^2 \sim \bigoplus_{j=0}^{(\nu-4)/2} \bigoplus_{k=1}^{j+1} [k(k+1)]^{\binom{\nu-1}{j+1-k}-\binom{\nu-1}{j-k}}$$

$$\oplus \bigoplus_{j=(\nu-2)/2}^{\nu-3} \bigoplus_{k=1}^{\nu-2-j} [k(k+1)]^{\binom{\nu-1}{\nu-2-j-k}-\binom{\nu-1}{\nu-3-j-k}} \oplus \bigoplus_{j=(\nu-1)/2}^{\nu-3} [0]^{\binom{\nu}{\nu/2}}$$

$$\sim \bigoplus_{j=0}^{(\nu-4)/2} \bigoplus_{k=1}^{j+1} [k(k+1)]^{2\left(\binom{\nu-1}{j+1-k}-\binom{\nu-1}{j-k}\right)} \oplus [0]^{\binom{\nu}{(\nu-1)/2}}$$

(by interchanging the order of summation in the first summand)

$$\sim \bigoplus_{k=1}^{(\nu-2)/2} \bigoplus_{j=k-1}^{(\nu-4)/2} [k(k+1)]^{2\left(\binom{\nu-1}{j+1-k}-\binom{\nu-1}{j-k}\right)} \oplus [0]^{\binom{\nu}{\nu/2}}$$
$$\sim \bigoplus_{k=0}^{(\nu-2)/2} [k(k+1)]^{2\binom{\nu-1}{(\nu-2)/2-k}}.$$

Therefore for ν even, we have

$$P(\nu) \sim I_{2^{\nu-1}} \oplus P(\nu-1)^2 \sim [1]^{2^{\nu-1}} \oplus \bigoplus_{k=0}^{(\nu-2)/2} [k(k+1)]^{2\binom{\nu-1}{(\nu-2)/2-k}},$$

giving the desired diagonal form. □

As an immediate corollary to our theorem, we obtain the following:

COROLLARY 2.8. *Let ν be an even integer. We have $P(\nu) \sim P(\nu-1) \oplus P(\nu-1)$.*

Proof. This follows directly from Theorem 2.7 by noting that $P(\nu)$ and $P(\nu-1) \oplus P(\nu-1)$ have the same diagonal forms. □

Another useful corollary is the following:

COROLLARY 2.9. *Let ν be a positive integer. We have* $\ker P(\nu) \cong \mathbb{Z}^{\binom{\nu}{[\nu/2]}}$.

Proof. This was proved in Theorem 2.7. □

We close this section by giving an alternate diagonal form for $P(\nu)$, which will be useful in our computations in Section 3.

THEOREM 2.10. *Let $\nu \in \mathbb{Z}^+$ and let X be a ν-set. The $2^\nu \times 2^\nu$ proper inclusion incidence matrix has as an alternate diagonal form the $2^\nu \times 2^\nu$ diagonal matrix*

$$\Delta(\nu) = \begin{cases} [0]^{\binom{\nu}{(\nu-1)/2}} \oplus \bigoplus_{k=1}^{(\nu+1)/2} [k]^{\binom{\nu+1}{(\nu+1)/2-k}}, & \nu \text{ odd}, \\ [0]^{\binom{\nu}{\nu/2}} \oplus \bigoplus_{k=1}^{\nu/2} [k]^{2\binom{\nu}{\nu/2-k}}, & \nu \text{ even}. \end{cases} \tag{2.20}$$

Proof. Recall that $P(\nu)$ is similar to $L(\nu)$, which is equivalent to

$$\begin{array}{c} \\ \binom{\nu-1}{1} \\ \binom{\nu-1}{2} \\ \vdots \\ \binom{\nu}{\nu} \\ \binom{\nu}{0} \end{array} \begin{array}{c} \begin{array}{ccccc} \binom{\nu}{0} & \binom{\nu}{1} & \cdots & \binom{\nu}{\nu-1} & \binom{\nu}{\nu} \end{array} \\ \begin{pmatrix} W_{0,1}(\nu)^t & & & & 0 \\ & W_{1,2}(\nu)^t & & & \\ & & \ddots & & \\ & & & W_{\nu-1,\nu}(\nu)^t & \\ 0 & & & & 0 \end{pmatrix} \end{array}. \tag{2.21}$$

So the desired diagonal form for $P(\nu)$ can be obtained by summing together the diagonal forms provided in [17, Theorem 2] for the matrices $W_{j,j+1}(\nu)^t$, $0 \leqslant j \leqslant \nu - 1$. For $j + (j + 1) \leqslant \nu$ we have $\binom{\nu}{j} \leqslant \binom{\nu}{j+1}$ and $W_{j,j+1}(\nu)^t$ has as a diagonal form the matrix

$$\begin{array}{c} \binom{\nu}{j} \\ \binom{\nu}{j+1} - \binom{\nu}{j} \end{array} \overset{\binom{\nu}{j}}{\begin{pmatrix} \bigoplus_{k=1}^{j+1} [k]^{\binom{\nu}{j+1-k} - \binom{\nu}{j-k}} \\ 0 \end{pmatrix}}. \tag{2.22}$$

For $j + (j + 1) > \nu$ we have $\binom{\nu}{j} > \binom{\nu}{j+1}$ and $W_{j,j+1}(\nu)^t = W_{\nu-j-1,\nu-j}(\nu)$, so that $W_{j,j+1}(\nu)^t$ has as a diagonal form the matrix

$$\begin{array}{c} \\ \binom{\nu}{j+1} \end{array} \begin{array}{cc} \binom{\nu}{j+1} & \binom{\nu}{j} - \binom{\nu}{j+1} \\ \left(\bigoplus_{k=1}^{\nu-j} [k]^{\binom{\nu}{\nu-j-k} - \binom{\nu}{\nu-j-1-k}} \right. & \left. 0 \right) \end{array}. \tag{2.23}$$

Using (2.22), (2.23) and methods similar to those used in the proof of Theorem 2.7, we obtain the diagonal form $\Delta(\nu)$. We omit details. □

3. The Gysin Sequence and $K^*(N/\Gamma)$

Let Γ be the integer Heisenberg group of rank $2n + 1$ for a fixed $n \in \mathbb{Z}^+$. Recall that Γ can be viewed as the set of triples

$$\{ (r, s, t) \mid r \in \mathbb{Z}, s, t \in \mathbb{Z}^n \}$$

with group operation given by

$$(r, s, t) \cdot (r', s', t') = \left(r + r' + \sum_{i=1}^{n} t_i s'_i, s + s', t + t' \right). \tag{3.1}$$

