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Generators of matrix algebras in dimension 2 and 3

Helmer Aslaksen^{a,*}, Arne B. Sletsjøe^b^a Department of Mathematics, National University of Singapore, Singapore 117543, Singapore^b Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, 0316 Oslo, Norway

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Abstract

Let K be an algebraically closed field of characteristic zero and consider a set of 2×2 or 3×3 matrices. Using a theorem of Shemesh, we give conditions for when the matrices in the set generate the full matrix algebra.

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1. Introduction

Let K be an algebraically closed field of characteristic zero, and let $M_n = M_n(K)$ be the algebra of $n \times n$ matrices over K . Given a set $S = \{A_1, \dots, A_p\}$ of $n \times n$ matrices, we would like to have conditions for when the A_i generate the algebra M_n . In other words, determine whether every matrix in M_n can be written in the form $P(A_1, \dots, A_p)$, where P is a noncommutative polynomial. (We identify scalars with scalar matrices so the constant polynomials give the scalar matrices.) The case $n = 1$ is of course trivial, and when $p = 1$, the single matrix A_1 generates a commutative subalgebra. We therefore assume that $n, p \geq 2$. This question has been studied by many authors, see for example the extensive bibliography in [9]. We will give some results in the case of $n = 2$ or 3. We would like to thank the referees and the editor for making nontrivial improvements to the paper.

* Corresponding author.

E-mail address: aslaksen@math.nus.sg (H. Aslaksen), arnebs@math.uio.no (A.B. Sletsjøe).

URLs: <http://www.math.nus.edu.sg/aslaksen/> (H. Aslaksen), <http://www.math.uio.no/arnebs/> (A.B. Sletsjøe).

24 **2. General observations**

25 Let \mathcal{A} be the algebra generated by S . If we could show that the dimension of \mathcal{A} as a vector space
26 is n^2 , it would follow that $\mathcal{A} = M_n$. This can sometimes be done when we know a linear spanning
27 set $\mathcal{B} = \{B_1, \dots, B_q\}$ of \mathcal{A} . Let M be the $n^2 \times q$ matrix obtained by writing the matrices in \mathcal{B}
28 as column vectors. We would like to show that $\text{rank } M = n^2$. Since M is an $n^2 \times n^2$ matrix and
29 $\text{rank } M = \text{rank } (MM^*)$, it suffices to show that $\det(MM^*) \neq 0$. Unfortunately, the size of \mathcal{B} may
30 be big [11]. In this paper we will combine this method with results of Shemesh and Spencer and
31 Rivlin to get some simple results for $n = 2$ or 3 .

32 The starting point is the following well-known consequence of Burnside’s Theorem.

33 **Lemma 1.** *Let $\{A_1, \dots, A_p\}$ be a set of matrices in M_n where $n = 2$ or 3 . The A_i ’s generate M_n
34 if and only if they do not have a common eigenvector or a common left-eigenvector.*

35 We can therefore use the following theorem due to Shemesh [12].

36 **Theorem 2.** *Two $n \times n$ matrices, A and B , have a common eigenvector if and only if*

$$\sum_{k,l=1}^{n-1} [A^k, B^l]^* [A^k, B^l]$$

37 *is singular.*

38 Adding scalar matrices to the A_i ’s does not change the subalgebra they generate, so we some-
39 times assume that our matrices lie in $sl_n = \{M \in M_n | \text{tr } M = 0\}$. We also sometimes identify
40 matrices in M_n with vectors in K^{n^2} , and if $N_1, \dots, N_{n^2} \in M_n$, then $\det(N_1, \dots, N_{n^2})$ denotes
41 the determinant of the $n^2 \times n^2$ matrix whose j th column is N_j , written as $(N_{j1}, \dots, N_{jn})^t$, where
42 N_{jk} is the k th row of N_j for $k = 1, 2, \dots, n$. We write the scalar matrix aI as a . When we say
43 that a set of matrices generate M_n , we are talking about M_n as an algebra, while when we say that
44 a set of matrices form a basis of M_n , we are talking about M_n as a vector space.

45 **3. The 2×2 case**

46 The following theorem is well-known, but we include a proof since it illustrated a technique
47 we will use in the 3×3 case. Notice that the proof gives us an explicit basis for M_2 .

