

Quivers and the Invariant Theory of Levi Subgroups

HELMER ASLAKSEN, ENG-CHYE TAN, AND CHEN-BO ZHU

Department of Mathematics, National University of Singapore

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We develop a theory of invariants using the formalism of quivers, generalizing earlier results attributed to Procesi. As an application, let H be the Levi component of a parabolic subgroup of a classical Lie group G with Lie algebra \mathfrak{g} . We describe a finite set of generators for $\mathcal{P}[\mathfrak{g}]^H$, the space of H -invariant polynomials on \mathfrak{g} , as well as the H -invariants in the universal enveloping algebra, $\mathcal{U}(\mathfrak{g})^H$, thus generalizing the results of Klink and Ton-That, and Zhu. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let \mathbb{C} be the field of complex numbers. We begin by formulating our problem in the context of quivers. We shall define a *weighted quiver* as a pair (Q, w) where Q is a quiver with n points $\{1, 2, \dots, n\}$ and finitely many arrows $\phi_{i,j}$ from vertices i to j and where $w = (w_i) \in \{0^+, 0^-, 1, -1\}^n$ is a set of weights attached to the vertices. Let V_Q and E_Q be the set of vertices and arrows of Q , respectively. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ be a dimension vector of Q . We then associate to (Q, w, α) a group $G_x = G_{Q,w,\alpha}$ defined by

$$G_x = \prod_i G_{x_i}, \tag{1.1}$$

where

$$G_{x_i} = \begin{cases} GL(x_i, \mathbb{C}), & \text{if } w_i = 0^+ \text{ or } 0^- , \\ O(x_i, \mathbb{C}), & \text{if } w_i = 1, \\ Sp(x_i, \mathbb{C}), & \text{if } w_i = -1. \end{cases} \tag{1.2}$$

Note that if $G_{x_i} = Sp(x_i, \mathbb{C})$, then α_i needs to be even. For convenience, we shall refer to a weighted quiver simply as a quiver.

Let $A = (A_{\phi_{i,j}})$ be a representation of Q of dimension α , and let $M = M(Q, w, \alpha)$ be the representation space, i.e.,

$$M(Q, w, \alpha) = \bigoplus_{\phi_{i,j} \in E_Q} M_{i,j}, \tag{1.3}$$

where $M_{i,j} = M\varphi_{i,j} = \text{Hom}(\mathbb{C}^{z_i}, \mathbb{C}^{z_j})$ and $A_{\phi_{i,j}} \in M_{i,j}$. Here \mathbb{C}^{z_i} is the defining module of G_{x_i} , except that when $w_i = 0$, we let \mathbb{C}^{z_i} be the contragredient of the defining module of $G_{x_i} = GL(z_i, \mathbb{C})$. For each arrow $\phi_{i,j} \in E_Q$, the action of G_x on $M_{i,j}$ is as follows

$$((g_{x_1}, \dots, g_{x_n}) \psi)(x) = g_{x_i}(\psi(g_{x_j}^{-1}(x))), \quad x \in \mathbb{C}^{z_i}, \psi \in M_{i,j}.$$

(Note that our definition of a representation actually gives a representation of the opposite quiver in the standard literature, but we have adopted this convention to facilitate our exposition in Sections 3 and 4.)

Such quivers often occur naturally. One example is to consider a parabolic subgroup P of a complex classical Lie group G with Lie algebra \mathfrak{g} . Write the Levi decomposition of P as $P = HN$, and let the Lie algebra of the unipotent radical N be denoted by \mathfrak{n} . Then the actions of the Levi component H on \mathfrak{g} , \mathfrak{n} and $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ all give rise to subrepresentations of some weighted quivers. It is thus natural to consider the following.

Problem. To describe a finite set of generators for $\mathcal{P}[M(Q, w, \alpha)]^{G_x}$, the G_x -invariant polynomials over $M(Q, w, \alpha)$.

A harder problem will be to obtain a minimal set of generators and their relations. This has been treated in some special cases by [3, 4] and in low-dimensional cases by [13]. (The papers [3, 4] also treat relative invariants.)

We first observe that we can produce invariants in the following natural way. A cycle σ of Q is a finite sequence of arrows $\phi_1 \phi_2 \cdots \phi_s$ such that the head of ϕ_j equals the tail of ϕ_{j+1} ($j = 1, 2, \dots, s-1$) and the head of ϕ_s equals the tail of ϕ_1 . We write $\sigma = \phi_1 \phi_2 \cdots \phi_s$. We can attach to σ a G_x -invariant polynomial function $\text{Tr } \sigma$ on M by setting

$$\text{Tr } \sigma = \text{Tr}(A_{\phi_1} A_{\phi_2} \cdots A_{\phi_s}), \tag{1.4}$$

where $A_{\phi_j} \in M_{\phi_j}$. If $w = (0^+, \dots, 0^+)$, Procesi (unpublished; we thank H. Kraft for pointing this out to us) has shown that these generate the G_x -invariants of M .

If $w \neq (0^+, \dots, 0^+)$, there are other invariants. To describe those extra invariants, we associate to each quiver Q its *partial double* \tilde{Q} defined as follows.

Let $Q^\#$ be the quiver with vertices $\{1^\#, 2^\#, \dots, n^\#\}$ and an arrow

$\phi_{i^*,j^*} \in E_{Q^*}$ if and only if there is an arrow $\phi_{j,i} \in E_Q$. We shall consider Q^* as a weighted quiver, where $w_{i^*} = w_i$ if $w_i = \pm 1$, and $w_{i^*} = 0^-$ if $w_i = 0^+$ and vice versa. We then identify in $Q \sqcup Q^*$, the disjoint union of Q and Q^* , those pairs of vertices i and i^* such that $w_i = \pm 1$. The resulting quiver is the partial double \tilde{Q} . Observe that if all the weights of Q are zero, then \tilde{Q} is the disjoint union of Q and its opposite quiver Q^{op} obtained by reversing the arrows. Also, if all the weights of Q are either 1 or -1 , then \tilde{Q} is a quiver with the same vertices $\{1, 2, \dots, n\}$, but with a reverse arrow added for each arrow in Q .

From a given representation of our weighted quiver Q , we can construct a representation of \tilde{Q} as follows. For a vector space V , we use the notation

$$V^* = \begin{cases} V, & \text{if } V \text{ is a formed space,} \\ V^*, & \text{otherwise.} \end{cases} \tag{1.5}$$

Thus there is always a pairing between V and V^* . For two vector spaces V and W , let the pairings between V and V^* be denoted by $\langle \cdot, \cdot \rangle_1$ and W with W^* be denoted by $\langle \cdot, \cdot \rangle_2$. We construct a natural map $\#$ from $\text{Hom}(V, W)$ to $\text{Hom}(W^*, V^*)$ by the recipe

$$\langle \phi v, w \rangle_2 = \langle v, \phi^* w \rangle_1, \quad v \in V, w \in W^*, \phi \in \text{Hom}(V, W). \tag{1.6}$$

Of course, this is simply the adjoint map if neither V nor W is a formed space, and it is the transpose (respectively symplectic transpose) map if both V and W are equipped with symmetric (respectively skewsymmetric) forms. It is then clear that

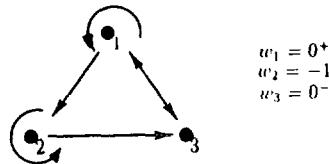
$$M(\tilde{Q}, w, \alpha) = \left\{ \bigoplus_{\phi_{ij} \in E_Q} (A_{ij} \oplus A_{ij}^*) \mid A_{ij} \in M_{ij} \right\} \tag{1.7}$$

gives a representation of \tilde{Q} , by the assignment

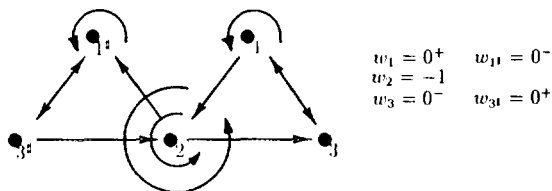
$$\phi_{ij} \rightarrow A_{ij}, \quad \phi_{j^*,i^*} \rightarrow (A_{ij})^*.$$

We can still consider cycles in \tilde{Q} , and the corresponding G_x invariants as in (1.4) using the representation of \tilde{Q} in $M(\tilde{Q}, w, \alpha)$.

EXAMPLE 1.



The partial double \tilde{Q} is



For instance, the cycle $\sigma = \phi_{12}\phi_{21}^*\phi_{13}^*\phi_{32}\phi_{22}\phi_{23}\phi_{31}$ in \tilde{Q} gives rise to a G_x -invariant in $\mathcal{P}[M(Q, w, \alpha)]$, namely

$$\text{Tr } \sigma = \text{Tr } A_{12}A_{12}^*A_{31}^*A_{23}^*A_{22}A_{23}A_{31}.$$

where $A_{ij} \in M_{\phi_{ij}}$.

Our first result is the following

THEOREM 1. *The G_x -invariant polynomials of $M(Q, w, \alpha)$ are generated by the functions $\text{Tr } \sigma$, where σ runs through all cycles of \tilde{Q} with the following property.*

(F) *For every vertex i or i^* of \tilde{Q} the cycle σ passes through i or i^* at most α_i^2 times.*

We shall discuss the proof of Theorem 1 in Section 2. In the same spirit, we will prove analogues of Theorem 1 in the following situations.