We can embed Γ in $\mathrm{SL}(n + 2, \mathbb{Z})$ by setting

$$(r, s_1, \ldots, s_n, t_1, \ldots, t_n) \mapsto \begin{pmatrix} 1 & t_1 & \cdots & t_n & r \\ & 1 & & 0 & s_1 \\ & & \ddots & & \vdots \\ & & & 1 & s_n \\ 0 & & & & 1 \end{pmatrix} \in \mathrm{SL}(n + 2, \mathbb{Z}).$$

By allowing the parameters r, s_i, t_i, $1 \leqslant i \leqslant n$, to take on real rather than integer values, we obtain the $(2n+1)$-dimensional simply connected Heisenberg Lie group, denoted by N. The group Γ sits inside N as a lattice, and the homogeneous space

N/Γ is a compact nilmanifold, which is a classifying space for Γ. In addition, the space N/Γ can be viewed as a principal $\mathbb{T}$-bundle over $\mathbb{T}^{2n}$ as follows. Let L be the subgroup of N generated by Γ and $[N,N]$, i.e., $L = \{(r,s,t) \mid r \in \mathbb{R}, s,t \in \mathbb{Z}\}$. Since $\Gamma \subset L$, we have the following fibration.

$$\begin{array}{ccc} \mathbb{T} = L/\Gamma & \xrightarrow{i} & N/\Gamma \\ & & \downarrow \pi \\ & & N/[N,N] \,\big/\, \Gamma/\Gamma \cap [N,N] = \mathbb{T}^{2n}. \end{array} \tag{3.2}$$

This fibration corresponds to the following short exact sequence of fundamental groups.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \xrightarrow{r\mapsto(r,0,0)} & \Gamma & \longrightarrow & \mathbb{Z}^{2n} & \longrightarrow & 1 \\ & & \cong & & \cong & & \cong & & \\ 1 & \longrightarrow & \pi_1(\mathbb{T}) & \longrightarrow & \pi_1(N/\Gamma) & \longrightarrow & \pi(\mathbb{T}^{2n}) & \longrightarrow & 1. \end{array} \tag{3.3}$$

In particular, $(E = N/\Gamma, \pi, \mathbb{T}^{2n})$ is a principal circle bundle on N/Γ, since the action of $[N,N] = \mathbb{R}$ on N/Γ factors through $[N,N]/\Gamma \cap [N,N] = \mathbb{T}$.

The circle bundle $(E, \pi, \mathbb{T}^{2n})$ gives rise via transition functions to an element of the sheaf cohomology group $\check{H}^1(\mathbb{T}^{2n}, \mathcal{S}) = \check{H}^1(\mathbb{T}^{2n}, \mathcal{U}(1))$ and, hence, we can form the associated complex line bundle $(\tilde{E}, \tilde{\pi}, \mathbb{T}^{2n})$ to which one can associate an element $\lambda(\tilde{E}) \in K^0(\mathbb{T}^{2n})$, via the canonical embedding of complex vector bundles over $\mathbb{T}^{2n}$ into $K^0(\mathbb{T}^{2n})$.

Choosing the usual metric on $\mathbb{C}$, it follows from the construction of $(\tilde{E}, \tilde{\pi}, \mathbb{T}^{2n})$ that the sphere bundle $(S(\tilde{E}), \tilde{\pi}, \mathbb{T}^{2n})$ is exactly $(E, \pi, \mathbb{T}^{2n})$ and, hence, we can use the Gysin exact sequence in K-theory [8, IV.1.13, p. 187] to compute the K-groups for $E = N/\Gamma$:

$$\begin{array}{ccccc} & K^0(\mathbb{T}^{2n}) & \xrightarrow{\alpha_0^*} & K^0(\mathbb{T}^{2n}) & \\ \nearrow & & & & \searrow \\ K^1(N/\Gamma) & & & & K^0(N/\Gamma) \\ \nwarrow & & & & \swarrow \\ & K^1(\mathbb{T}^{2n}) & \xleftarrow{\alpha_1^*} & K^1(\mathbb{T}^{2n}). & \end{array} \tag{3.4}$$

where the maps α_j^*, $j = 0, 1$, are given by the product with $1 - \lambda(\tilde{E}) \in K^0(\mathbb{T}^{2n})$ (recall $K^*(\mathbb{T}^{2n})$ has a graded ring structure). Since $K^*(\mathbb{T}^{2n})$ is torsion free, our problem thus becomes a problem in multilinear algebra involving the ring structure of $K^*(\mathbb{T}^{2n})$ and the computation of $1 - \lambda(\tilde{E})$ as an endomorphism of $K^*(\mathbb{T}^{2n})$.

It is well-known that the ring $K^*(\mathbb{T}^m)$ is isomorphic to the exterior algebra over $\mathbb{Z}$ on m generators, $\Lambda^*_{\mathbb{Z}}\{e_1, \ldots, e_m\}$ ([15], p.185; [4]). Indeed, in this case the Chern character $ch\colon K^*(\mathbb{T}^m) \to H^*(\mathbb{T}^m, \mathbb{Q})$ is integral and gives an isomorphism $ch_0\colon K^0(\mathbb{T}^m) \to H^{\text{even}}(\mathbb{T}^m, \mathbb{Z})$ and $ch_1\colon K^1(\mathbb{T}^m) \to H^{\text{odd}}(\mathbb{T}^m, \mathbb{Z})$ (see [7, Theorem A7] for a proof) where $H^*(\mathbb{T}^m, \mathbb{Z})$ under cup product, is well-known to be isomorphic to $\Lambda^*_{\mathbb{Z}} H^1(\mathbb{T}^m, \mathbb{Z}) \cong \Lambda^*_{\mathbb{Z}}\{e_1, \dots, e_m\}$ with $H^k(\mathbb{T}^m, \mathbb{Z}) \cong \Lambda^k_{\mathbb{Z}}\{e_1, \dots, e_m\}$. Denote by ϕ_i the corresponding isomorphisms

$$\phi_i \colon K^i(\mathbb{T}^{2n}) \to \bigoplus_{r=0}^{n-i} \Lambda^{2r+i}_{\mathbb{Z}}\{e_1, \dots, e_{2n}\}, \qquad i = 0, 1. \tag{3.5}$$

In order to carry out the computations implicit in the sequence (3.4), we need to compute $ch(1 - \lambda(\tilde{E})) = 1 - ch(\lambda(\tilde{E})) \in \Lambda^{\text{even}}_{\mathbb{Z}}\{e_1, \dots, e_m\}$. We will do this in Proposition 3.2.