48 **Theorem 3.** *Let $A, B \in M_2$. A and B generate M_2 if and only if $[A, B]$ is invertible.*

49 **Proof.** A direct computation shows that

$$\det(I, A, B, AB) = -\det(I, A, B, BA) = \det[A, B].$$

50 Hence

$$\det(I, A, B, [A, B]) = 2\det[A, B]. \tag{1}$$

51 But if $I, A, B, [A, B]$ are linearly independent, then the dimension of \mathcal{A} as a vector space is 4,
52 so A and B generate M_2 . \square

We call $[M, N, P] = [M, [N, P]]$ a double commutator. The characteristic polynomial of A can be written as

$$x^2 - x \operatorname{tr} A + ((\operatorname{tr} A)^2 - \operatorname{tr} A^2)/2.$$

It follows that the discriminant of the characteristic polynomial of A can be written as

$$\operatorname{disc}(A) = 2 \operatorname{tr} A^2 - (\operatorname{tr} A)^2.$$

Lemma 4. *Let $A, B, C \in M_2$ and suppose that no two of them generate M_2 . Then A, B, C generate M_2 if and only if the double commutator $[A, B, C] = [A, [B, C]]$ is invertible.*

Proof. A direct computation shows that

$$\det(I, A, B, C)^2 = -\det[A, [B, C]] - \operatorname{disc}(A)\det[B, C]. \quad (2)$$

But if I, A, B, C are linearly independent, then A, B and C generate M_2 . \square

Notice that the above proof gives us an explicit basis for M_2 . We can now give a complete solution for the case $n = 2$.

Theorem 5. *The matrices $A_1, \dots, A_p \in M_2$ generate M_2 if and only if at least one of the commutators $[A_i, A_j]$ or double commutators $[A_i, A_j, A_k] = [A_i, [A_j, A_k]]$ is invertible.*

Proof. If $p > 4$, the matrices are linearly dependent, so we can assume that $p \leq 4$. Suppose that A_1, A_2, A_3, A_4 generate M_2 , but that no proper subset of them generates M_2 . Then the four matrices are linearly independent, and we can write the identity I as a linear combination of them. If the coefficient of A_4 in this expression is nonzero, then A_1, A_2, A_3, I span and therefore generate M_2 , so A_1, A_2, A_3 generate M_2 . Thus, if A_1, \dots, A_p generate M_2 , we can always find a subset of three of these matrices that generate M_2 . The result now follows from Theorem 3 and Lemma 4. \square

4. Two 3×3 matrices

In the case of two 3×3 matrices, we have the following well-known theorem.

Theorem 6. *Let $A, B \in M_3$. If $[A, B]$ is invertible, then A and B generate M_3 .*

For $M \in M_3$, we define $H(M)$ to be the linear term in the characteristic polynomial of M . Hence

$$H(M) = ((\operatorname{tr} M)^2 - \operatorname{tr} M^2)/2,$$

which is equal to the sum of the three principal minors of degree two of M . Notice that $H(M)$ is invariant under conjugation, and that if $[A, B]$ is singular, then $[A, B]$ is nilpotent if and only if $H([A, B]) = 0$.

The following theorem shows that if $[A, B]$ is invertible and $H([A, B]) \neq 0$, then we can give an explicit basis for M_3 .

Theorem 7. Let $A, B \in M_3$. Then

$$\det(I, A, A^2, B, B^2, AB, BA, [A, [A, B]], [B, [B, A]]) = 9 \det[A, B]H([A, B]), \quad (3)$$

so if $\det[A, B] \neq 0$ and $H([A, B]) \neq 0$, then

$$\{I, A, A^2, B, B^2, AB, BA, [A, [A, B]], [B, [B, A]]\}$$

form a basis for M_3 .

The proof of (3) is by direct computation. Notice that this can be thought of as a generalization of (1) and (2).

We can also use Shemesh's Theorem to characterize pairs of generators for M_3 .

Theorem 8. The two 3×3 matrices A and B generate M_3 if and only if both

$$\sum_{k,l=1}^2 [A^k, B^l]^* [A^k, B^l] \quad \text{and} \quad \sum_{k,l=1}^2 [A^k, B^l] [A^k, B^l]^*$$

are invertible.

5. Three or more 3×3 matrices

We start with the following theorem due to Laffey [8].

Theorem 9. Let \mathcal{S} be a set of generators for M_3 . If \mathcal{S} has more than four elements, then M_3 can be generated by a proper subset of \mathcal{S} .

It is therefore sufficient to consider the cases $p = 3$ or 4. Following the approach outlined earlier, we start by finding a linear spanning set. Using the polarized Cayley–Hamilton Theorem, Spencer and Rivlin [13,14] deduced the following theorem.

