NORTH-HOLLAND

Free Resolutions of Generic Symmetric Matrices

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ABSTRACT

We give an elementary construction of a finite free resolution of $k[P]/I_{n-1}$, where $k[P]$ is the ring of polynomial functions in the entries of a generic symmetric $n \times n$ matrix P, and I_{n-1} is the ideal generated by the $n-1$ minors of P.

1. INTRODUCTION

Let k be a field of characteristic zero, and let $G = O(m, k)$. Let $V = U^n$, where U is the standard module of G. Consider the action of G on $k[V]$, the space of polynomial functions on V. Denote the algebra of G invariants in $k[V]$ by $k[V]^G$. We can identify $k[V]$ with the polynomial ring $k[X]$, where X is an $m \times n$ matrix of indeterminates. If P is an $n \times n$ symmetric matrix of indeterminates, it is well known [5, Theorems 2.9.A and 2.17.A] that

$$k[V]^G \simeq k[X^t X] \simeq k[P]/I_{m+1},$$

where $I_{m+1} \subset k[P]$ is the ideal generated by the $(m+1) \times (m+1)$ minors of P.

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Recall that for a module M over a commutative algebra R, a finite free resolution of M is an exact sequence of R modules

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 = M \to 0,$$

where each F_j is a finitely generated free R module.

Our main result is to give a finite free resolution of $k[P]$ modules of $k[V]^G$ when $m = n - 2$. After finishing the first version of this paper, we discovered that this theorem was known (see Goto and Tachibana [1] and Józefiak [4]). However, since our approach is elementary and global, we feel that it has independent interest.

Let $\mathfrak{sl}(n, k[P])$, $\mathfrak{so}(n, k[P])$, and $\mathfrak{sm}(n, k[P])$ be the $n \times n$ matrices with entries in $k[P]$ that have trace equal to zero, are skew-symmetric, or are symmetric, respectively. Let m_{ij} be the ij minor of P, and let $M = (m_{ij})$ be the adjoint of P.

THEOREM 1. *Let P be an $n \times n$ symmetric matrix of indeterminates, and define the following maps:*

$$\phi_1(Y) = YP,$$

$$\phi_2(S) = PS + S^tP,$$

$$\phi_3(X) = \operatorname{tr}(XM) \qquad \text{where } M \text{ is the adjoint of } P,$$

$$\pi(f) = f + I_{n-1}.$$

Then the following is a finite free resolution of $k[P]/I_{n-1}$:

$$0 \to \mathfrak{so}(n, k[P]) \overset{\phi_1}{\to} \mathfrak{sl}(n, k[P]) \overset{\phi_2}{\to} \mathfrak{sm}(n, k[P])$$

$$\overset{\phi_3}{\to} k[P] \overset{\pi}{\to} k[P]/I_{n-1} \to 0. \tag{1}$$

Recall that for a graded module $M = \Sigma_{d \geqslant 0} M_d$ over a graded k algebra $R = \Sigma_{d \geqslant 0} R_d$, the Poincaré series $P_M(t)$ of M is defined to be

$$P_M(t) = \sum_{d \geqslant 0} (\dim_k M_d) t^d.$$

The following corollary is now immediate.

COROLLARY 2. *For $G = O(n-2)$ the Poincaré series of $k[V]^G \simeq k[P]/I_{n-1}$ is*

$$\frac{1 - \frac{n(n+1)}{2}t^{n-1} + (n^2-1)t^n - \frac{n(n-1)}{2}t^{n+1}}{(1-t)^{n(n+1)/2}} = \frac{\sum_{i=0}^{n-2}\binom{i+2}{2}t^i}{(1-t)^{n(n+1)/2-3}}. \tag{2}$$

In particular the determinantal variety defined by the ideal I_{n-1} has dimension $n(n+1)/2 - 3$ and degree

$$\sum_{i=0}^{n-2}\binom{i+2}{2} = \frac{n(n^2-1)}{6}.$$

The results in [1] and [4] are stated over a commutative ring with unit. From analyzing our proof it is clear that we do not need a field, and that we can relax the requirement about the characteristic being equal to zero. But we need the characteristic to be different from 2 [see the statement after (5)]. Because of our many explicit divisibility arguments, we also need the ring to be a unique factorization domain. Our proof is quite explicit, and the price we have to pay is less generality in the choice of rings to work with.