We recall the following result of Massey, specialized to our context ([1]). Let Δ be a countable discrete group, let B be a connected CW-complex of type $K(\Delta, 1)$, and suppose that (E, π, B) is a principal $\mathbb{T}$-bundle over B such that the fundamental group $\pi_1(E)$ is a central extension of $\pi_1(B) = \Delta$ by $\pi_1(\mathbb{T}) = \mathbb{Z}$. Then the characteristic class c_1 of the bundle (E, π, B) as an element of $\check{H}^2(B, \mathbb{Z}) \cong \check{H}^1(B, \mathcal{S})$ (that is, the first obstruction to a cross-section), can be identified by the group cohomology class $\kappa \in H^2(\Delta, \mathbb{Z})$ determined by the central group extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(E) \longrightarrow \pi_1(B) \cong \Delta \longrightarrow 0$$

via the canonical isomorphism $\lambda_* : H^2(\Delta, \mathbb{Z}) \longrightarrow \check{H}^2(B, \mathbb{Z})$ which is discussed in Lemma 2.6 of [13], for example. In our case, $\pi_1(E) = \pi_1(N/\Gamma) = \Gamma$, $\pi_1(B) = \Delta \cong \mathbb{Z}^{2n}$, and, using the notation of Section 3 in [2], $\kappa \in H^2(\mathbb{Z}^{2n}, \mathbb{Z})$ is defined by $M = \sum_{i=1}^n E_{n+i,i}$ (here E_{jk} denotes the elementary matrix with 1 in the (j, k)th spot, 0's elsewhere). Thus we may write $\kappa = \sum_{i=1}^n [\sigma_{E_{n+i,i}}]$ (see [2] for notation). From Section 3 of [2] it is known that $\{[\sigma_{E_{jk}}] : 1 \leqslant k < j \leqslant n\}$ is a basis for $H^2(\mathbb{Z}^{2n}, \mathbb{Z})$, and it is easily checked that $\lambda_*([\sigma_{E_{jk}}]) = e_k \wedge e_j \in \Lambda^2_{\mathbb{Z}}\{e_1, \dots, e_{2n}\} \cong H^2(\mathbb{T}^{2n}, \mathbb{Z})$ (the line bundle corresponding to $e_k \wedge e_j$ is the pull-back of the standard Heisenberg non-trivial line bundle on $\mathbb{T}^2$ to $\mathbb{T}^{2n} = \prod_{i=1}^{2n} \mathbb{T}$ via projection onto the kth and jth coordinates). It follows that the characteristic class of $(N/\Gamma, \pi, \mathbb{T}^{2n})$, or what is the same thing, the first Chern class of the complex line bundle $(\tilde{E}, \pi, \mathbb{T}^{2n})$, is given by $c_1(\tilde{E}) = \lambda_*([\sigma_M]) = \lambda_*(\sum_{i=1}^n [\sigma_{E_{n+i,i}}]) = \sum_{i=1}^n e_i \wedge e_{n+i} \in H^2(\mathbb{T}^{2n}, \mathbb{Z})$. We have thus proved:

PROPOSITION 3.1. *Let Γ be the standard integer lattice in the $(2n+1)$-dimensional simply connected Heisenberg Lie group N. Then the characteristic class in $H^2(\mathbb{T}^{2n}, \mathbb{Z}) = \Lambda^2_{\mathbb{Z}}\{e_1, \dots, e_{2n}\}$ and hence the first Chern class defined by the complex line bundle $(\tilde{E}, \tilde{\pi}, \mathbb{T}^{2n})$ associated to $(N/\Gamma, \pi, \mathbb{T}^{2n})$ is defined by $c_1(\tilde{E}) = \sum_{i=1}^n e_i \wedge e_{n+i}$.*

We now use Proposition 3.1 to deduce:

PROPOSITION 3.2. *Let N and Γ be as in Proposition 3.1, let $(\tilde{E}, \tilde{\pi}, \mathbb{T}^{2n})$ be the complex line bundle associated to the principal $\mathbb{T}$ bundle $(N/\Gamma, \pi, \mathbb{T}^{2n})$, and let $\lambda(\tilde{E})$ be the corresponding representative in $K^0(\mathbb{T}^{2n})$. Then the formula for $ch(\lambda(\tilde{E}))$ in*

$H^{\text{even}}(\mathbb{T}^{2n}, \mathbb{Z}) = \Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, \ldots, e_{2n}\}$ *is given by*

$$\begin{aligned} ch(\lambda(\tilde{E})) &= \sum_{j=0}^{n} \frac{\left(\sum_{i=1}^{n} e_i \wedge e_{n+i}\right)^j}{j!} \\ &= 1 + \sum_{i=1}^{n} e_i \wedge e_{n+i} + \sum_{1 \leqslant i_1 < i_2 \leqslant n} (e_{i_1} \wedge e_{n+i_1}) \wedge (e_{i_2} \wedge e_{n+i_2}) \\ &\quad + \cdots + \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{n-1} \leqslant n} (e_{i_1} \wedge e_{n+i_1}) \wedge \cdots \wedge (e_{i_{n-1}} \wedge e_{n+i_{n-1}}) \\ &\quad + (e_1 \wedge e_{n+1}) \wedge (e_2 \wedge e_{n+2}) \wedge \cdots \wedge (e_n \wedge e_{2n}). \end{aligned}$$

Proof. The first equality follows from the formula for the Chern character of complex line bundles given in p.196 of [12], that is, $ch(\lambda(\tilde{E})) = \exp(c_1(\tilde{E}))$. The second formula follows by expanding the first expression. □

From the proposition we immediately obtain:

THEOREM 3.3. *Let* Γ *be the standard integer lattice in the* $(2n+1)$*-dimensional simply connected Heisenberg Lie group* N*. The* K*-groups of* N/Γ *can be computed from the exact sequence*

$$\begin{array}{ccccc} & \Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, \ldots, e_{2n}\} & \overset{\alpha_*^0}{\rightarrow} & \Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, \ldots, e_{2n}\} & \\ \nearrow & & & & \searrow \\ K^1(N/\Gamma) & & & & K^0(N/\Gamma) \\ \nwarrow & & & & \swarrow \\ & \Lambda_{\mathbb{Z}}^{\text{odd}}\{e_1, \ldots, e_{2n}\} & \overset{\alpha_*^1}{\leftarrow} & \Lambda_{\mathbb{Z}}^{\text{odd}}\{e_1, \ldots, e_{2n}\} & \end{array} \tag{3.6}$$

where the maps $\alpha_*^i \colon \Lambda_{\mathbb{Z}}^*\{e_1, \ldots, e_{2n}\} \longrightarrow \Lambda_{\mathbb{Z}}^*\{e_1, \ldots, e_{2n}\}$ *are defined by*

$$\alpha_*^i(\gamma) = -\gamma \wedge \left(\sum_{j=1}^{n} \left(\sum_{1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant n} (e_{i_1} \wedge e_{n+i_1}) \wedge \cdots \wedge (e_{i_j} \wedge e_{n+i_j}) \right) \right),$$

for $\gamma \in \Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, \ldots, e_{2n}\}$ *if* $i = 0$*, and for* $\gamma \in \Lambda_{\mathbb{Z}}^{\text{odd}}\{e_1, \ldots, e_{2n}\}$ *if* $i = 1$*. In particular,*

$$K^0(N/\Gamma) \cong \operatorname{coker} \alpha_*^0 \oplus \ker \alpha_*^1,$$
$$K^1(N/\Gamma) \cong \operatorname{coker} \alpha_*^1 \oplus \ker \alpha_*^0.$$

Proof. Using the Chern character $ch \colon K^*(\mathbb{T}^{2n}) \longrightarrow \Lambda_{\mathbb{Z}}^*\{e_1, \ldots, e_{2n}\}$, the Gysin sequence for K-theory given in Diagram (3.4) becomes exactly the Diagram (3.6), and by Proposition 3.2, the maps α_*^i, $i = 0, 1$ of Diagram (3.4) become the maps stated in the theorem, using the fact that $ch(1_{K^*}) = 1_{\Lambda_{\mathbb{Z}}^*}$. We obtain the splitting for

$K^i(N/\Gamma)$, $i = 0, 1$, by using the fact the $\Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, \dots e_{2n}\}$ and $\Lambda_{\mathbb{Z}}^{\text{odd}}\{e_1, \dots e_{2n}\}$ are finitely generated free Abelian groups, so that the respective subgroups $\ker \alpha_*^i$, $i = 0, 1$, would have the same property. □

If we let L_0 and L_1 denote the restrictions to $\Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, \dots, e_{2n}\}$ and $\Lambda_{\mathbb{Z}}^{\text{odd}}\{e_1, \dots, e_{2n}\}$, respectively, of the transformation $L\colon \Lambda_{\mathbb{Z}}^*\{e_1, \dots, e_{2n}\} \to \Lambda_{\mathbb{Z}}^*\{e_1, \dots, e_{2n}\}$ defined by the wedge product with $\sum_{i=1}^n e_i \wedge e_{i+n}$, then by using the isomorphisms $\{\phi_i \mid i = 0, 1\}$ and Proposition 3.2, Diagram (3.4) is transformed into

$$\begin{array}{ccccc} & \bigoplus_{r=0}^{n} \Lambda_{\mathbb{Z}}^{2r}\{e_1, \dots, e_{2n}\} & \xrightarrow{[I-\exp(L_0)]} & \bigoplus_{r=0}^{n} \Lambda_{\mathbb{Z}}^{2r}\{e_1, \dots, e_{2n}\} & \\ \nearrow & & & & \searrow \\ K^1(N/\Gamma) & & & & K^0(N/\Gamma) \\ \nwarrow & & & & \swarrow \\ & \bigoplus_{r=0}^{n-1} \Lambda_{\mathbb{Z}}^{2r+1}\{e_1, \dots, e_{2n}\} & \xleftarrow{[I-\exp(L_1)]} & \bigoplus_{r=0}^{n-1} \Lambda_{\mathbb{Z}}^{2r+1}\{e_1, \dots, e_{2n}\}. & \end{array} \tag{3.7}$$

We can easily check that L and the restrictions L_0 and L_1 are nilpotent transformations of degree n so that

$$\exp(L_i) = \sum_{j=0}^{n} \frac{L_i^j}{j!}, \quad i = 0, 1. \tag{3.8}$$

We now write

$$P = -[I - \exp L] = \exp L - I\colon \Lambda_{\mathbb{Z}}^*\{e_1, \dots, e_{2n}\} \to \Lambda_{\mathbb{Z}}^*\{e_1, \dots, e_{2n}\},$$

and let P_0 and P_1 denote the restrictions of P to $\Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, \dots, e_{2n}\}$ and $\Lambda_{\mathbb{Z}}^{\text{odd}}\{e_1, \dots, e_{2n}\}$, respectively. Since

$$\begin{aligned} &\operatorname{coker}(I - \exp L_i) = \operatorname{coker} P_i, \quad i = 0, 1, \\ &\ker(I - \exp L_i) = \ker P_i, \qquad i = 0, 1, \end{aligned}$$

and $\ker P_i$ for $i = 0, 1$ are finitely generated free Abelian groups, diagram (3.7) gives the following splitting for the K-groups of N/Γ.

$$K^0 \cong \operatorname{coker} P_0 \oplus \ker P_1, \qquad K^1 \cong \operatorname{coker} P_1 \oplus \ker P_0. \tag{3.9}$$

Our intention is to use the diagonal forms for $\{P(\nu) \mid \nu \in \mathbb{Z}^+ \cup \{0\}\}$ established in the previous section to calculate $\{\operatorname{coker} P_i \mid i = 0, 1\}$ and $\{\ker P_i \mid i = 0, 1\}$, thus calculating $K^i(N/\Gamma)$ for $i = 0, 1$. To do this we first decompose $\Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, \dots, e_{2n}\}$ and $\Lambda_{\mathbb{Z}}^{\text{odd}}\{e_1, \dots, e_{2n}\}$ into the direct sums of subgroups that are P_0- and P_1-invariant. We first review for the reader's convenience some results on the decomposition of $\Lambda_{\mathbb{Z}}^*\{e_1, \dots, e_{2n}\}$ that can be found in [11].

DEFINITION 3.4. Let $C \subseteq \{1, \ldots, 2n\}$. We say that C is *pair-free* if $\{i, i+n\} \not\subseteq C$, for all $1 \leqslant i \leqslant n$.