(a) Consider the invariant theory of several copies of general linear groups embedded diagonally in G_x .

(b) Allow M_{i^*} to be one of the canonical $\#$ -eigenspace in $\text{Hom}((\mathbb{C}^{\alpha_i})^{\#}, \mathbb{C}^{\alpha_i})$.

These results are given in Theorems 2 and 5 in Section 2. In principle, our technique will give a theory of invariants in terms of quivers if we understand the invariant theory of the groups involved. For instance, we can allow some of the G_{x_i} to be $SO(\alpha_i, \mathbb{C})$ and obtain analogues of Theorem 1, generalising results in [1].

In Section 3, we apply our results to the algebra $\mathcal{P}[\mathfrak{g}]^H$, where H is the Levi component of a parabolic subgroup of a classical Lie group G with Lie algebra \mathfrak{g} . This is parallel to the results of an earlier paper [14] on the construction of a spanning set of K -invariant polynomials in $\mathcal{P}[\mathfrak{g}]$, where K is a maximal compact subgroup of G . In fact, the results of [14] are contained in Section 2.

In Section 4, we explore the relationship between the H -invariant polynomials and H -invariant differential operators, and we give a construction

for a finite set of generators for $\mathcal{U}(\mathfrak{g})^H$, which is one of the main contributions of this article (see Theorem 13).

After submitting this paper, we received a preprint from Tuong Ton-That [7] which deals with similar problems, using different methods.

2. INVARIANT THEORY OF WEIGHTED QUIVERS

Our proof uses the polarization method in invariant theory. Basically, we can reduce the problem of finding polynomial invariants to finding multilinear invariants. If V is a vector space with a group G acting on it, we have an action of G on the polynomial algebra $\mathcal{P}[V]$ that respects the natural grading $\mathcal{P}[V] = \sum_j \mathcal{P}^j[V]$. Then the map

$$\underbrace{((V \otimes V \cdots \otimes V)^*)^G}_{j \text{ copies}} \rightarrow (\mathcal{P}^j[V])^G$$

$$\phi \rightarrow f_\phi, \quad \text{where } f_\phi(x) = \phi(x, \dots, x), x \in V,$$

is surjective. Thus, to find $\mathcal{P}[V]^G$, we only need to find the G -invariant linear functionals on $V^{\otimes j} = V \otimes V \otimes \cdots \otimes V$ (j copies).

Proof of Theorem 1. Recall the definition of $W^\#$ in Section 1. We have the isomorphism $\text{Hom}(W, V) \simeq W^\# \otimes V$ given by

$$(w^\# \otimes v)(w) = \langle w, w^\# \rangle_2 v. \tag{2.1}$$

Consider

$$\begin{aligned} \mathcal{S} &= \bigotimes_{\phi_{i,j} \in E_Q} M_{i,j}^{\otimes n_{ij}} \\ &\simeq \bigotimes_{\phi_{i,j} \in E_Q} ((\mathbb{C}^{z_i})^\# \otimes \mathbb{C}^{z_i})^{\otimes n_{ij}} \\ &= \bigotimes_{i \in V_Q} (((\mathbb{C}^{z_i})^\#)^{\otimes n_i} \otimes (\mathbb{C}^{z_i})^{\otimes \widehat{n}_i}) \\ &= \bigotimes_{i \in V_Q} \mathcal{S}_i, \end{aligned} \tag{2.2}$$

where \mathcal{S}_i is defined by the last equation, the n_{ij} are arbitrary non-negative integers, $n_i = \sum_{j=1}^n n_{ij}$ and $\widehat{n}_i = \sum_{j=1}^n n_{ji}$. Since all the G_{z_i} are reductive, it is clear that

$$(\mathcal{S}^\#)^{G_z} = \bigotimes_{i \in V_Q} (\mathcal{S}_i^\#)^{G_{z_i}}. \tag{2.3}$$

Our next step is to study $(\mathcal{S}_i^\#)^{G_{z_i}}$.

Case A. $G_{\alpha_i} = GL(\alpha_i, \mathbb{C})$. The First Fundamental Theorem of Invariant Theory says that $(\mathcal{S}_i^*)^{G_{\alpha_i}}$ is nontrivial if $n_i = \widehat{n}_i$ and any G_{α_i} -invariant linear functional on \mathcal{S}_i is a linear combination of the invariants

$$\mu_\rho(\psi_1 \otimes \cdots \otimes \psi_{n_i} \otimes v_1 \otimes \cdots \otimes v_{n_i}) = \prod_{k=1}^{n_i} \psi_k(v_{\rho(k)}), \tag{2.4}$$

where $\rho \in S_{n_i}$, the symmetric group on n_i letters.

Case B. $G_{\alpha_i} = O(\alpha_i, \mathbb{C})$ or $Sp(\alpha_i, \mathbb{C})$. The first Fundamental Theorem of Invariant Theory says that $(\mathcal{S}_i^*)^{G_{\alpha_i}}$ is nontrivial if $n_i + \widehat{n}_i$ is even and any G_{α_i} -invariant linear functional on \mathcal{S}_i is a linear combination of the invariants

$$\begin{aligned} &\mu_{Y,\phi}((\psi_1 \otimes \cdots \otimes \psi_{n_i}) \otimes (\psi_{n_i+1} \otimes \cdots \otimes \psi_{n_i+\widehat{n}_i})) \\ &= \prod_{j \in Y} \langle \psi_j, \psi_{\phi(j)} \rangle, \end{aligned} \tag{2.5}$$

where Y is any subset of $\{1, \dots, n_i + \widehat{n}_i\}$ with $(n_i + \widehat{n}_i)/2$ elements and ϕ is any bijective map from Y to $\{1, \dots, n_i + \widehat{n}_i\} - Y$. Essentially, this says that the invariants are generated by all possible pairings of $\psi_1 \otimes \cdots \otimes \psi_{n_i + \widehat{n}_i}$.

By virtue of (2.3), any G_{α_i} -invariant in \mathcal{S}^* is a linear combination of products of invariants of the form (2.4) or (2.5). The next step is to describe these invariants in matrix notation, using the isomorphism given by (2.2).

Case A. $G_{\alpha_i} = GL(\alpha_i, \mathbb{C})$. Without loss of generality, we can consider

$$\prod_{i=1}^n \prod_{k=1}^{n_i} \psi_{i,k}(v_{i,\rho_i(k)}). \tag{2.6}$$

We shall use the first indices to determine which copy of M a component $\psi \otimes v$ sits in, i.e., in our convention $A_{\phi_j}(l) = \psi_{j,l} \otimes v_{i,l}$ occurs in the l th copy of $M_{ij}^{\otimes n_{ij}}$. We start with the pairing $\psi_{1,1}(v_{1,\rho_1(1)})$. There exist a j_1 such that the product $\psi_{j_1,\rho_1(1)} \otimes v_{1,\rho_1(1)}$ occurs in some copy of $M_{1,j_1}^{\otimes n_{1,j_1}}$. Now $v_{j_1,\rho_{j_1}(\rho_1(1))}$ is paired with $\psi_{j_1,\rho_1(1)}$ in (2.6), and in turn for some j_2 , the term $\psi_{j_2,\rho_{j_2}(\rho_1(1))} \otimes v_{j_1,\rho_{j_1}(\rho_1(1))}$ occurs in the $\rho_{j_1}\rho_1(1)$ th copy of $M_{j_1,j_2}^{\otimes n_{j_1,j_2}}$. Repeating in this way, we obtain a sequence of integers j_1, j_2, \dots, j_k with $j_{k+1} = 1$ for some k . Implicit in this sequence is a cycle $\phi_{j_1}\phi_{j_2} \cdots \phi_{j_k}$ in Q , and we see that

$$\begin{aligned} &\prod_{l=1}^k \psi_{j_l,\rho_{j_{l-1}} \cdots \rho_1(1)}(v_{j_l,\rho_{j_l}\rho_{j_{l-1}}\rho_{j_{l-2}} \cdots \rho_1(1)}) \\ &= \text{Tr}(A_{\phi_{j_1}}(\rho_1(1)) \cdots A_{\phi_{j_{k-1}j_k}}(\rho_{j_k-1} \cdots \rho_1(1)) \\ &\quad \times A_{\phi_{j_k}}(\rho_{j_k} \cdots \rho_1(1))). \end{aligned} \tag{2.7}$$

So we can write (2.5) in terms of products arising from cycles as in (2.7).

Case B. $G_{\alpha_i} = O(\alpha_i, \mathbb{C})$ or $Sp(\alpha_i, \mathbb{C})$. This is different from Case A, since there is an additional set of invariants arising from the fact that ψ_i may

contract among themselves (cf. (2.5)), instead of contracting only with v_j as in the GL case. We appeal to the map $\#$ which has the effect of permuting the first and second factor in $\mathbb{C}^{x_i} \otimes \mathbb{C}^{x_j}$ up to a multiple of $+1$ or -1 (see Lemma 3). Because of the way our partial double \tilde{Q} is constructed, a moment's reflection and a similar argument as in the GL case give the desired result.

