Journal of Pure and Applied Algebra 93 (1994) 135–146
North-Holland

Determining summands in tensor products of Lie algebra representations

Helmer Aslaksen*

Department of Mathematics, National University of Singapore, Singapore 0511, Singapore

Communicated by C.A. Weibel
Received 28 September 1991
Revised 15 June 1992

Abstract

Aslaksen, H., Determining summands in tensor products of Lie algebra representations, Journal of Pure and Applied Algebra 93 (1994) 135–146.

We give some results that enable us to find certain summands in tensor products of Lie algebra representations. We concentrate on the splitting of tensor squares into their symmetric and antisymmetric parts. Our results are valid for any Lie algebra of arbitrary rank, but we do not attempt to give the complete decomposition.

1. Introduction

The problem of decomposing tensor products of finite-dimensional representations of finite-dimensional, simple, complex Lie algebras occurs frequently in both mathematics and physics. There are several methods available, but most of them are practically useful only if the rank is small. In this paper we will focus on two problems.

(1) In some applications we need at least partial results for cases with high or even arbitrary rank [8].

(2) In some applications the splitting of a tensor square into its symmetric and anti-symmetric parts, $\lambda \otimes \lambda = S^2\lambda \oplus \Lambda^2\lambda$, is crucial [8].

A good reference for problems of the first kind is the fundamental paper by Dynkin [2]. He gives several rules that allows us to immediately identify some summands. (Cahn [1] gives a very readable presentation of the relevant results from Dynkin's

Correspondence to: H. Aslaksen, Department of Mathematics, National University of Singapore, Singapore 0511, Singapore. Email: mathelmr@nusunix.nus.sg.

* Partially sponsored by NSF grant DMS 87-01609 and a grant from the Norwegian Research Council for Science and Humanities (NAVF).

SSDI 0022-4049(93)E0101-9

paper.) In Section 3 we will extend some of Dynkin's results. In Section 4 we will prove some similar results about the $\lambda \otimes \lambda = S^2\lambda \oplus \Lambda^2\lambda$ decomposition using the method of successive subtractions.

This paper grew out of a suggestion by Wu-yi Hsiang at Berkeley to generalize Wolf's classification of the isotropy irreducible homogeneous spaces [8] to the spaces with 2 or 3 irreducible summands in the isotropy group representation. It turned out that these had already been classified by Krämer [5] using results from his earlier papers [3, 4]. These results [3, 4] have the same aim as the results in this paper and are more powerful in the sense that they in general will determine more summands. My results, however, are more direct and elementary and will in some cases give more precise information.

2. Notation

We will use the following notation. $\mathfrak{g}$ will denote a finite-dimensional simple complex Lie algebra with root system Δ. We pick a basis of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and for every $\alpha \in \Delta^+$ we can find $X_\alpha \in \mathfrak{g}^\alpha$ and $Y_\alpha \in \mathfrak{g}^{-\alpha}$ such that X_α, Y_α and $h_\alpha = [X_\alpha, Y_\alpha]$ span a subalgebra isomorphic to $\mathfrak{sl}(2)$.

We will often denote an irreducible representation ϕ by its highest weight λ. We denote the space λ acts on by $V(\lambda)$ (or simply V) and use $\Delta(\lambda)$ to denote the weight system of λ. For $\mu \in \Delta(\lambda)$ we write $V(\mu, \lambda)$ (or simply $V(\mu)$) for the weight space corresponding to μ. We will denote the conjugate representation by λ^*. We set $E_\alpha = \phi(X_\alpha)$.

A nonzero $v \in V$ is called an *extreme vector* of the (possibly reducible) representation ϕ if

$$E_{\alpha_i} v = 0 \quad \text{for } i = 1, \dots, n.$$

The set

$$\{E_{-\beta_s} \cdots E_{-\beta_1} v \mid \beta_1, \dots, \beta_s \in \Pi\}$$

then generates an irreducible subspace of V.

We will call

$$\lambda_i = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}$$

the *i*th *Dynkin coefficient*. The Dynkin coefficients will be written over the corresponding dots in the Dynkin diagram or placed inside brackets. We will call ω_{α_i} the fundamental weight corresponding to α_i if $(\omega_{\alpha_i})_j = \delta_{ij}$.