By the pigeonhole principle, if C is pair-free, then $|C| \leqslant n$. Clearly any subset of $\{1, \ldots, 2n\}$ can be written uniquely as $C \cup P$, where C is pair-free and

$$P = \{i_1, i_1 + n, i_2, i_2 + n, \ldots, i_k, i_k + n\},$$
$$0 \leqslant k \leqslant n, \quad 1 \leqslant i_j \leqslant n, \quad 1 \leqslant j \leqslant k,$$

consists entirely of pairs.

NOTATION 3.5. For any pair-free subset C of $\{1, \ldots, 2n\}$, let

$$D_C = \{\, j \in \{1, \ldots, n\} \mid \{j, j+n\} \cap C = \emptyset\,\}.$$

By construction, $|D_C| = n - |C|$.

NOTATION 3.6. Let $C = \{i_1, \ldots, i_k\}$ be a fixed pair-free subset of $\{1, \ldots, 2n\}$, and $J = \{j_1, \ldots, j_p\}$ an arbitrary subset of D_C. We define $C_J \in \Lambda_{\mathbb{Z}}^{k+2p}\{e_1, \ldots, e_{2n}\}$ by

$$C_J = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \wedge e_{j_1} \wedge e_{j_1+n} \wedge \cdots \wedge e_{j_p} \wedge e_{j_p+n}. \tag{3.10}$$

For a fixed pair-free set C of order k and $p \geqslant 0$, let

$$B^{k+2p}(C) = \{C_J \mid J \subseteq D_C,\ |J| = p\},$$

and let $V^{k+2p}(C)$ denote the subgroup of $\Lambda_{\mathbb{Z}}^{k+2p}\{e_1, \ldots, e_{2n}\}$ generated by $B^{k+2p}(C)$. A counting argument shows that the rank of $V^{k+2p}(C)$ is equal to $\binom{n-k}{p}$. For fixed j, $0 \leqslant j \leqslant 2n$, it follows from [11, Equation (1.8)] that

$$\Lambda_{\mathbb{Z}}^{j}\{e_1, \ldots, e_{2n}\} = \bigoplus_{p=0}^{[j/2]} \bigoplus_{\substack{C \text{ pair-free} \\ |C|=j-2p}} V^{(j-2p)+2p}(C). \tag{3.11}$$

We remark that in order for $V^j(C)$ to be a nontrivial summand of $\Lambda_{\mathbb{Z}}^{j}\{e_1, \ldots, e_{2n}\}$, we need

$$|C| \leqslant j, \qquad |C| \equiv j \mod 2, \qquad j \leqslant 2n - |C|. \tag{3.12}$$

The last inequality follows from the fact that for $j > 2n - |C|$, we have $B^j(C) = \emptyset$, so that $V^j(C) = 0$.

Using (3.12), we easily obtain

$$\Lambda^{\text{even}} \cong \bigoplus_{r=0}^{n} \Lambda_{\mathbb{Z}}^{2r}\{e_1, \ldots, e_{2n}\} \cong \bigoplus_{r=0}^{n} \bigoplus_{p=0}^{r} \bigoplus_{\substack{C \text{ pair-free} \\ |C|=2r-2p}} V^{2r}(C)$$
$$\cong \bigoplus_{\substack{C \text{ pair-free} \\ |C| \text{ even}}} \bigoplus_{p=0}^{n-|C|} V^{|C|+2p}(C), \tag{3.13}$$

and similarly

$$\Lambda^{\text{odd}} \cong \bigoplus_{r=0}^{n-1} \Lambda_Z^{2r+1}\{e_1, \dots, e_{2n}\} \cong \bigoplus_{\substack{\text{C pair-free} \\ |C| \text{ odd}}} \bigoplus_{p=0}^{n-|C|} V^{|C|+2p}(C). \tag{3.14}$$

Fixing a pair-free subset C of $\{1, \dots, 2n\}$, we let

$$V(C) = \bigoplus_{p=0}^{n-|C|} V^{|C|+2p}(C). \tag{3.15}$$

PROPOSITION 3.7. *Let $n \in \mathbb{Z}^+$. We have*

$$\Lambda^{\text{even}} \cong \bigoplus_{\substack{C \text{ pair-free} \\ |C| \text{ even}}} V(C), \tag{3.16}$$

$$\Lambda^{\text{odd}} \cong \bigoplus_{\substack{C \text{ pair-free} \\ |C| \text{ odd}}} \bigoplus_{p=0}^{n-|C|} V^{|C|+2p}(C). \tag{3.17}$$

The rank of $V(C)$ is equal to $2^{n-|C|}$, and $V(C)$ is L_0- and P_0-invariant for $|C|$ even and L_1- and P_1-invariant for $|C|$ odd.

Proof. Equations (3.16) and (3.17) follow immediately from Equations (3.13) and (3.14) and Definition (3.15). Moreover,

$$\text{rank } V(C) = \bigoplus_{p=0}^{n-|C|} \text{rank } V^{k+2p}(C) = \bigoplus_{p=0}^{n-|C|} \binom{n-|C|}{p} = 2^{n-|C|}.$$

For a fixed pair-free subset C of $\{1, \dots, 2n\}$, it was explained in [11, Lemma 1.3] that it follows from [16, Chapter 1, p. 21] that $L\colon\ V^j(C) \to V^{j+2}(C)$ for all $j \geqslant |C|$, so that $L\colon\ V(C) \to V(C)$ and $P = \exp(L) - I\colon\ V(C) \to V(C)$, giving the desired result. □

Now let $L(C)$ and $P(C)$ denote the restrictions of L and P, respectively, to the subgroup $V(C)$ of $\Lambda_Z^*\{e_1, \dots, e_{2n}\}$. From now on we will write $\nu = n - |C|$ and identify $V(C)$ with $\mathbb{Z}^{2^\nu}$. Recall that $V(C)$ has as a basis $\cup_{p=0}^{n-|C|} B^{|C|+2p}(C)$, that is, it has as a basis the elements $\{ C_J \mid J \subseteq D_C \}$ defined in Notation 3.6. Therefore the basis for $V(C)$ can be identified with the set of all subsets of the ν-set D_C.

Following [16, p. 17], we easily compute that

$$L(C_J) = \sum_{i \in D_C - J} C_{J \cup \{i\}}. \tag{3.18}$$

Thus if we give the matrix $L(C)$ a $2^\nu \times 2^\nu$ matrix representation (L_{IJ}) corresponding to the basis above, we obtain

$$L_{IJ} = \begin{cases} 1, & \text{if } J \subseteq I \quad \text{and} \quad |I| = |J| + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.19}$$

We thus see that the matrix representation for $L(C)$ depends only on the order $\nu = n - |C|$ of D_C and not on C itself, and that without loss of generality we can identify $L(C)$ with the incidence matrix $L(\nu)$ studied in Section 2. Similarly, $P(C) = \exp(L(C)) - I$ can be identified with the proper inclusion incidence matrix $P(\nu)$ of Section 2.