The finiteness condition (F) is a consequence of results due to Procesi and Raszmyslov (see [8, 9]), which say that the invariants of several $n \times n$ matrices are generated by the invariants of degree $\leq n^2$. ■

Let us now consider the following situation. Suppose that there are subsets $J_1 = \{i_1, \dots, i_k\}$ and $J_2 = \{\tilde{i}_1, \dots, \tilde{i}_k\}$ of V_Q with $\alpha_{i_j} = \alpha_{\tilde{i}_j}$, $w_{i_j} = 0^+$ and $w_{\tilde{i}_j} = 0^-$ for $1 \leq j \leq k$. We want to describe the invariants in $\mathcal{P}[M(Q, w, \alpha)]$ of

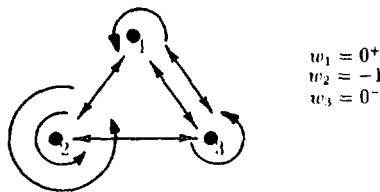
$$H_{(J_1, J_2)} = (\prod_{i \notin J_1 \cup J_2} G_{\alpha_i}) \times GL(\alpha_{i_1}, \mathbb{C}) \times \dots \times GL(\alpha_{i_k}, \mathbb{C}) \quad (2.8)$$

embedded in G_α using the diagonal embedding (for $1 \leq l \leq k$)

$$GL(\alpha_{i_j}, \mathbb{C}) \hookrightarrow GL(\alpha_{i_j}, \mathbb{C}) \times GL(\alpha_{\tilde{i}_j}, \mathbb{C}).$$

Denote by $\tilde{Q}_{(J_1, J_2)}$ the quiver obtained from \tilde{Q} by identifying the vertices i_j with \tilde{i}_j^* and \tilde{i}_j with i_j^* . We will call $\tilde{Q}_{(J_1, J_2)}$ the *collapsing* of \tilde{Q} with respect to the pair (J_1, J_2) . Note that if $J_1 = J_2 = \emptyset$, then $\tilde{Q}_{(J_1, J_2)}$ is just \tilde{Q} . The underlying representation space of $\tilde{Q}_{(J_1, J_2)}$ is the same as that of \tilde{Q} .

EXAMPLE 2. We illustrate using Example 1 again. Given a dimension vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, where $\alpha_1 = \alpha_3$, we can take $J_1 = \{1\}$, $J_2 = \{3\}$ and consider the invariants of $H_{(J_1, J_2)}$ in $\mathcal{P}[M(Q, w, \alpha)]$. The collapsed quiver $\tilde{Q}_{(J_1, J_2)}$ is as follows.



Note that we have identified vertices 1^* and 3^* in Example 1 with vertices 3 and 1, respectively. Also observe that there are two arrows from vertices 1 to 3 and vice versa. Since the two quivers have the same representation space, the description of $\text{Tr } \sigma$ arising from a cycle σ in $\tilde{Q}_{(J_1, J_2)}$ is exactly the same as in Example 1.

The following theorem can be proved by using exactly the same technique as in the proof of Theorem 1, and we shall thus omit it.

THEOREM 2. *Let the notations be as in the above, and let $H_{(J_1, J_2)} \subset G_\alpha$ be given as in (2.8). The $H_{(J_1, J_2)}$ -invariant polynomials of $M(Q, w, \alpha)$ are generated by the functions $\text{Tr } \sigma$, where σ runs through all cycles of $\tilde{Q}_{(J_1, J_2)}$ with the following property.*

(F) *For every vertex i or i^* of $\tilde{Q}_{(J_1, J_2)}$, the cycle σ passes through i or i^* at most α_i^2 times.*

The two sets of arrows in Example 2 suggests a finer version of Theorem 2. Recall the isomorphism $\text{Hom}(W, V) \simeq W^* \otimes V$ given by (2.1). Similarly, we have $\text{Hom}(V^*, W^*) \simeq V \otimes W^*$. The map $\#$ from $\text{Hom}(W, V)$ to $\text{Hom}(V^*, W^*)$ thus induces a map from $W^* \otimes V$ to $V \otimes W^*$, which we still denote by $\#$. The next lemma follows from a simple computation.

LEMMA 3. *The map*

$$\#: W^* \otimes V \rightarrow V \otimes W^*$$

satisfies

$$(w \otimes v)^\# = \varepsilon_V(v \otimes w), \quad w \in W^*, v \in V,$$

where $\varepsilon_V = -1$ if $\langle \cdot, \cdot \rangle_1$ is skewsymmetric and $+1$ otherwise.

If $W = V^*$, the map $\#$ is an involution, i.e., $\#^2 = 1_{\text{Hom}(V^*, V)}$, the identity map on $\text{Hom}(V^*, V)$. Let

$$\text{Hom}(V^*, V) = \text{Hom}^+(V^*, V) \oplus \text{Hom}^-(V^*, V)$$

be the decomposition of $\text{Hom}(V^*, V)$ into the $+1$ and (-1) -eigenspace of $\#$. Then both eigenspaces are invariant under the isometry group G of the form on V if V is a formed space and $GL(V)$ otherwise. Moreover, in the first case $\text{Hom}^-(V^*, V)$ can be identified with the Lie algebra of G .

Now we extend Theorem 2 to allow M_{i, i^*} to be any one of the canonical G_{α_i} -invariant subspaces $\text{Hom}^+((\mathbb{C}^{\alpha_i})^\#, \mathbb{C}^{\alpha_i})$ or $\text{Hom}^-((\mathbb{C}^{\alpha_i})^\#, \mathbb{C}^{\alpha_i})$ in $\text{Hom}((\mathbb{C}^{\alpha_i})^\#, \mathbb{C}^{\alpha_i})$. Here we assume that an arrow from i to i^* exists in $\tilde{Q}_{(J_1, J_2)}$. This can happen under the following situations:

- (a) If $w_i = \pm 1$ and there is an arrow $\phi_{i, i} \in E_Q$. Note that $i^* = i \in E_{\tilde{Q}}$.
- (b) Suppose that for some $i \in J_1, \tilde{i} \in J_2$, there is an arrow $\phi_{i, \tilde{i}} \in E_Q$. Then under the identification of i with \tilde{i}^* and \tilde{i} with i^* , there is an arrow $\phi_{i, i^*} \in E_{\tilde{Q}_{(J_1, J_2)}}$.

We need the following simple result.

LEMMA 4. Let G be a reductive group acting on a vector space V . If $V_1 \subset V$ is a G -invariant subspace, then the restriction map

$$r_{V|V_1}: \mathcal{P}[V]^G \rightarrow \mathcal{P}[V_1]^G$$

is surjective.

Proof. Since G is reductive, there exists a G -invariant subspace V_2 , such that $V = V_1 \oplus V_2$. Then we have,

$$\begin{aligned} \mathcal{P}(V) &= \mathcal{P}(V_1) \otimes \mathcal{P}(V_2) = \mathcal{P}(V_1) \otimes (\mathbb{C} \oplus V_2^* \mathcal{P}(V_2)) \\ &= \mathcal{P}(V_1) \oplus V_2^* \mathcal{P}(V). \end{aligned}$$

Again, since G is reductive, we have,

$$\mathcal{P}(V)^G = \mathcal{P}(V_1)^G \oplus (V_2^* \mathcal{P}(V))^G,$$

and the lemma follows. ■

Let $I = \{i_1, i_2, \dots, i_k\} \subset V_{\mathcal{Q}(J_1, J_2)}$ be such that there are arrows from i_l to $i_l^\#$ for $1 \leq l \leq k$. Instead of (1.3), we now consider

$$M_{(J_1, J_2), I}(Q, w, \alpha) = \bigoplus_{\phi_{i,j} \in E_{\mathcal{Q}(J_1, J_2)}} M_{i,j},$$

where

$$M_{i,j} = \begin{cases} \text{Hom}^+((\mathbb{C}^{x_i})^\#, \mathbb{C}^{x_i}) \text{ or } \text{Hom}^-((\mathbb{C}^{x_i})^\#, \mathbb{C}^{x_i}), \\ \text{if } \{i, j\} = \{i_l, i_l^\#\} \text{ with } i_l \in I, \\ \text{Hom}(\mathbb{C}^{x_j}, \mathbb{C}^{x_i}), \\ \text{otherwise.} \end{cases}$$

In such cases, to describe the invariants of $H_{(J_1, J_2)}$ in $\mathcal{P}[M_{(J_1, J_2), I}(Q, w, \alpha)]$, we shall need to remove redundant arrows in our quiver $\mathcal{Q}_{(J_1, J_2)}$ appropriately. More precisely, we define the deleting $\tilde{\mathcal{Q}}_{(J_1, J_2), I}$ of $\mathcal{Q}_{(J_1, J_2)}$ with respect to I to be the quiver obtained by taking away one of two arrows from i_l to $i_l^\#$ in $\mathcal{Q}_{(J_1, J_2)}$ for $i_l \in I$. As an example, consider Example 2 with $I = \{1, 2, 3\}$. Then $\tilde{\mathcal{Q}}_{(J_1, J_2), I}$ is simply the quiver obtained from $\mathcal{Q}_{(J_1, J_2)}$ by removing one of the arrows from 1 to 3, around 2 and from 3 to 1. In view of Lemma 4, we have the following modified version of Theorem 2.