2. EXACTNESS AT $\mathfrak{sl}(n, k[P])$

The following theorem proves that the sequence (1) is exact at $\mathfrak{sl}(n, k[P])$.

THEOREM 3. *Let P be an $n \times n$ generic symmetric matrix of indeterminates, and let $S \in \mathfrak{sl}(n, k[P])$. Assume that PS is skew-symmetric, i.e.,*

$$PS + S^tP = 0. \tag{3}$$

Then S can be written in the form

$$S = YP, \qquad \textit{where} \quad Y \in \mathfrak{so}(n, k[P]). \tag{4}$$

Proof. We can think of the p_{ij} as representing points in a space of dimension $n(n+1)/2$. Let $E_1, \dots, E_{n(n-1)/2}$ be some ordering of the symmetrized matrix units $E_{ij} + E_{ji}$ with $1 \leqslant i,\ j \leqslant n$, and let $p_1, \dots, p_{n(n-1)/2}$ be the corresponding ordering of the p_{ij} with $i < j$. For $i = 0, \dots, n(n-1)/2$ let D_i be the subspace defined by $p_{i+1} = \cdots = p_{n(n-1)/2} = 0$ for $i < n(n-1)/2$, and define $D_{n(n-1)/2}$ to be the whole space. We observe that $P_0 = P|_{D_0} = \operatorname{diag}(p_{11}, \dots, p_{nn})$ and that for $i > 0$ we have $P_i = P|_{D_i} = P_0 + p_1 E_1 + \cdots + p_i E_i$.

We will prove (4) by restricting (3) to D_i and then using induction on i. Suppose that

$$P_0 Z + Z^t P_0 = 0. \tag{5}$$

Then $2p_{ii} z_{ii} = 0$ for all i, so $z_{ii} = 0$ for all i. (Here we need the characteristic to be different from 2.) Moreover, $p_{ii} z_{ij} + p_{jj} z_{ji} = 0$ for $i \neq j$, which implies that $z_{ij} = p_{jj} y_{ij}$ and $z_{ji} = p_{ii} y_{ji}$, where $y_{ij} + y_{ji} = 0$. This shows that

$$\text{if} \quad P_0 Z + Z^t P_0 = 0, \quad \text{then} \quad Z = Y P_0, \quad \text{where } Y \text{ is skew-symmetric.} \tag{6}$$

In particular,

$$S_0 = S|_{D_0} = Y_0 P_0, \qquad \text{where } Y_0 \text{ is skew-symmetric.} \tag{7}$$

We will now consider the restriction of (3) to D_1. We can assume that S is homogeneous of degree d. We can now write

$$\begin{aligned} P_1 = P|_{D_1} = P_0 + p_1 E_1 \quad \text{and} \quad S_1 &= S|_{D_1} \\ &= S_0 + p_1 S_{11} + \cdots + p_1^d S_{1d}, \end{aligned} \tag{8}$$

where S_{1i} does not contain p_1. (Notice that S_{1d} may be zero.) If we plug (8) into the restriction of (3) to D_1 we get

$$\begin{aligned} 0 &= P_1 S_1 + S_1^t P_1 \\ &= (P_0 + p_1 E_1)(S_0 + p_1 S_{11} + \cdots + p_1^d S_{1d}) \\ &\quad + (S_0^t + p_1 S_{11}^t + \cdots + p_1^d S_{1d}^t)(P_0 + p_1 E_1) \\ &= P_0 S_0 + S_0^t P_0 + p_1(E_1 S_0 + S_0^t E_1 + P_0 S_{11} + S_{11}^t P_0) + \cdots \\ &\quad + p_1^d(E_1 S_{1,d-1} + S_{1,d-1}^t E_1 + P_0 S_{1d} + S_{1d}^t P_0) \\ &\quad + p_1^{d+1}(E_1 S_{1d} + S_{1d}^t E_1). \end{aligned}$$