If we write

$$L_0 = \bigoplus_{\substack{C \text{ pair-free} \\ |C| \text{ even}}} L(C), \qquad L_1 = \bigoplus_{\substack{C \text{ pair-free} \\ |C| \text{ odd}}} L(C),$$

$$P_0 = \bigoplus_{\substack{C \text{ pair-free} \\ |C| \text{ even}}} P(C), \qquad P_1 = \bigoplus_{\substack{C \text{ pair-free} \\ |C| \text{ odd}}} P(C),$$

and use the fact that for fixed k, $0 \leqslant k \leqslant n$, there are exactly $2^k\binom{n}{k}$ pair-free subsets of order k, we obtain

$$\begin{aligned} L_0 &= \bigoplus_{r=0}^{[n/2]} L(n-2r)^{2^{2r}\binom{n}{2r}}, \quad L_1 = \bigoplus_{r=0}^{[(n-1)/2]} L(n-(2r+1))^{2^{2r+1}\binom{n}{2r+1}}, \\ P_0 &= \bigoplus_{r=0}^{[n/2]} P(n-2r)^{2^{2r}\binom{n}{2r}}, \quad P_1 = \bigoplus_{r=0}^{[(n-1)/2]} P(n-(2r+1))^{2^{2r+1}\binom{n}{2r+1}}. \end{aligned} \tag{3.20}$$

Hence, using Equations (3.9) and (3.20) we obtain

$$\begin{aligned} K^0(N/\Gamma) \;\cong\; & \bigoplus_{r=0}^{[n/2]} \operatorname{coker}(P(n-2r))^{2^{2r}\binom{n}{2r}} \\ & \oplus \bigoplus_{r=0}^{[(n-1)/2]} \ker(P(n-(2r+1)))^{2^{2r+1}\binom{n}{2r+1}}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} K^1(N/\Gamma) \;\cong\; & \bigoplus_{r=0}^{[(n-1)/2]} \operatorname{coker}(P(n-(2r+1)))^{2^{2r+1}\binom{n}{2r+1}} \\ & \oplus \bigoplus_{r=0}^{[n/2]} \ker(P(n-2r))^{2^{2r}\binom{n}{2r}}. \end{aligned} \tag{3.22}$$

Equations (3.21) and (3.22) together with the results of Section 2 will allow us to calculate $K^0(N/\Gamma)$ and $K^1(N/\Gamma)$ without too much difficulty. We first recall the following combinatorial lemma from [11, Lemma 1.5].

LEMMA 3.8. *For* $0 \leqslant k \leqslant 2n$,

$$\sum_{p=0}^{[k/2]} 2^{k-2p} \binom{n}{k-2p} \binom{n-k+2p}{p} = \binom{2n}{k}.$$

Proof. The proof, which follows from a counting argument, is derived from the identity

$$\Lambda_{\mathbb{Z}}^k\{e_1, \ldots, e_{2n}\} = \bigoplus_{p=0}^{[k/2]} \bigoplus_{\substack{C \text{ pair-free} \\ |C|=k-2p}} V^k(C). \qquad \square$$

We can now prove our main theorem.

THEOREM 3.9. *Let* $n \in \mathbb{Z}^+$ *and let* Γ *be the standard integer lattice in the* $(2n+1)$*-dimensional simply connected Heisenberg Lie group* N*. The topological* K*-groups for* N/Γ *are given by*

$$K^0(N/\Gamma) \cong \bigoplus_{r=0}^{n} H^{2r}(N/\Gamma, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-1-2k}}, & n \text{ odd}, \\ \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{n/2} \mathbb{Z}_k^{2\binom{2n}{n-2k}}, & n \text{ even}, \end{cases}$$

$$K^1(N/\Gamma) \cong \bigoplus_{r=0}^{n} H^{2r+1}(N/\Gamma, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_k^{2\binom{2n}{n-2k}}, & n \text{ odd}, \\ \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-2)/2} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-1-2k}}, & n \text{ even}. \end{cases}$$

These formulas can also be written as

$$K^i(N/\Gamma) \cong \begin{cases} \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{[(n-1)/2]} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-1-2k}}, & n-i \equiv 1 \mod 2, \\ \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{[n/2]} \mathbb{Z}_k^{2\binom{2n}{n-2k}}, & n-i \equiv 0 \mod 2. \end{cases}$$

Proof. We use Equations (3.21) and (3.22) and the diagonal forms for $P(\nu)$ obtained in Section 2. We first concentrate on the case when ν is odd. In that case,

$[(n-1)/2] = (n-1)/2$ and

$$
\begin{aligned}
K^0(N/\Gamma) &= \bigoplus_{r=0}^{(n-1)/2} \operatorname{coker}(P(n-2r))^{2^{2r}\binom{n}{2r}} \oplus \bigoplus_{r=0}^{(n-1)/2} \ker(P(n-(2r+1)))^{2^{2r+1}\binom{n}{2r+1}} \\
&\cong \bigoplus_{l=0}^{(n-1)/2} \operatorname{coker}(P(2l+1))^{2^{n-(2l+1)}\binom{n}{n-(2l+1)}} \oplus \bigoplus_{l=0}^{(n-1)/2} \ker(P(2l))^{2^{n-2l}\binom{n}{n-2l}} \\
&\qquad \text{(by Theorem 2.7 and Corollary 2.9)} \\
&\cong \bigoplus_{l=0}^{(n-1)/2} (\mathbb{Z}^{\binom{2l+1}{l}})^{2^{n-(2l+1)}\binom{n}{n-(2l+1)}} \oplus \bigoplus_{l=1}^{(n-1)/2} \bigoplus_{k=1}^{l} \mathbb{Z}_{k(k+1)}^{\binom{2l+1}{l-k}2^{n-(2l+1)}\binom{n}{n-(2l+1)}} \\
&\qquad \oplus \bigoplus_{l=0}^{(n-1)/2} (\mathbb{Z}^{\binom{2l}{l}})^{2^{n-2l}\binom{n}{n-2l}} \\
&\cong \mathbb{Z}^{\sum_{l=0}^{(n-1)/2} \binom{2l+1}{l}2^{n-(2l+1)}\binom{n}{n-(2l+1)}+\binom{2l}{l}2^{n-2l}\binom{n}{n-2l}} \\
&\qquad \oplus \bigoplus_{l=1}^{(n-1)/2} \bigoplus_{k=l}^{(n-1)/2} \mathbb{Z}_{k(k+1)}^{\binom{2l+1}{l-k}2^{n-(2l+1)}\binom{n}{n-(2l+1)}} \\
&\qquad \text{(by Lemma 3.8)} \\
&\cong \mathbb{Z}^{\binom{2n}{n-1}+\binom{2n}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_{k(k+1)}^{\sum_{l=0}^{(n-(2k+1))/2} \binom{2l+2k+1}{l}2^{n-(2k+1)-2l}\binom{n}{n-(2k+1)-2l}} \\
&\qquad \text{(by Lemma 3.8)} \\
&\cong \mathbb{Z}^{\binom{2n}{n-1}+\binom{2n}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-(2k+1)}}.
\end{aligned}
$$