Theorem 10. Let $A, B, C \in M_3$. Define

$$S(A) = \{A, A^2\}$$

$$T(A, B) = \{AB, A^2B, AB^2, A^2B^2, A^2BA, A^2B^2A\}$$

$$S(A_1, A_2) = T(A_1, A_2) \cup T(A_2, A_1)$$

$$T(A, B, C) = \{ABC, A^2BC, BA^2C, BCA^2, A^2B^2C, CA^2B^2, ABCA^2\}$$

$$S(A_1, A_2, A_3) = \bigcup_{\sigma \in S_3} T(A_{\sigma(1)}, A_{\sigma(2)}, A_{\sigma(3)}).$$

1. The subalgebra generated by A and B is spanned by

$$I \cup S(A) \cup S(B) \cup S(A, B).$$

2. The subalgebra generated by A, B and C is spanned by

$$I \cup S(A) \cup S(B) \cup S(A, B) \cup S(A, B, C).$$

These spanning sets are not optimal. They include words of length 5. Paz [10] has proved that M_n can be generated by words of length $\lceil (n^2 + 2)/3 \rceil$. For M_3 this gives words of length 4. The general bound has been improved by Pappacena [11].

We next give a version of Shemesh's Theorem for three 3×3 matrices.

Theorem 11. *The matrices $A, B, C \in M_3$ have a common eigenvector if and only the matrix*

$$M(A, B, C) = \sum_{\substack{M \in S(A), \\ N \in S(B)}} [M, N]^* [M, N] + \sum_{\substack{M \in S(A), \\ N \in S(C)}} [M, N]^* [M, N] \\ + \sum_{\substack{M \in S(B), \\ N \in S(C)}} [M, N]^* [M, N] + \sum_{\substack{M \in S(A, B), \\ N \in S(C)}} [M, N]^* [M, N]$$

is singular.

Proof. Let \mathcal{A} be the algebra generated by A, B, C . Set

$$V = \bigcap_{\substack{M \in S(A), \\ N \in S(B)}} \ker[M, N] \bigcap_{\substack{M \in S(A), \\ N \in S(C)}} \ker[M, N] \bigcap_{\substack{M \in S(B), \\ N \in S(C)}} \ker[M, N] \bigcap_{\substack{M \in S(A, B), \\ N \in S(C)}} \ker[M, N].$$

We claim that V is invariant under \mathcal{A} . Let $v \in V$ and consider $\mathcal{A}v$. We know from Theorem 10 that any element of \mathcal{A} is a linear combination of terms of the form

$$p(A, B)C^i q(A, B)C^j r(A, B)$$

with $p(A, B), q(A, B), r(A, B) \in I \cup S(A) \cup S(B) \cup S(A, B)$. Since

$$v \in \ker[S(A, B), S(C)] \cap \ker[S(A), S(C)] \cap \ker[S(B), S(C)],$$

we get

$$p(A, B)C^i q(A, B)C^j r(A, B)v = p(A, B)C^i q(A, B)r(A, B)C^j v \\ = p(A, B)C^{i+j} q(A, B)r(A, B)v \\ = p(A, B)q(A, B)r(A, B)C^{i+j} v = C^{i+j} p(A, B)q(A, B)r(A, B)v.$$

In the same way we use the fact that $v \in [S(A), S(B)]$ to sort the terms of the form $p(A, B)q(A, B)r(A, B)v$, so that we finally get

$$\mathcal{A}v = \left\{ \sum a_{ijk} C^i B^j A^k v \mid 0 \leq i, j, k \leq 2, a_{ijk} \in K \right\}.$$

Using the above technique, it follows easily that $\mathcal{A}v \subset V$ and that V is \mathcal{A} invariant. Hence we can restrict \mathcal{A} to V , but since the elements of \mathcal{A} commute on V , they have a common eigenvector, and we can finish as in the proof of Theorem 2. \square

From this we deduce the following theorem.

Theorem 12. *Let $A, B, C \in M_3$. Then A, B, C generate M_3 if and only if both $M(A, B, C)$ and $M(A^t, B^t, C^t)$ are invertible.*

For the case of four matrices, we can prove the following theorem.

Theorem 13. The matrices $A_1, A_2, A_3, A_4 \in M_3$ have a common eigenvector if and only the matrix

$$M(A_1, A_2, A_3, A_4) = \sum_{\substack{i,j=1, \\ i < j}}^4 \left(\sum_{\substack{M \in S(A_i), \\ N \in S(A_j)}} [M, N]^* [M, N] \right) \\ + \sum_{\substack{i,j=1, \\ i < j}}^3 \left(\sum_{\substack{M \in S(A_i, A_j), \\ N \in S(A_4)}} [M, N]^* [M, N] \right) + \sum_{\substack{M \in S(A_1, A_2), \\ N \in S(A_3)}} [M, N]^* [M, N] \\ + \sum_{\substack{M \in S(A_1, A_2, A_3), \\ N \in S(A_4)}} [M, N]^* [M, N].$$

is singular.

Proof. Similar to the proof of Theorem 11. \square

From this we deduce the following theorem.

Theorem 14. Let $A, B, C, D \in M_3$. Then A, B, C, D generate M_3 if and only if both $M(A, B, C, D)$ and $M(A^t, B^t, C^t, D^t)$ are invertible.

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