THEOREM 5. Let the notations be as above. The $H_{(J_1, J_2)}$ -invariant polynomials of $M_{(J_1, J_2), I}(Q, w, \alpha)$ are generated by the functions $\text{Tr } \sigma$, where σ runs through all cycles of $\tilde{\mathcal{Q}}_{(J_1, J_2), I}$ with the following property.

(F) For every vertex i or i^* of $\tilde{Q}_{(J_1, J_2), I}$ the cycle σ passes through i or i^* at most α_i^2 times.

3. INVARIANT THEORY OF LEVI SUBGROUPS

Let G be a classical Lie group and \mathfrak{g} be its Lie algebra. If $P = HN$ is a parabolic subgroup of G with Levi component H , it is easy to see that

$$\mathcal{P}[\mathfrak{g}]^H \simeq \mathcal{P}[\mathfrak{g}_{\mathbb{C}}]^{H_{\mathbb{C}}}$$

as vector spaces. Thus we will only consider complex groups.

Consider the groups $G = GL(n, \mathbb{C})$, $O(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$. Let V be the defining module of G , and (\cdot, \cdot) be the associated nondegenerate quadratic form on V if $G = O(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$. Let

$$L = \begin{cases} n & \text{if } G = GL(n, \mathbb{C}) \text{ or } Sp(2n, \mathbb{C}), \\ \lfloor n/2 \rfloor & \text{if } G = O(n, \mathbb{C}). \end{cases}$$

For $G = GL(n, \mathbb{C})$, we choose a standard basis $\{e_1, e_2, \dots, e_L\}$ on V . For $G = O(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$, we choose a standard basis $\{e_1, \dots, e_L\}$ for a maximal isotropic subspace of the form (\cdot, \cdot) . Then the standard flag (or isotropic flag) of G is

$$\{0\} \subset \{e_1\} \subset \{e_1, e_2\} \subset \dots \subset \{e_1, \dots, e_L\}.$$

The parabolic subgroups of G are then given, up to conjugation in G , by the stabilisers of subflags of the standard flag. Write

$$V_k = \text{span}\{e_1, \dots, e_k\},$$

where $1 \leq k \leq L$. A subflag of the standard flag can then be written as a sequence

$$\{0\} \subset V_{\alpha_1} \subset V_{\alpha_1 + \alpha_2} \subset \dots \subset V_{\alpha_1 + \alpha_2 + \dots + \alpha_l} \subset V_L. \tag{3.1}$$

Let α_l be defined by

$$\begin{cases} \alpha_1 + \dots + \alpha_l = n & \text{if } G = GL(n, \mathbb{C}) \text{ or } Sp(2n, \mathbb{C}), \\ 2\alpha_1 + \dots + 2\alpha_{l-1} + \alpha_l = n & \text{if } G = O(n, \mathbb{C}). \end{cases}$$

For simplicity we shall denote the flag in (3.1) by the sequence $\alpha_1, \alpha_2, \dots, \alpha_l$.

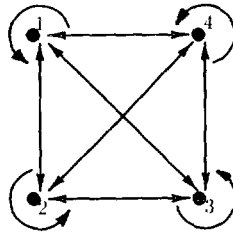
Let the Levi component of the stabiliser of this flag be denoted by $H_{\alpha_1, \alpha_2, \dots, \alpha_l}$. It is not difficult to see that

$$H_{\alpha_1, \alpha_2, \dots, \alpha_l} = \begin{cases} GL(\alpha_1, \mathbb{C}) \times \dots \times GL(\alpha_{l-1}, \mathbb{C}) \times GL(\alpha_l, \mathbb{C}), \\ \quad \text{if } G = GL(n, \mathbb{C}), \\ GL(\alpha_1, \mathbb{C}) \times \dots \times GL(\alpha_{l-1}, \mathbb{C}) \times O(\alpha_l, \mathbb{C}), \\ \quad \text{if } G = O(n, \mathbb{C}), \\ GL(\alpha_1, \mathbb{C}) \times \dots \times GL(\alpha_{l-1}, \mathbb{C}) \times Sp(2\alpha_l, \mathbb{C}), \\ \quad \text{if } G = Sp(2n, \mathbb{C}). \end{cases}$$

Certainly $H_{\alpha_1, \dots, \alpha_2, \dots, \alpha_l}$ acts on $\mathcal{P}(\mathfrak{g})$, the polynomial algebra on the Lie algebra \mathfrak{g} of G . This action arises from the restriction of the Adjoint action of G on \mathfrak{g} . We shall associate to each flag a quiver (Q, w) such that for an appropriate dimension vector α and some (J_1, J_2) , the Levi component $H_{\alpha_1, \dots, \alpha_l} \simeq H_{(J_1, J_2)} \subset G_\alpha$ acts on the representation space $M(Q, w, \alpha)$, which is isomorphic to the above action of $H_{\alpha_1, \dots, \alpha_l}$ on \mathfrak{g} . Let us treat each case individually.

Case A. $GL(n, \mathbb{C})$. Corresponding to the flag (3.1), there is a quiver (Q, w) , which has l points $\{1, 2, \dots, l\}$ with arrows $\{\phi_{ij} \mid 1 \leq i, j \leq l\}$ and weight $w = (0^+, 0^+, \dots, 0^+)$. If we take the dimension vector $\alpha = (\alpha_1, \dots, \alpha_l)$, then $G_\alpha = H_{\alpha_1, \dots, \alpha_l}$ acts on the representation space $M(Q, w, \alpha) = \bigoplus_{i,j=1}^n \text{Hom}(\mathbb{C}^{\alpha_i}, \mathbb{C}^{\alpha_j})$. Applying Theorem 1 to this quiver, we can deduce the results of [6].

A diagram illustrating this quiver for $l=4$ is as follows.



Case B. For ease of notation, let

$$\varepsilon = \begin{cases} 1, & \text{if } G = O(n, \mathbb{C}), \\ -1, & \text{if } G = Sp(2n, \mathbb{C}). \end{cases}$$

Let $\{e_1, \dots, e_L, f_1, \dots, f_L\}$ be a maximal set of independent vectors in V such that

$$(e_i, e_j) = 0 = (f_i, f_j) \quad \text{and} \quad (e_i, f_j) = \delta_{ij}.$$

The Levi component of the stabiliser of the flag in (3.1) the decomposition

$$V = X_{x_1} \oplus \cdots \oplus X_{x_{l-1}} \oplus X_{x_l} \oplus \bar{X}_{x_{l-1}} \oplus \cdots \oplus \bar{X}_{x_1},$$

where

$$\begin{aligned} X_{x_j} &= \text{span}\{e_{x_1 + \cdots + x_{j-1} + 1}, \dots, e_{x_1 + \cdots + x_j}\}, & 1 \leq j \leq l-1, \\ \bar{X}_{x_j} &= \text{span}\{f_{x_1 + \cdots + x_{j-1} + 1}, \dots, f_{x_1 + \cdots + x_j}\}, & 1 \leq j \leq l-1, \end{aligned}$$

and X_{x_l} is the orthogonal complement of $X_{x_1} \oplus \cdots \oplus X_{x_{l-1}} \oplus \bar{X}_{x_{l-1}} \oplus \cdots \oplus \bar{X}_{x_1}$ in V with respect to the form (\cdot, \cdot) . Observe that for $1 \leq i \leq l-1$, the space $X_{x_i}^*$ can be identified with \bar{X}_{x_i} via the pairing

$$x(y) = (x, y), \quad x \in \bar{X}_{x_i}, y \in X_{x_i}. \tag{3.2}$$

The same pairing also identifies $X_{x_i}^*$ with X_{x_i} .

Let us describe our quiver (Q, w) . It has $2l-1$ points $\{1, 2, \dots, 2l-1\}$ with arrows

$$\begin{aligned} &\{\phi_{ij} \mid 1 \leq i \leq l-1, 1 \leq j \leq 2l-i\} \\ &\cup \{\phi_{ij} \mid 2l-j \leq i \leq 2l-1, 1 \leq j \leq l\}, \end{aligned} \tag{3.3}$$

and weight $w = (w_i)$, where

$$\begin{cases} w_i = 0^+, & w_{l+i} = 0^-, & \text{for } 1 \leq i \leq l-1, \\ w_l = \varepsilon. \end{cases} \tag{3.4}$$

Then as a vector space, the Lie algebra \mathfrak{g} of G has the following decomposition

$$\mathfrak{g} = \sum_{\phi_{ij} \in E_Q} M_{ij}, \tag{3.5}$$

where

$$M_{ij} = \begin{cases} \text{Hom}(X_{x_i}, X_{x_j}), & \text{if } 1 \leq i \leq l-1, 1 \leq j \leq l, \\ \text{Hom}(\bar{X}_{x_{2l-j}}, X_{x_i}), & \text{if } 1 \leq i \leq l-1, l+1 \leq j < 2l-i, \\ \text{Hom}^-(\bar{X}_{x_i}, X_{x_j}), & \text{if } 1 \leq i \leq l-1, j = 2l-i, \\ \text{Hom}(X_{x_i}, \bar{X}_{x_{2l-i}}), & \text{if } 2l-i < j \leq 2l-1, 1 \leq j \leq l, \\ \text{Hom}^-(X_{x_j}, \bar{X}_{x_i}), & \text{if } i = 2l-j, 1 \leq j \leq l-1, \\ \text{Hom}^-(X_{x_i}, X_{x_i}), & \text{if } i = j = l. \end{cases} \tag{3.6}$$