We can now use (7), and the coefficient of p_1 then becomes

$$P_0(S_{11} - Y_0E_1) + \left(E_1Y_0 + S_{11}^t\right)P_0 = 0,$$

so it follows from (6) that

$$S_{11} = Y_0E_1 + Y_1P_0, \qquad \text{where } Y_1 \text{ is skew-symmetric.}$$

Plugging this into the coefficient for p_1^2, we get that $S_{12} - Y_1E_1$ is also a solution of (5), so

$$S_{12} = Y_1E_1 + Y_2P_0, \qquad \text{where } Y_2 \text{ is skew-symmetric.}$$

Continuing like this, we get that

$$S_{1i} = Y_{i-1}E_1 + Y_iP_0, \qquad \text{where } Y_i \text{ is skew-symmetric,}$$

for $i = 0, \dots, d$. But when $i = d$ we see that S_{1d} has degree zero, so $Y_d = 0$. We finally plug this into the coefficient of p_1^{d+1}, and we get that

$$E_1S_{1d} + S_{1d}^tE_1 = E_1Y_{d-1}E_1 - E_1Y_{d-1}E_1 = 0,$$

which shows that the last equation is compatible with our solution. Hence

$$\begin{aligned} S_1 &= S_0 + p_1S_{11} + \cdots + p_1^dS_{1d} \\ &= Y_0P_0 + p_1(Y_0E_1 + Y_1P_0) + p_1^2(Y_1E_1 + Y_2P_0) + \cdots + p_1^dY_{d-1}E_1 \\ &= Y_0(P_0 + p_1E_1) + p_1Y_1(P_0 + p_1E_1) + \cdots + p_1^{d-1}Y_{d-1}(P_0 + p_1E_1) \\ &= \left(Y_0 + p_1Y_1 + \cdots\ p_1^{d-1}Y_{d-1}\right)P_1 = T_1P_1, \end{aligned}$$

where T_1 is skew-symmetric. This proves the restriction of (4) to D_1. The general induction step is similar, and the theorem follows. ■

3. EXACTNESS AT $\mathfrak{S}\mathfrak{m}(n, k[P])$

The exactness of the sequence (1) at $\mathfrak{S}\mathfrak{m}(n, k[P])$ follows from the next theorem.

THEOREM 4. *Let P be an $n \times n$ generic symmetric matrix of indeterminates, let $X \in \mathfrak{S}\mathfrak{m}(n, k[P])$, and let M be the adjoint of P. If*

$$\operatorname{tr} XM = 0, \tag{9}$$

then X can be written in the form

$$X = PS + S^tP, \qquad \textit{where} \quad S \in \mathfrak{sl}(n, k[P]). \tag{10}$$

Proof. To simplify our formulas we will write

$$A^s = A + A^t.$$

We will prove the theorem by using induction on the size n of the matrices. The result is trivially true for 1×1 matrices.

From now on we will assume that the result holds for all symmetric $(n-1) \times (n-1)$ matrices. In particular, the statements of Propositions 5 and 6 depend on this hypothesis.

Let C_i denote the subspace defined by setting $p_{1,n} = \cdots = p_{n-1,n} = 0$. Set

$$P_0 = P|_{C_0} = \begin{pmatrix} P' & 0 \\ 0 & p_{nn} \end{pmatrix},$$

where

$$P' = \begin{pmatrix} p_{11} & \cdots & p_{1,n-1} \\ \vdots & & \vdots \\ p_{1,n-1} & \cdots & p_{n-1,n-1} \end{pmatrix}.$$

We will denote the minors of P' by m'_{ij} and its adjoint by M'. Then

$$M_0 = \text{Adjoint}(P_0) = M|_{C_0} = \begin{pmatrix} p_{nn}M' & 0 \\ 0 & m_{nn} \end{pmatrix}.$$

We will break the proof into several propositions and lemmas.