This establishes the formula for $K^0(N/\Gamma)$ when n is odd.

We now calculate $K^1(N/\Gamma)$ for n odd.

$$
\begin{aligned}
K^1(N/\Gamma) &= \bigoplus_{r=0}^{(n-1)/2} \operatorname{coker}(P(n-(2r+1)))^{2^{2r+1}\binom{n}{2r+1}} \oplus \bigoplus_{r=0}^{(n-1)/2} \ker(P(n-2r)))^{2^{2r}\binom{n}{2r}} \\
&\cong \bigoplus_{l=0}^{(n-1)/2} \operatorname{coker}(P(2l))^{2^{n-2l}\binom{n}{n-2l}} \oplus \bigoplus_{l=0}^{(n-1)/2} \ker(P(2l+1))^{2^{n-(2l+1)}\binom{n}{n-(2l+1)}} \\
&\qquad \text{(by Theorem 2.10 and Corollary 2.9)} \\
&\cong \bigoplus_{l=0}^{(n-1)/2} (\mathbb{Z}^{\binom{2l}{l}})^{2^{n-2l}\binom{n}{n-2l}} \oplus \bigoplus_{l=1}^{(n-1)/2} \bigoplus_{k=1}^{l} \mathbb{Z}_k^{2\binom{2l}{l-k}2^{n-2l}\binom{n}{n-2l}}
\end{aligned}
$$

$$
\begin{aligned}
&\oplus \bigoplus_{l=0}^{(n-1)/2} (\mathbb{Z}^{\binom{2l+1}{l}})^{2^{n-(2l+1)}\binom{n}{n-(2l+1)}} \\
\cong\ & \mathbb{Z}^{\sum_{l=0}^{(n-1)/2}\left[\binom{2l}{l}2^{n-2l}\binom{n}{n-2l}+\binom{2l+1}{l}2^{n-(2l+1)}\binom{n}{n-(2l+1)}\right]} \\
&\oplus \bigoplus_{l=1}^{(n-1)/2} \bigoplus_{k=l}^{(n-1)/2} \mathbb{Z}_k^{2\binom{2l}{l-k}2^{n-2l}\binom{n}{n-2l}} \\
&\text{(by Lemma 3.8)} \\
\cong\ & \mathbb{Z}^{\binom{2n}{n}+\binom{2n}{n-1}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_k^{2\sum_{l=0}^{(n-(2k+1))/2}\binom{2l+2k}{l}2^{n-2k-2l}\binom{n}{n-2k-2l}} \\
&\text{(by Lemma 3.8)} \\
\cong\ & \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_k^{2\binom{2n}{n-2k}}.
\end{aligned}
$$

This establishes our formula for $K^1(N/\Gamma)$ when n is odd. For n even, formulas for $K^0(N/\Gamma)$ and $K^1(N/\Gamma)$ are computed similarly, with Theorem 2.10 being used for $K^0(N/\Gamma)$ and Theorem 2.7 for $K^1(N/\Gamma)$. We leave the verification to the reader.

To complete the proof of the theorem, it remains to show that

$$K^0(N/\Gamma) \cong \bigoplus_{r=0}^{n} H^{2r}(N/\Gamma, \mathbb{Z}), \qquad K^1(N/\Gamma) \cong \bigoplus_{r=0}^{n} H^{2r+1}(N/\Gamma, \mathbb{Z}).$$

For this it is possible to use the isomorphism $H^i(N/\Gamma, \mathbb{Z}) \cong H^i(\Gamma, \mathbb{Z})$ and the formulas for $H^i(\Gamma, \mathbb{Z})$ established in [11] and then calculate directly. However, since we wish this paper to be self-contained, we indicate a different method.

The Gysin sequence in cohomology [14] for circle bundles can be used to calculate $H^*(N/\Gamma)$ as follows. Consider the diagram

$$
\begin{array}{ccccc}
H^{k-2}(\mathbb{T}^{2n}, \mathbb{Z}) & \overset{L_{k-2}}{\to} & H^k(\mathbb{T}^{2n}, \mathbb{Z}) & \to & H^k(N/\Gamma, \mathbb{Z}) \\
 & & & \swarrow & \\
 & & H^{k-1}(\mathbb{T}^{2n}, \mathbb{Z}) & \overset{L_{k-1}}{\to} & H^{k+1}(\mathbb{T}^{2n}, \mathbb{Z}),
\end{array}
\tag{3.23}
$$

where $L_j\colon H^j(\mathbb{T}^{2n}, \mathbb{Z}) \to H^{j+2}(\mathbb{T}^{2n}, \mathbb{Z})$ is the the cup product with the characteristic class $[c] \in H^2(\mathbb{T}^{2n}, \mathbb{Z})$. By identifying $H^0(\mathbb{T}^{2n}, \mathbb{Z})$ with $\Lambda^j_{\mathbb{Z}}\{e_1, \ldots, e_{2n}\}$ and summing the sequence in (3.23) over even and odd values of k, we obtain the following six-term exact sequence for $H^{\text{even}}(N/\Gamma, \mathbb{Z})$ and $H^{\text{odd}}(N/\Gamma, \mathbb{Z})$, where the maps L_0 and L_1 were discussed in the paragraph following Theorem 3.3.