The dimension vector of this representation is

$$\alpha = \begin{cases} (\alpha_1, \dots, \alpha_{l-1}, \alpha_l, \alpha_{l-1}, \dots, \alpha_1), & \text{if } G = O(n, \mathbb{C}), \\ (\alpha_1, \dots, \alpha_{l-1}, 2\alpha_l, \alpha_{l-1}, \dots, \alpha_1), & \text{if } G = Sp(2n, \mathbb{C}). \end{cases} \quad (3.7)$$

Note that the involution $\#$ of $\text{Hom}(V, V)$ given by Eq. (1.6) preserves $\text{Hom}(\bar{X}_{x_i}, X_{x_i})$, $\text{Hom}(X_{x_i}, \bar{X}_{x_i})$, $1 \leq i \leq l-1$, and $\text{Hom}(X_{x_l}, X_{x_l})$. Therefore lines 2, 4, 6 of (3.6) make sense. Moreover, under the identification $\text{Hom}(\bar{X}_{x_i}, X_{x_i}) \simeq \text{Hom}(X_{x_i}^*, X_{x_i})$ (using (3.2)), the $\#$ map becomes the $-\varepsilon_*$ map and so we have

$$\text{Hom}^-(\bar{X}_{x_i}, X_{x_i}) \simeq \begin{cases} \text{Hom}^-(X_{x_i}^*, X_{x_i}), & \text{if } G = O(n, \mathbb{C}), \\ \text{Hom}^+(X_{x_i}^*, X_{x_i}), & \text{if } G = Sp(2n, \mathbb{C}). \end{cases} \quad (3.8)$$

Likewise for $\text{Hom}^-(X_{x_i}, \bar{X}_{x_i})$.

Let

$$\begin{aligned} J_1 &= \{l-i \mid 1 \leq i \leq (l-1)\}, \\ J_2 &= \{l+i \mid 1 \leq i \leq (l-1)\}, \\ I &= \{1, \dots, l\}, \end{aligned} \quad (3.9)$$

then $H = H_{x_1, \dots, x_l} \simeq H_{(J_1, J_2)} \subset G_x$ acts on the representation space $M_{(J_1, J_2), l}(Q, w, \alpha)$. More precisely, let

$$G_0 = G|_{x_{x_l}} \simeq \begin{cases} O(\alpha_l, \mathbb{C}), & \text{if } G = O(n, \mathbb{C}), \\ Sp(2\alpha_l, \mathbb{C}), & \text{if } G = Sp(2n, \mathbb{C}), \end{cases}$$

then

$$\begin{aligned} H &= H_{(J_1, J_2)} \simeq GL(\alpha_1, \mathbb{C}) \times \dots \times GL(\alpha_{l-1}, \mathbb{C}) \times G_0 \\ &\hookrightarrow GL(\alpha_1, \mathbb{C}) \times \dots \times GL(\alpha_{l-1}, \mathbb{C}) \times G_0 \\ &\quad \times GL(\alpha_{l-1}, \mathbb{C}) \times \dots \times GL(\alpha_1, \mathbb{C}) \end{aligned}$$

as follows $(g_1, \dots, g_l) \in H \hookrightarrow (g_1, \dots, g_{l-1}, g_l, g_{l-1}, \dots, g_1) \in G_x$.

Remark. In Lie group literature, the imbedding is usually given by

$$(g_1, \dots, g_l) \hookrightarrow (g_1, \dots, g_{l-1}, g_l, (g'_{l-1})^{-1}, \dots, (g'_1)^{-1}) \in G_x.$$

In this article we have $w_{l+1} = \dots = w_{2l-1} = 0$. Thus the last $(l-1)$ components of G_x act contragradiently.

To look for the H invariants, we first form the partial double \tilde{Q} with vertices $i, i^\#$ with $1 \leq i \leq 2l-1$, $i \neq l$ and $l = l^\#$. Then we identify in \tilde{Q} the vertices

$$(l-i) \leftrightarrow (l+i)^\#, \quad (l+i) \leftrightarrow (l-i)^\#, \quad (3.10)$$

where $1 \leq i \leq (l-1)$. This is $\tilde{Q}_{(J_1, J_2)}$. Finally we delete one of the two arrows from i to $i^\# = 2l-i$ in $\tilde{Q}_{(J_1, J_2)}$ to obtain $\tilde{Q}_{(J_1, J_2), l}$. We give an example in matrix notations.

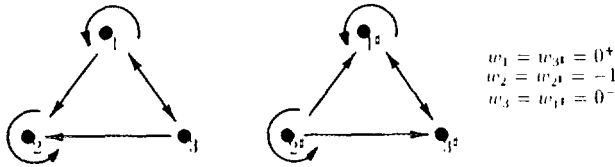
EXAMPLE 3. Let $M_{\alpha, \beta}(\mathbb{C})$ denote the space of $\alpha \times \beta$ matrices over \mathbb{C} . Consider $G = Sp(2n, \mathbb{C})$ and $l = 2$. The Lie algebra \mathfrak{g} may be written in the matrix form,

$$\begin{pmatrix} A & B & X \\ -J_{x_2} C' & S & J_{x_2} B' \\ Y & C & -A' \end{pmatrix},$$

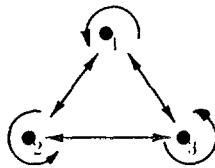
where $A \in M_{x_1, x_1}(\mathbb{C})$, $B \in M_{x_1, 2x_2}(\mathbb{C})$, $X = X' \in M_{x_1, x_1}(\mathbb{C})$, $S = J_{x_2} S' J_{x_2}^{-1} \in M_{2x_2, 2x_2}(\mathbb{C})$, $C \in M_{x_1, 2x_2}(\mathbb{C})$, $Y = Y' \in M_{x_1, x_1}(\mathbb{C})$, and

$$J_{x_2}^{-1} = \begin{pmatrix} 0 & I_{x_2} \\ -I_{x_2} & 0 \end{pmatrix}.$$

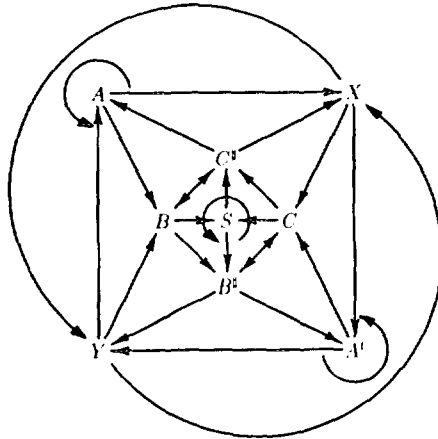
The Lie algebra of the Levi component H is embedded as the diagonal blocks $(A, S, -A')$ in \mathfrak{g} . A diagram illustrating the partial double of the quiver is as follows.



The resulting collapsed quiver describing the invariants of H is as follows.



To read off the H -invariants from this quiver, it is convenient to construct a certain "derived" diagram. In the following picture, $B^* = -J_{x_2} B'$ and $C^* = J_{x_2}^- C'$. Cycles in this diagram give rise to H -invariants. For instance, $\text{Tr}(AXA'YBC^*)$ is a H -invariant.



4. GENERALISED CASIMIR OPERATORS

Let \mathfrak{g}_0 be a real Lie algebra with adjoint group G_0 and $\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$, $G = (G_0)_{\mathbb{C}}$ be the respective complexifications. Let \mathcal{I} (respectively \mathcal{J}) be the ideals in the tensor algebra $T(\mathfrak{g}) = \sum_{n \geq 0} T^n(\mathfrak{g}) = \sum_{n \geq 0} \otimes^n \mathfrak{g}$ generated by $x \otimes y - y \otimes x$ (respectively $x \otimes y - y \otimes x - [x, y]$) for every $x, y \in \mathfrak{g}$. The quotient algebras

$$\mathcal{S}(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I} \quad \text{and} \quad \mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{J}$$

are the symmetric algebra on \mathfrak{g} and the universal enveloping algebra of \mathfrak{g} , respectively. Clearly $\mathcal{S}(\mathfrak{g}) = \sum_{n \geq 0} \mathcal{S}^n(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g}) = \sum_{n \geq 0} \mathcal{U}^n(\mathfrak{g})$ acquire the natural grading from $T(\mathfrak{g})$. Since G acts on \mathfrak{g} via the Adjoint representation, G acts on both $\mathcal{S}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$, preserving their gradings.

A map relating $\mathcal{S}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{g})$ is the symmetrization map

$$\Phi: X_1 X_2 \cdots X_n \in \mathcal{S}(\mathfrak{g}) \rightarrow \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(n)} \in \mathcal{U}(\mathfrak{g}), \quad (4.1)$$

which is a linear isomorphism. It is a well known result that Φ is in fact a G -equivariant isomorphism. In particular, for a subgroup H of G , Φ is a H -module isomorphism.