PROPOSITION 5. *If Z_0 is symmetric and satisfies*

$$\operatorname{tr} Z_0 M_0 = 0,$$

then we can write

$$Z_0 = (P_0 S_0)^s + R_0,$$

where $\operatorname{tr} S_0 = 0$ *and*

$$R_0 = \begin{pmatrix} 0 & \alpha \\ \alpha^t & 0 \end{pmatrix}$$

for some $(n-1)$*-vector column* α*. Moreover, we can choose* S_0 *so that* α_i *does not contain any* p_{nn} *and* $(S_0)_{n-1,n} = 0$.

Proof. We write

$$Z_0 = (z_{ij}) = \begin{pmatrix} Z' & Z_n \\ Z_n^t & z_{nn} \end{pmatrix}.$$

Then

$$\operatorname{tr} Z_0 M_0 = p_{nn} \operatorname{tr} Z'M' + z_{nn} m_{nn} = 0,$$

which shows that $z_{nn} = p_{nn} w$, while we get no information about z_{in} for $i \leqslant n-1$. Observe that

$$\left[P_0 \begin{pmatrix} T & 0 \\ 0 & -\operatorname{tr} T \end{pmatrix} \right]^s = \begin{pmatrix} (P'T)^s & 0 \\ 0 & -2p_{nn} \operatorname{tr} T \end{pmatrix}$$

and

$$\left[P_0 \begin{pmatrix} 0 & 0 \\ u^t & 0 \end{pmatrix} \right]^s = \begin{pmatrix} 0 & p_{nn} u \\ p_{nn} u^t & 0 \end{pmatrix}.$$

By subtracting

$$\left[P_0 \begin{pmatrix} -\dfrac{w}{2(n-1)} I_{n-1} & 0 \\ 0 & w/2 \end{pmatrix} \right]^s$$

we can assume that $z_{nn} = 0$. (If the characteristic is $n-1$, we can still achieve this.) But then we get $\operatorname{tr} Z'M' = 0$, so we can use our assumption that Theorem 4 is true for symmetric $(n-1) \times (n-1)$ matrices to conclude that $Z' = (P'S')^s$, where $\operatorname{tr} S' = 0$. (If Z depends on p_{nn}, we can treat

p_{nn} as a constant, so we can still apply the induction hypothesis.) If we write $z_{in} = p_{nn}u_i + \alpha_i$ for $i \leqslant n - 1$, where α_i does not contain any p_{nn}, we get the desired expression for Z_0, namely

$$Z_0 = (P_0 S_0)^s + R_0, \tag{11}$$

where

$$S_0 = \begin{pmatrix} S' & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -\dfrac{w}{2(n-1)} I_{n-1} & 0 \\ 0 & w/2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ u^t & 0 \end{pmatrix} \tag{12}$$

and

$$R_0 = \begin{pmatrix} 0 & \alpha \\ \alpha^t & 0 \end{pmatrix},$$

where α_i does not contain p_{nn}. This completes the proof of Proposition 5. ■

We will now restrict to C_1. We have

$$P_1 = P|_{C_1} = \begin{pmatrix} P' & & 0 \\ & & p_{n-1,n} \\ 0 & p_{n-1,n} & p_{nn} \end{pmatrix} = P_0 + p_{n-1,n} E_1,$$

where $E_1 = E_{n-1,n} + E_{n,n-1}$. Then

$$M_1 = \text{Adjoint}(P_1) = M|_{C_1} = M_0 + p_{n-1,n} M_{11} + p_{n-1,n}^2 M_{12},$$

where

$$M_{11} = \begin{pmatrix} 0 & u \\ u^t & 0 \end{pmatrix} \qquad \text{with} \quad u_i = -m'_{i,n-1}$$

and

$$M_{12} = \begin{pmatrix} M'_{12} & 0 \\ 0 & 0 \end{pmatrix},$$

where $M'_{12} = (m'_{n-1,n-1;i,j})$ is an $(n-2) \times (n-2)$ matrix. Here $m'_{i,j;i',j'}$ denotes $(-1)^{i+j+i'+j'}$ times the determinant of the matrix obtained by removing rows i and i' and columns j and j' of P'. We will also write