$$\begin{array}{ccccc} & \bigoplus_{r=0}^{n} \Lambda_{\mathbb{Z}}^{2r}\{e_1, \ldots, e_{2n}\} & \xrightarrow{L_0} & \bigoplus_{r=0}^{n} \Lambda_{\mathbb{Z}}^{2r}\{e_1, \ldots, e_{2n}\} & \\ \nearrow & & & & \searrow \\ \bigoplus_{r=0}^{n} H^{2r+1}(N/\Gamma, \mathbb{Z}) & & & & \bigoplus_{r=0}^{n} H^{2r}(N/\Gamma, \mathbb{Z}) \\ \nwarrow & & & & \swarrow \\ & \bigoplus_{r=0}^{n-1} \Lambda_{\mathbb{Z}}^{2r+1}\{e_1, \ldots, e_{2n}\} & \xleftarrow{L_1} & \bigoplus_{r=0}^{n-1} \Lambda_{\mathbb{Z}}^{2r+1}\{e_1, \ldots, e_{2n}\}. & \end{array} \tag{3.24}$$

Thus, as in the case for the K-groups, diagram (3.24) implies that

$$\bigoplus_{r=0}^{n} H^{2r}(N/\Gamma, \mathbb{Z}) \cong \operatorname{coker} L_0 \oplus \ker L_1, \tag{3.25}$$

$$\bigoplus_{r=0}^{n} H^{2r+1}(N/\Gamma, \mathbb{Z}) \cong \operatorname{coker} L_1 \oplus \ker L_0. \tag{3.26}$$

By Equation (3.20) together with Proposition 2.6, we see that $\operatorname{coker} L_i \cong \operatorname{coker} P_i$ for $i = 0, 1$, and $\ker L_i \cong \ker P_i$ for $i = 0, 1$, so that Equations (3.25), (3.26) and (3.9) imply that

$$K^0(N/\Gamma) \cong \bigoplus_{r=0}^{n} H^{2r}(N/\Gamma, \mathbb{Z}), \qquad K^1(N/\Gamma) \cong \bigoplus_{r=0}^{n} H^{2r+1}(N/\Gamma, \mathbb{Z}),$$

as desired. □

We can now obtain our formula for the group $K^*(N/\Gamma) \cong K^0(N/\Gamma) \oplus K^1(N/\Gamma)$ that is valid for all $n \in \mathbb{Z}^+$.

COROLLARY 3.10. *Let $n \in \mathbb{Z}^+$ and let Γ denote the integer Heisenberg group of rank $2n+1$ sitting inside the Heisenberg Lie group N. As an Abelian group,*

$$K^*(N/\Gamma) \cong \bigoplus_{k=0}^{2n+1} H^k(N/\Gamma, \mathbb{Z}) \cong \bigoplus_{k=0}^{[(n+1)/2]} \mathbb{Z}_k^{\binom{2n+2}{n+1-2k}}.$$

Proof. This time we perform the calculation for n even, leaving the case n odd to the reader. By Theorem 3.9, for n even we have

$$\begin{aligned}
&K^0(N/\Gamma) \oplus K^1(N/\Gamma)\\
&\quad\cong \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{n/2} \mathbb{Z}_k^{2\binom{2n}{n-2k}} \oplus \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-2)/2} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-1-2k}}\\
&\quad\cong \mathbb{Z}^{\binom{2n+2}{n+1}} \oplus \bigoplus_{k=1}^{n/2-1} \mathbb{Z}_k^{\left[\binom{2n}{n-1-2k}+\binom{2n}{n-2k}\right]+\left[\binom{2n}{n-2k}+\binom{2n}{n+1-2k}\right]} \oplus \mathbb{Z}_{n/2}^{2n+2}\\
&\quad\cong \mathbb{Z}^{\binom{2n+2}{n+1}} \oplus \bigoplus_{k=1}^{n/2-1} \mathbb{Z}_k^{\binom{2n+2}{n+1-2k}}\\
&\quad\cong \bigoplus_{k=0}^{(n+1)/2} \mathbb{Z}_k^{\binom{2n+2}{n+1-2k}}. \qquad \square
\end{aligned}$$

We can also use our main theorem to prove the result about twisted discrete Heisenberg group C^*-algebras mentioned in the introduction. By a result of the third author and I. Raeburn [13, Theorem 2.3 and Corollary 2.10], if Γ is a cocompact subgroup of a solvable simply connected Lie group G, and if σ is a multiplier on Γ that is homotopic to the identity multiplier, and if we denote by $C^*(\Gamma, \sigma)$ the twisted group C^*-algebra associated to the pair (Γ, σ), then

$$K_i(C^*(\Gamma, \sigma)) \cong K^{i+\dim G}(G/\Gamma) \quad \text{for } i = 0, 1. \tag{3.27}$$

For the integer Heisenberg group defined in (3.1), a structure result in one of our earlier works, [9, Theorem 2.1], shows that every multiplier on Γ is homotopic to the trivial multiplier. Thus we obtain the following corollary.

COROLLARY 3.11. *Let $n \in \mathbb{Z}^+$, let Γ be the integer Heisenberg group of rank $2n + 1$, and let σ be any multiplier on Γ. The K-groups for the twisted group C^*-algebra $C^*(\Gamma, \sigma)$ are given by the following formulae.*

$$K_0(C^*(\Gamma, \sigma)) \cong \begin{cases} \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_k^{2\binom{2n}{n-2k}}, & \textit{for } n \textit{ odd},\\ \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-2)/2} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-2k-1}}, & \textit{for } n \textit{ even}, \end{cases}$$

$$K_1(C^*(\Gamma, \sigma)) \cong \begin{cases} \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{(n-1)/2} \mathbb{Z}_{k(k+1)}^{\binom{2n}{n-2k-1}}, & \textit{for } n \textit{ odd},\\ \mathbb{Z}^{\binom{2n+1}{n}} \oplus \bigoplus_{k=1}^{n/2} \mathbb{Z}_k^{2\binom{2n}{n-2k}}, & \textit{for } n \textit{ even}. \end{cases}$$

Proof. This is a direct application of Theorem 3.9 and Equation (3.27). Note that taking $\sigma = 1$, this corollary gives a closed formula for K-groups of the ordinary group C^*-algebra $C^*(\Gamma)$.

Remark. In [10, Section 1], a procedure was outlined, which if implemented, would allow us to construct a much wider class of lattices $\Gamma_0 \subseteq N$ with associated multipliers σ_0 in Γ_0 such that $C^*(\Gamma_0, \sigma_0)$ is KK-equivalent to $C^*(\Gamma)$, for Γ the standard integer lattice in N. Thus Corollary 3.11 also gives the K-groups for all twisted group C^*-algebras of this form.

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