In Section 3, we have described $\mathcal{P}(\mathfrak{g})^H \simeq \mathcal{S}(\mathfrak{g}^*)^H$, where $H = H_{(J_1, J_2)}$ is the Levi component of a parabolic subgroup of $G = GL(n, \mathbb{C})$, $O(n, \mathbb{C})$ or $Sp(2n, \mathbb{C})$. Recall that if $\sigma = \phi_1 \cdots \phi_l$ is a cycle in the quiver $\tilde{Q}_{(J_1, J_2), I}$, where (Q, w, α) is associated to the parabolic subgroup indexed by $\alpha_1, \dots, \alpha_l$ and J_1, J_2, I are defined in Case B of Section 3, we can construct an invariant of H in $\mathcal{P}(\mathfrak{g})$ using

$$\text{Tr } \sigma = \text{Tr}(A_{\phi_1} \cdots A_{\phi_l}). \tag{4.2}$$

Note that we can identify \mathfrak{g}^* with \mathfrak{g} in a G -equivariant way using the trace form (which is a multiple of the Killing form). We refer the reader to Eq. (4.13) for the explicit description of this isomorphism. Let us denote by $\text{Tr}^* \sigma$ the element in $\mathcal{S}(\mathfrak{g})$ corresponding to $\text{Tr } \sigma$ under the identification. To construct a set of generators (we call them *generalised Casimir operators*) in $\mathcal{U}(\mathfrak{g})^H$, we construct the top degree of $\Phi(\text{Tr}^* \sigma)$ as σ runs through all cycles of the quiver $\tilde{Q}_{(J_1, J_2), I}$. We proceed to construct these elements explicitly.

Let $M_{\alpha, \beta}(\mathbb{C})$ denote the space of $\alpha \times \beta$ matrices over \mathbb{C} . Suppose a matrix group $H = \prod_{i=1}^n H_i$, where $H_i \subset GL(\alpha_i, \mathbb{C})$ acts on $M = \bigoplus M_{\alpha_i, \beta_i}(\mathbb{C})$ via one of the following actions.

$$(h_1, \dots, h_n) \cdot Z = \sigma(h_i) Z \tau(h_j) = \begin{cases} h_i Z h_j^{-1}, \\ h_i Z h_j', \\ (h_i')^{-1} Z h_j^{-1}, \\ (h_i')^{-1} Z h_j', \end{cases} \tag{4.3}$$

where $h_i \in H_i$ and $Z \in M_{\alpha_i, \beta_j}(\mathbb{C})$ are represented as matrices of appropriate order. We shall simply call such actions ($L(\sigma(h_i)), R(\tau(h_j))$) actions, where $L(\sigma(h_i))$ refers to left multiplication by $\sigma(h_i)$ and $R(\tau(h_j))$ refers to right multiplication by $\tau(h_j)$. The corresponding derived actions for (4.3) are easily computed to be as follows.

$$(X_1, \dots, X_n) \cdot Z = \begin{cases} X_i Z - Z X_j, \\ X_i Z + Z X_j', \\ -X_i' Z - Z X_j, \\ -X_i' Z + Z X_j', \end{cases} \tag{4.4}$$

where $X_i \in \mathfrak{h}_i$, the Lie algebra of H_i , and $Z \in M_{\alpha_i, \beta_j}(\mathbb{C})$.

For simplicity, let us assume that H is a reductive subgroup of a group G with Lie algebra decomposition $\mathfrak{g} = \mathfrak{m} = \bigoplus M_{ij}$ where M_{ij} is a subspace of $M_{\alpha_i, \beta_j}(\mathbb{C})$. Also suppose that the restriction of the adjoint action to H preserves M_{ij} and that the action adjoint on M_{ij} is given by one of the

actions described in (4.3). Suppose $\{Z_{st} \mid 1 \leq s \leq \alpha_i, 1 \leq t \leq \alpha_j\}$ is a set of elements in M_{ij} with the relations

$$(X_1, \dots, X_n) \cdot Z_{st} = \sum_l A(X_i)_{sl} Z_{lt} + \sum_l B(X_j)_{tl} Z_{sl}, \tag{4.5}$$

where $A(X_i) = (A(X_i)_{st})$ and $B(X_j) = (B(X_j)_{ts})$ are constant matrices of appropriate order depending only on X_i and X_j , respectively. We shall write (4.5) formally as

$$X \cdot Z = [X, Z] = A(X)Z + ZB(X), \tag{4.6}$$

where $Z = (Z_{st})$ and call the system of commutation relations satisfied by $\{Z_{st}\}$ as in (4.6) an $(A(X), B(X))$ system for the action of \mathfrak{h} on M_{ij} . In general, we allow Z_{st} to be in $\mathcal{U}(\mathfrak{g})$ and we will simply say that Z is an $(A(X), B(X))$ system for \mathfrak{h} on $\mathcal{U}(\mathfrak{g})$ if (4.6) holds.

EXAMPLE 4. As an example, a $(L(h_1), R(h'_2))$ action on $M_{\alpha_1, \alpha_2}(\mathbb{C})$, where $h_1 \in GL(\alpha_1, \mathbb{C})$ and $h_2 \in GL(\alpha_2, \mathbb{C})$, gives rise to the set of commutation relations

$$[(E_{ij}, F_{kl}), Z_{st}] = \delta_{js} Z_{it} + \delta_{it} Z_{sk},$$

where $E_{ij} \in \mathfrak{gl}(\alpha_1, \mathbb{C})$, $F_{kl} \in \mathfrak{gl}(\alpha_2, \mathbb{C})$ and $Z_{ts} \in M_{\alpha_1, \alpha_2}(\mathbb{C})$ are all matrix units. So if $X_1 = \sum a_{kl} E_{kl} \in \mathfrak{gl}(\alpha_1, \mathbb{C})$ and $X_2 = \sum b_{kl} F_{kl} \in \mathfrak{gl}(\alpha_2, \mathbb{C})$, then

$$[(X_1, X_2), Z_{st}] = \sum_l a_{ls} Z_{lt} + \sum_l b_{tl} Z_{st}, \tag{4.7}$$

that is, (4.7) is an $(A(X), B(X))$ system for the action of $\mathfrak{gl}(\alpha_1, \mathbb{C}) \oplus \mathfrak{gl}(\alpha_2, \mathbb{C})$ on $M_{\alpha_1, \alpha_2}(\mathbb{C})$, where $A(X)' = (a_{kl})$ and $B(X) = (b_{kl})$.

LEMMA 6. If $\{Z_{ts}\}$ is an $(A(X), -A(X))$ system for $X \in \mathfrak{h}$ on $\mathcal{U}(\mathfrak{g})$, then $\text{Tr } Z = \sum_t Z_{tt} \in \mathcal{U}(\mathfrak{g})$ is a H -invariant.

Proof. This is immediate. ■

LEMMA 7. If $\{V_{kl}\}$ is an $(A(X), B(X))$ system for an \mathfrak{h} action on $\mathcal{U}(\mathfrak{g})$ and $\{W_{kl}\}$ is an $(-B(X), C(X))$ system for the same \mathfrak{h} action on $\mathcal{U}(\mathfrak{g})$, then $\{U_{kl} = \sum_s V_{ks} W_{st}\}$ is an $(A(X), C(X))$ system for $X \in \mathfrak{h}$.

Proof. Formally, it goes as follows.

$$[X, V] = A(X)V + VB(X)$$

and

$$[X, W] = -B(X)W + WC(X)$$

gives

$$\begin{aligned} [X, VW] &= (A(X) V + VB(X)) W + V(-B(X) W + WC(X)) \\ &= A(X)(VW) + (VW) C(X). \end{aligned}$$

These computations are valid because $\mathcal{U}(\mathfrak{g})$ is associative. \blacksquare

Remark. In what follows, we shall sometimes perform similar formal computations which can be justified by writing out explicitly using index notations.

COROLLARY 8. *With the same assumptions as above, consider*

$$U_{kl} = \sum_{i_2, \dots, i_s} Z(1)_{ki_2} Z(2)_{i_2 i_3} \cdots Z(s)_{i_s l},$$

where each $\{Z(t)_{ab}\}$ belongs to a $(A_t(X), B_t(X))$ system for a $(L(\sigma_t(h)), R(\tau_t(h)))$ action of H on $M_{i_t, i_t} \subset M_{x_{i_t}, x_{i_t}}(\mathbb{C})$. If $i_{t+1} = j_t$ and $B_t(X) + A_{t+1}(X) = 0$ on M_{i_t, j_t} for $1 \leq t \leq s-1$, then $\{U_{kl}\}$ is a $(A_1(X), B_s(X))$ system.

Remark. Let E_{ij} be the matrix units of $\mathfrak{gl}(n, \mathbb{C})$ and $E = (E_{ij})$. Then Corollary 8 generalises a lemma of Gelfand [12, p. 159] which he used to show that for any positive integer k , the elements

$$\text{Tr } E^k = \sum_{i_1, i_2, \dots, i_k} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_k i_1}$$

are in the center of the universal enveloping algebra of $\mathfrak{gl}(n, \mathbb{C})$.