$$\det P_1 = a_0 + a_1 p_{n-1,n} + a_2 p_{n-1,n}^2.$$

Since

$$P_1 M_1 = M_1 P_1 = (\det P_1) I,$$

we have

$$(P_0 + p_{n-1,n} E_1)(M_0 + p_{n-1,n} M_{11} + p_{n-1,n}^2 M_{12})$$
$$= a_0 + a_1 p_{n-1,n} + a_2 p_{n-1,n}^2,$$

and we get the following fundamental relations:

$$\begin{aligned} P_0 M_0 &= a_0 = M_0 P_0, \\ E_1 M_0 + P_0 M_{11} &= a_1 = M_{11} P_0 + M_0 E_1, \\ E_1 M_{11} + P_0 M_{12} &= a_2 = M_{12} P_0 + M_{11} E_1, \\ E_1 M_{12} &= 0 = M_{12} E_1. \end{aligned} \tag{13}$$

Next we prove the following proposition.

PROPOSITION 6. *If Z is symmetric and satisfies*

$$\operatorname{tr} Z M_1 = 0,$$

then we can write

$$Z = (P_1 S_1)^s$$

where $\operatorname{tr} S_1 = 0$.

Proof. Writing

$$Z = Z_0 + p_{n-1,n} Z_{11} + \cdots + p_{n-1,n}^d Z_{1d},$$

we get

$$\operatorname{tr}(Z_0 + p_{n-1,n} Z_{11} + \cdots + p_{n-1,n}^d Z_{1d})(M_0 + p_{n-1,n} M_{11} + p_{n-1,n}^2 M_{12})$$
$$= 0,$$

or

$$
\begin{aligned}
\operatorname{tr} Z_0 M_0 &= 0, \\
\operatorname{tr}(Z_0 M_{11} + Z_{11} M_0) &= 0, \\
\operatorname{tr}(Z_0 M_{12} + Z_{11} M_{11} + Z_{12} M_0) &= 0, \\
\operatorname{tr}(Z_{1,k-2} M_{12} + Z_{1,k-1} M_{11} + Z_{1k} M_0) &= 0 \qquad \text{for} \quad 3 \leqslant k \leqslant d, \\
\operatorname{tr}(Z_{1,d-1} M_{12} + Z_{1d} M_{11}) &= 0, \\
\operatorname{tr} Z_{1d} M_{12} &= 0.
\end{aligned}
\tag{14}
$$

We know already that Z_0 is of the form (11), so we will try to determine Z_{11}. Using (13) we get

$$
\begin{aligned}
&\operatorname{tr}(Z_0 M_{11} + Z_{11} M_0) \\
&\quad = \operatorname{tr}\left\{\left[(P_0 S_0)^s + R_0\right] M_{11} + Z_{11} M_0\right\} \\
&\quad = \operatorname{tr}\left(M_{11} P_0 S_0 + S_0^t P_0 M_{11} + Z_{11} M_0 + R_0 M_{11}\right) \\
&\quad = \operatorname{tr}\left[(a_1 I - M_0 E_1) S_0 + S_0^t (a_1 I - E_1 M_0) + Z_{11} M_0 + R_0 M_{11}\right] \\
&\quad = \operatorname{tr}\left[\left(-E_1 S_0 - S_0^t E_1 + Z_{11}\right) M_0\right] + \operatorname{tr} R_0 M_{11} = 0.
\end{aligned}
\tag{15}
$$

At this stage we need a little lemma.

LEMMA 7. *If Z is symmetric and satisfies* $\operatorname{tr} ZM_1 = 0$, *then we can write*

$$Z = (P_1 S)^s + Z'$$

where $\operatorname{tr} S = 0$ *and* $(Z')_{ii}$ *is divisible by* p_{ii}.