We note a straightforward result.

LEMMA 9. *Suppose a Lie subalgebra \mathfrak{h} of $\mathfrak{gl}(\alpha, \mathbb{C}) \oplus \mathfrak{gl}(\beta, \mathbb{C})$ acts on $M_{\alpha, \beta}(\mathbb{C})$ in the following way*

$$(X_1, X_2) \cdot M = X_1 M + M X_2, \tag{4.8}$$

where $X_1, X_2 \in \mathfrak{gl}(\alpha, \mathbb{C})$, $(X_1, X_2) \in \mathfrak{h}$ and $M \in M_{\alpha, \beta}(\mathbb{C})$. Let E_{ij} be the matrix units in $M_{\alpha, \beta}(\mathbb{C})$. Then $E = (E_{ij})$ is a (X'_1, X'_2) system for $(X_1, X_2) \in \mathfrak{h}$.

LEMMA 10. *Let \mathfrak{h} be a Lie subalgebra of $\mathfrak{gl}(\alpha, \mathbb{C}) \oplus \mathfrak{gl}(\beta, \mathbb{C})$. Assume that the set of matrices $\{Z_{ij}\}$ in $M_{\alpha, \beta}(\mathbb{C})$ is an $(A(X), B(X))$ system for $X \in \mathfrak{h}$. Let $S \in GL(\beta, \mathbb{C})$, $T \in GL(\alpha, \mathbb{C})$ be two constant matrices. Then $Z^\# = SZ^T T$, i.e., $(Z^\#)_{ij} = \sum_{k, l} S_{ik} Z_{lk} T_{lj}$, is an $(SB(X)' S^{-1}, T^{-1} A(X)' T)$ system for $X \in \mathfrak{h}$.*

Remark. If $Z \in M_{\alpha, \beta}(\mathbb{C})$, the matrix version of canonical $\#$ map (see (1.6)) is of the following form

$$M^\# = S M^T T \in M_{\beta, \alpha}(\mathbb{C}) \tag{4.9}$$

for some invertible constant matrices S and T . The matrix S or its inverse S^{-1} (similarly for T and T^{-1}) can either be I (the identity matrix) or one of the following

$$\begin{aligned}
 J_x^\pm &= \begin{bmatrix} 0 & I_{[\alpha/2]} \\ \pm I_{[\alpha/2]} & 0 \end{bmatrix}, & \text{if } \alpha \text{ is even,} \\
 J_x^+ &= \begin{bmatrix} 0 & 0 & I_{[\alpha/2]} \\ 0 & 1 & 0 \\ I_{[\alpha/2]} & 0 & 0 \end{bmatrix}, & \text{if } \alpha \text{ is odd.}
 \end{aligned}
 \tag{4.10}$$

Proof. We have

$$\begin{aligned}
 X \cdot Z^\# &= X \cdot (SZ'T) \\
 &= S(X \cdot Z)' T \\
 &= S(A(X)Z + ZB(X))' T \\
 &= S(Z'A(X)' + B(X)'Z') T \\
 &= Z^\#(T^{-1}A(X)' T) + (SB(X)' S^{-1}) Z^\#. \quad \blacksquare
 \end{aligned}$$

PROPOSITION 11. *Suppose a Lie subalgebra \mathfrak{h} of $\mathfrak{gl}(\alpha, \mathbb{C}) \oplus \mathfrak{gl}(\alpha, \mathbb{C})$ acts on $M_{\alpha, \alpha}(\mathbb{C})$ in the following way*

$$(X_1, X_2) \cdot M = X_1 M + M X_2, \tag{4.11}$$

where $X_1, X_2 \in \mathfrak{gl}(\alpha, \mathbb{C})$, $(X_1, X_2) \in \mathfrak{h}$ and $M \in M_{\alpha, \alpha}(\mathbb{C})$.

(a) *For $M \in M_{\alpha, \alpha}(\mathbb{C})$, let $M^\# = SM'T$. Further assume that \mathfrak{h} preserves one of the following subspaces*

$$V^\pm = \{M \in M_{\alpha, \alpha}(\mathbb{C}) \mid M \pm M^\# = 0\}. \tag{4.12}$$

Then for any $(A(X), B(X))$ system $Z = (Z_{ij})$ of \mathfrak{h} on $M_{\alpha, \alpha}(\mathbb{C})$, $Z^\#$ is again an $(A(X), B(X))$ system for $X \in \mathfrak{h}$ on $M_{\alpha, \alpha}(\mathbb{C})$.

(b) *In addition to the conditions in (a), further assume that $S = T^{-1} = J^{-1}$, where $J^2 = \pm I_\alpha$ and let $E = (E_{ij})$ be the system of matrix units of $M_{\alpha, \alpha}(\mathbb{C})$. Also let $F = \frac{1}{2}(E \mp E^\#)$. Then $F_{ij} \in V^\pm$ and F is an (X'_1, X'_2) system for $(X_1, X_2) \in \mathfrak{h}$ on V^\pm satisfying*

$$\text{Tr}(F_{ij} M) = M_{ji} \tag{4.13}$$

for any $M = (M_{ij}) \in V^\pm$.

Remark. Using the above theorem, we can choose an $(X, -X)$ system F for the adjoint of $X \in \mathfrak{g}$ as follows.

$$F = \begin{cases} E & \text{if } G = GL(n, \mathbb{C}), \\ \frac{1}{2}(E - (J_n^+)^{-1} E' J_n^+) & \text{if } G = O(n, \mathbb{C}), \\ \frac{1}{2}(E - (J_{2n})^{-1} E' J_{2n}) & \text{if } G = Sp(2n, \mathbb{C}). \end{cases} \quad (4.14)$$

Here E_{ij} are the appropriate matrix units. Thus $\text{Tr } F^k$ will be in $\mathcal{Z}\mathcal{U}(\mathfrak{g})$, the centre of $\mathcal{U}(\mathfrak{g})$.

Proof. (a) It follows from an elementary computation that \mathfrak{h} leaves V^\pm invariant if and only if

$$A(X) \mp SB(X)' S^{-1} = 0 \quad \text{and} \quad B(X) \mp T^{-1}A(X)' T = 0.$$

Lemma 10 then gives the required result.

(b) We first need to check that $F_{ij} = \frac{1}{2}(E_{ij} \mp (J^{-1})_{ik} E_{jk}(J)_{lj}) \in V^\pm$. A simple computation gives $(J^{-1})_{ik} E_{jk}(J)_{lj} = J E_{jl} J^{-1}$. Since $J^2 = \pm I$, the latter term is equal to $J^{-1} E_{jl} J = E_{ij}^\#$. Therefore

$$F_{ij} = \frac{1}{2}(E_{ij} \mp E_{ij}^\#) \in V^\pm.$$

Finally, we have for $M \in V^\pm$,

$$\begin{aligned} \text{Tr}(F_{ij} M) &= \frac{1}{2} \text{Tr}(E_{ij} M \mp E_{ij}^\# M) \\ &= \frac{1}{2} \text{Tr}(E_{ij} M \mp E_{ij} M^\#) \\ &= \frac{1}{2} \text{Tr}(E_{ij} M + E_{ij} M) = M_{ji}. \end{aligned}$$

Part (b) follows. ■

Recall that given a parabolic subgroup P of $G = O(n, \mathbb{C}), Sp(2n, \mathbb{C})$ indexed by the sequence $\alpha_1, \alpha_2, \dots, \alpha_l$, we have constructed a weighted quiver (Q, w, α) such that the Lie algebra \mathfrak{g} is a representation. Namely we have

$$\mathfrak{g} = \bigoplus_{\phi_{ij} \in E_Q} M_{ij}$$

as a vector space. See Section 3 for the definition of M_{ij} . Moreover for some suitable $J_1, J_2, I \in V_Q$, the Levi component H of P is isomorphic to a subgroup $H_{(J_1, J_2)}$ of G_α .

Let $d = 2l - 1$. From the description of the weighted quiver (Q, w) in this case, we see that $E_{\tilde{Q}_{(J_1, J_2), l}}$, the set of arrows of $\tilde{Q}_{(J_1, J_2), l}$, corresponds exactly to the set of pairs (i, j) such that $1 \leq i, j \leq d$. Moreover, because of the way the vertices of $E_{\tilde{Q}_{(J_1, J_2), l}}$ are identified (see (3.10)), we have in fact an involution $\#$ on the set $\{1, 2, \dots, d\}$ given by $i^\# = d + 1 - i$. For $1 \leq i, j \leq d$ with $\phi_{ij} \in E_Q$, and $X^{ij} \in M_{ij}$, let $X^{j^\# i^\#} = -(X^{ij})^\#$. Here the $\#$ notation on the right hand side of the equation is the natural map defined in Eq. (1.6) composed with the identification of $X_{x_i}^*$ with \bar{X}_{x_i} , etc., as in (3.2). For example if $X^{ij} \in \text{Hom}(X_{x_j}, X_{x_i})$ with $1 \leq i, j \leq l - 1$, we have $(X^{ij})^\# \in \text{Hom}(X_{x_i}^*, X_{x_j}^*) \simeq \text{Hom}(\bar{X}_{x_i}, \bar{X}_{x_j})$. Using the standard basis implicit in the definition of X_{x_j} and \bar{X}_{x_j} , we will identify various X^{ij} 's as matrices of appropriate order.

It thus makes sense to form the following matrix

$$X = \begin{pmatrix} X^{11} & \dots & X^{1d} \\ \vdots & \vdots & \vdots \\ X^{d1} & \dots & X^{dd} \end{pmatrix}. \tag{4.15}$$

We observe that if we let

$$m = \begin{cases} n, & \text{if } G = O(n, \mathbb{C}), \\ 2n, & \text{if } G = Sp(2n, \mathbb{C}), \end{cases}$$

then

$$X \in \mathfrak{g} = \{ Y \in M_{m, m}(\mathbb{C}) \mid Y^t J_{x_1, \dots, x_l}^c + J_{x_1, \dots, x_l}^c Y = 0 \},$$

where

$$J_{x_1, \dots, x_l}^c = \begin{pmatrix} 0 & & & & & & & & I_{x_1} \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ \varepsilon I_{x_1} & \dots & & & & & & & 0 \end{pmatrix}. \tag{4.16}$$

Therefore we can identify the quiver representation $M(\tilde{Q}_{(J_1, J_2), l}, w, \alpha)$ with the above matrix realization of the Lie algebra \mathfrak{g} of G (with a possible discrepancy of a negative sign which is immaterial). Moreover, under this realization, the Lie algebra \mathfrak{h} of H is embedded as the diagonal blocks

be the division of F into $d \times d$ blocks determined by the partition

$$n = (\alpha_1) + \cdots + (\alpha_{l-1}) + (\alpha_l) + (\alpha_{l-1}) + \cdots + (\alpha_1), \quad \text{if } G = O(n, \mathbb{C}),$$

$$2n = (\alpha_1) + \cdots + (\alpha_{l-1}) + (2\alpha_l) + (\alpha_{l-1}) + \cdots + (\alpha_1), \quad \text{if } G = Sp(2n, \mathbb{C}).$$

LEMMA 12. Each F^{kl} is an $((A_k)^l, -(A_l)^l)$ system for the action of \mathfrak{h} on \mathfrak{g} .

Remark. In what follows, we shall use the convention of denoting the block position by a pair of superscript index, the row and column positions within each block by a pair of subscript index.

Proof. Let

$$A = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_d \end{pmatrix} \in \mathfrak{h} \quad \text{and} \quad X = \begin{pmatrix} X^{11} & \cdots & X^{1d} \\ \vdots & \vdots & \vdots \\ X^{d1} & \cdots & X^{dd} \end{pmatrix} \in \mathfrak{g}.$$

Recall that we have $\text{Tr}(F_{ij}X) = X_{ji}$, namely,

$$\text{Tr}(F_{ij}^{kl}X) = X_{ji}^{lk}.$$

We compute,

$$\text{Tr}((A \cdot F_{ij}^{kl})X) = -\text{Tr}(F_{ij}^{kl}(A \cdot X)) = -\text{Tr}(F_{ij}^{kl}(AX - XA)).$$

Since $A \in \mathfrak{h}$, $X \in \mathfrak{g}$, we have $AX - XA \in \mathfrak{g}$. Therefore,

$$\begin{aligned} \text{Tr}((A \cdot F_{ij}^{kl})X) &= -(AX - XA)_{ji}^{lk} = -(A_l X^{lk} - X^{lk} A_k)_{ji} \\ &= \sum_a X_{ja}^{lk} (A_k)_{ai} - \sum_a (A_l)_{ja} X_{ai}^{lk} \\ &= \sum_a \text{Tr}(F_{aj}^{kl}X) (A_k)_{ai} - \sum_a (A_l)_{ja} \text{Tr}(F_{ia}^{kl}X). \end{aligned}$$

By the nondegeneracy of the trace form on \mathfrak{g} , we conclude

$$A \cdot F_{ij}^{kl} = \sum_a F_{aj}^{kl} (A_k)_{ai} - \sum_a (A_l)_{ja} F_{ia}^{kl}.$$

Hence the lemma follows. \blacksquare

We continue with the construction of generating elements in $\mathcal{U}(\mathfrak{g})^H$. As observed in the previous discussion, an arrow in the quiver $\tilde{Q}_{(J_1, J_2), l}$ can be thought of as a pair (i, j) , or the corresponding (i, j) th entry of the $d \times d$

block defined above. For a cycle $\sigma = \phi_{i_1 i_2} \cdots \phi_{i_{s-1} i_s} \phi_{i_s i_1}$ in $\tilde{Q}_{(J_1, J_2), l}$, we can thus consider the noncommutative trace

$$\text{Tr } F^\sigma = \sum_{i_1, \dots, i_s} F_{i_1 i_2}^{i_1 i_2} F_{i_2 i_3}^{i_2 i_3} \cdots F_{i_s i_1}^{i_s i_1}. \tag{4.19}$$

Observe that from Proposition 11, $F_{ij}^{kl} \in \mathfrak{g}$, so the above noncommutative trace is an element of $\mathcal{U}(\mathfrak{g})$. By Lemma 12, $F^{i_j i_{j+1}}$ is an $((A_j)', -(A_{j+1})')$ system for the action of \mathfrak{h} . Therefore Corollary 8 and Lemma 6 imply that expression (4.19) is in $\mathcal{U}(\mathfrak{g})^H$. We will show that these are all that are needed to generate $\mathcal{U}(\mathfrak{g})^H$.

THEOREM 13. *Let H be the Levi component of a parabolic subgroup P of G indexed by the sequence $\alpha_1, \alpha_2, \dots, \alpha_r$, and let the weighted quiver (Q, w, α) and $J_1, J_2, l \subset V_Q$ be defined as in Section 3. Then the algebra $\mathcal{U}(\mathfrak{g})^H$ is generated by $\text{Tr } F^\sigma$, where $\sigma = \phi_1 \cdots \phi_s$ runs through all cycles of $\tilde{Q}_{(J_1, J_2), l}$ with the following property.*

(F) *For every vertex i of $\tilde{Q}_{(J_1, J_2), l}$ the cycle σ passes through i at most β_i^2 times. Here β_i is the i th-component of the dimension vector given in (3.7). In particular, $\mathcal{U}(\mathfrak{g})^H$ is finitely generated.*

Remarks. (a) Setting $G = GL(n, \mathbb{C})$, we can deduce the results of [6]. Moreover from the finiteness property (F), it is not difficult to deduce the well-known result that for the maximal parabolic subgroup P with Levi component $H \simeq GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C})$, $\mathcal{U}(\mathfrak{g})^H$ is Abelian and $\mathcal{U}(\mathfrak{g})^H = \mathcal{U}(\mathfrak{g})^G \mathcal{U}(\mathfrak{h})^H$. Note that $\alpha_2 = 1$ is crucial here.

(b) We could replace H by a maximal compact subgroup K of G and deduce an analogous result giving a finite set of generators for $\mathcal{U}(\mathfrak{g})^K$. For a classical group G , this was previously treated on a case by case basis in a preprint [14]. We point out that the method of this paper even applies to exceptional groups and certain reductive subgroups. Results along this direction will appear elsewhere.

Proof. We have seen that $\text{Tr } F^\sigma$ is H -invariant. To conclude that they generate $\mathcal{U}(\mathfrak{g})^H$, we use the symmetrization map which is a H -isomorphism between $\mathcal{S}(\mathfrak{g})^H$ and $\mathcal{U}(\mathfrak{g})^H$. Recall the H -invariant $\text{Tr } \sigma = \text{Tr } X_{\phi_1} \cdots X_{\phi_s}$ in $\mathcal{P}(\mathfrak{g}) \simeq \mathcal{S}(\mathfrak{g}^*)$ associated with a cycle $\sigma = \phi_1 \cdots \phi_s$ of $\tilde{Q}_{(J_1, J_2), l}$, and the corresponding invariant $\text{Tr}^* \sigma$ in $S(\mathfrak{g})$ under the G isomorphism of \mathfrak{g}^* with \mathfrak{g} induced by the trace form. In view of the fact $\text{Tr}(F_{ij} X) = X_{ji}$ for $X \in \mathfrak{g}$, we see that under the symmetrization map

$$\Phi(\text{Tr}^* \sigma) = \text{Tr } F^\sigma + \text{lower order terms in } \mathcal{U}(\mathfrak{g}) \tag{4.20}$$

Moreover for two cycles σ_1 and σ_2 of $\tilde{Q}_{(J_1, J_2), l}$, we have

$$\Phi(\mathrm{Tr}^* \sigma_1 \mathrm{Tr}^* \sigma_2) = \mathrm{Tr} F^{\sigma_1} \mathrm{Tr} F^{\sigma_2} + \text{lower order terms in } \mathcal{U}(\mathfrak{g}). \quad (4.21)$$

The second term on the right hand sides of equations (4.20) and (4.21) are in fact in $\mathcal{U}(\mathfrak{g})^H$ since the other terms are. Using the fact that $\mathrm{Tr}^* \sigma$, for $\sigma = \phi_1 \cdots \phi_s$ satisfying the finiteness property (F), generates $\mathcal{P}(\mathfrak{g})^H$, an induction argument completes the proof. ■

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