Proof. We first observe that if $(r, s) \neq (i, i)$, then $(M_1)_{rs}$ will have positive total degree in the ith row or ith column of P_1. If we combine this with $\operatorname{tr} ZM_1 = 0$, we see that we can write $(Z)_{ii}$ in the form $\Sigma s_r (P_1)_{ir}$. But if we let S be the matrix which has $(s_1, \dots, s_{i-1}, 0, s_{i+1}, \dots, s_n)$ in the ith column and zero elsewhere, the lemma follows. ∎

We will now continue the proof of Proposition 6. By Lemma 7 we can assume that $(Z)_{nn}$, and hence $(Z_{11})_{nn}$, is divisible by p_{nn}. We note that $\operatorname{tr} BM_0$ will have p_{nn} as a factor if $(B)_{nn} = 0$. But the nn entry of $E_1 S_0 + S_0^t E_1$ is $2(S_0)_{n-1,n}$, which is 0 by Proposition 5. Hence all the summands in the first term on the last line of (15) contain p_{nn} as a factor, while the second term does not contain any p_{nn}. Hence we have

$$\operatorname{tr}\left[\left(-E_1 S_0 - S_0^t E_1 + Z_{11}\right) M_0\right] = 0, \tag{16}$$

$$\operatorname{tr} R_0 M_{11} = 0. \tag{17}$$

We could now use (11) and (16) to get a formula for Z_{11}, but we will first show how we can use (17) to prove a generalization of (11). We can write (17) as

$$\sum_{i=1}^{n-1} \alpha_i m'_{i,n-1} = 0. \tag{18}$$

If we set

$$B = \begin{pmatrix} & & & \alpha_1 \\ & 0 & & \vdots \\ & & & \alpha_{n-2} \\ \alpha_1 & \cdots & \alpha_{n-2} & 2\alpha_{n-1} \end{pmatrix},$$

then it follows from (18) that $\operatorname{tr} R_0 M_{11} = 0$ is equivalent to $\operatorname{tr} BM' = 0$, so by induction we get $B = (P'S')^s$, where $\operatorname{tr} S' = 0$. Because of the special form of B, we can in fact choose S' to be of the form

$$\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$$

for some column $(n-2)$-vector γ. To do this, we will need the following lemma, which is a generalization of Theorem 3.

LEMMA 8. *Write*

$$P = \begin{pmatrix} P' & P_n \\ P_n^t & p_{nn} \end{pmatrix}.$$

Suppose that $S' \in \mathfrak{gl}(n-1, k[P])$ *and* $v \in k[P]^{n-1}$ *satisfy*

$$P'S' + vP_n^t = A, \tag{19}$$

where $A \in \mathfrak{so}(n-1, k[P])$. *Then we can express* S' *and* v *as*

$$S' = A'P' + \beta P_n^t \quad \text{and} \quad v = -\beta^t P'.$$

for some $A' \in \mathfrak{so}(n-1, k[P])$ *and* $\beta \in k[P]^{n-1}$.

Proof. The proof is similar to the proof of Theorem 3, so we will only outline the idea. We first restrict to D_0, i.e., P' diagonal. We then find that $S'|_{D_0} = A'_0 P'_0 + \beta_0 P_n^t$ and $v|_{D_0} = -\beta_0^t P'_0$. Then we restrict (19) to D_1, D_2, etc. and use induction. ∎

We will now deduce the following corollary.

COROLLARY 9. *Write*

$$P = \begin{pmatrix} P' & P_n \\ P_n^t & P_{nn} \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} S' & u \\ v^t & -\operatorname{tr} S' \end{pmatrix}.$$

If

$$(PS)^s = \begin{pmatrix} 0 & \alpha \\ \alpha^t & w \end{pmatrix},$$

then we can write

$$S = \begin{pmatrix} A' & \beta \\ -\beta^t & 0 \end{pmatrix} P + \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}$$

where A' *is skew-symmetric.*

(Notice that the first term will vanish when we multiply by P and symmetrize.)

Proof. If we multiply out, we see that $(P'S')^s + (vP_n^t)^s = 0$. This is equivalent to saying that

$$P'S' + vP_n^t = A, \tag{20}$$

where A is skew-symmetric. We must prove that we can then find A', β, and γ such that

$$S' = A'P' + \beta P_n^t, \tag{21}$$

$$v' = -\beta^t P', \tag{22}$$

$$u = A'P_n + \beta p_{nn} + \gamma, \tag{23}$$

$$-\operatorname{tr} S' = -\beta P_n. \tag{24}$$

But (21) and (22) follow from Lemma 8, (23) can be satisfied by choosing γ, and (24) follows from taking the trace of (21). This completes the proof of the corollary. ∎

By applying Corollary 9 to $(n-1)\times(n-1)$ matrices, we get that

$$B = \begin{pmatrix} & & & \alpha_1 \\ & 0 & & \vdots \\ & & & \alpha_{n-2} \\ \alpha_1 & \cdots & \alpha_{n-2} & 2\alpha_{n-1} \end{pmatrix} = \left[P'\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}\right]^s,$$

but this implies that

$$R_0 = \begin{pmatrix} 0 & \alpha \\ \alpha^t & 0 \end{pmatrix} = \left[P_0\begin{pmatrix} & 0 & \gamma \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]^s,$$

and we can merge R_0 with S_0. Since this does not change $(S_0)_{n-1} = 0$, the argument leading up to (16) still holds, except now $R_0 = 0$.

To sum up, we have shown that if Z_0 is symmetric and satisfies $\operatorname{tr} Z_0 M_0 = 0$, then $Z_0 = (P_0 S_0)^s + R_0$. But if we have the added condition that $\operatorname{tr} R_0 M_{11} = 0$, e.g. when $Z_0 = Z|_{C_0}$, where Z is a solution of $\operatorname{tr} ZM_1 = 0$, then we can choose S_0 so that $Z_0 = (P_0 S_0)^s$.

Returning to (16), we get that

$$Z_{11} = E_1 S_0 + S_0^t E_1 + P_0 S_{11}' + (S_{11}')^t P_0 + R_1,$$

where S_{11}', R_1 are as in Proposition 5. Substituting Z_0 and Z_{11} into the third equation of (14), and arguing as before, we get that $\operatorname{tr} R_1 M_{11} = 0$, so we can write

$$Z_{11} = E_1 S_0 + S_0^t E_1 + P_0 S_{11} + S_{11}^t P_0$$

by merging R_1 with S'_{11}. Continuing like this, we get

$$Z_{1i} = E_1 S_{1,i-1} + S^t_{1,i-1} E_1 + P_0 S_{1,i} + (S_{1,i})^t P_0 \qquad \text{for} \quad 1 \leqslant i \leqslant d.$$

We claim that in fact $S_{1,d} = 0$. We can assume without loss of generality that Z is homogeneous of degree d. But then $S_{1,i}$ is of degree $d - i - 1$, so $S_{1,d} = 0$. This shows that

$$\begin{aligned} Z &= (P_0 S_0)^s + p_{n-1,n}\left[(E_1 S_0)^s + (P_0 S_{11})^s\right] + \cdots \\ &\quad + p^{d-1}_{n-1,n}\left[(E_1 S_{1,d-2})^s + (P_0 S_{1,d-1})^s\right] + p^d_{n-1,n}(E_1 S_{1,d-1})^s \\ &= \left[P_1(S_0 + p_{n-1,n} S_{11} + \cdots + p^{d-1}_{n-1,n} S_{1,d-1})\right]^s \\ &= (P_1 S_1)^s. \end{aligned}$$

This completes the proof of Proposition 6.

The general induction step for the proof of Theorem 4 is in fact easier, since it is only in the first step that we need to worry about the R terms. We will outline the second step as an example. When we restrict M and X to C_2, we get $M_2 = M_1 + p_{n-2,n} M_{21} + p^2_{n-2,n} M_{22}$ and $X_2 = X_1 + p_{n-2,n} X_{21} + \cdots$. We then get equations corresponding to (14) except that M_0 is replaced by M_1, M_{1i} is replaced by M_{2i} and similarly for X. But this will then give equations of the form $\operatorname{tr} ZM_1 = 0$, so we will not get any R terms. This completes the proof of Theorem 4. ∎

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