We set $n(\mu, \lambda) = \dim V(\mu, \lambda)$, *i.e., the inner* multiplicity of μ as a weight of λ, while $m(\mu, \lambda \otimes \lambda')$ denotes the number of times μ occurs as a summand in $\lambda \otimes \lambda'$, i.e., the *outer* multiplicity of μ in $\lambda \otimes \lambda'$.

We will write $\lambda \subset \Lambda_1 \otimes \Lambda_2$ to denote that λ is a summand in the tensor product. If $\Lambda_1 = \Lambda_2$ we write

$$\lambda_s \oplus \lambda'_a \subset \Lambda \otimes \Lambda$$

to denote that λ lies in the symmetric part $S^2\Lambda$ and λ' lies in the antisymmetric part $\Lambda^2\Lambda$ of $\Lambda \otimes \Lambda$.

3. Dynkin's methods and some generalizations

In his paper from 1952 [2], Dynkin developed three very useful techniques for finding summands in tensor products: the second-highest weights rule, the method of subordination and the method of parts. We will prove some generalizations of the first two methods.

Let λ and Λ be two irreducible representations of $\mathfrak{g}$. We say that λ is subordinate to Λ, written $\lambda \leq^{\text{sub}} \Lambda$, if $\lambda_i \leq \Lambda_i$ for $i = 1, \dots, n$. If $\psi = \lambda^1 \oplus \cdots \oplus \lambda^s$ and $\phi = \Lambda^1 \oplus \cdots \oplus \Lambda^t$, then $\psi \leq^{\text{sub}} \phi$ if $s \leq t$ and $\lambda^i \leq^{\text{sub}} \Lambda^i$ for $i = 1, \dots, n$.

It is easy to see that $\psi \leq^{\text{sub}} \phi$ is equivalent to saying that there is a linear map $f: V(\phi) \to V(\psi)$ satisfying the following three conditions:

(1) f is onto,

(2) f takes extreme vectors to extreme vectors,

(3) $f(E_{-\alpha_i}v) = E_{-\alpha_i}f(v)$ for $i = 1, \dots, n$ and $v \in V(\phi)$.

Dynkin's method of subordination [2, Theorem 3.17] says that if $\lambda \leq^{\text{sub}} \Lambda$ and $\lambda' \leq^{\text{sub}} \Lambda'$, then $\lambda \otimes \lambda' \leq^{\text{sub}} \Lambda \otimes \Lambda'$. By elaborating on Dynkin's proof, we will show how the weights of the corresponding summands in the two products are related, and that the $S^2\lambda \oplus \Lambda^2\lambda$ decomposition is preserved.

Theorem 1. *If $\lambda \leq^{\text{sub}} \Lambda$ and $\lambda' \leq^{\text{sub}} \Lambda'$, then*

$$m(\Lambda + \Lambda' - \textstyle\sum k_i\alpha_i, \Lambda \otimes \Lambda') \geq m(\lambda + \lambda' - \textstyle\sum k_i\alpha_i, \lambda \otimes \lambda'),$$

and if $\lambda' = \lambda$ and $\Lambda' = \Lambda$, then

$$m(\Lambda + \Lambda' - \textstyle\sum k_i\alpha_i, S^2\Lambda) \geq m(\lambda + \lambda' - \textstyle\sum k_i\alpha_i, S^2\lambda),$$

$$m(\Lambda + \Lambda' - \textstyle\sum k_i\alpha_i), \Lambda^2\Lambda) \geq m(\lambda + \lambda' - \textstyle\sum k_i\alpha_i, \Lambda^2\lambda).$$

Proof. Assume that $f: V(\Lambda) \to V(\lambda)$ and $f': V(\Lambda') \to V(\lambda')$ satisfy conditions (1)–(3). Then $F = f \otimes f': V(\Lambda) \otimes V(\Lambda') \to V(\lambda) \otimes V(\lambda')$ is easily seen to satisfy (1)–(3), too. Now let y be an extreme vector of weight $\lambda + \lambda' - \sum k_i\alpha_i$. Then there must be an extreme vector x of $V(\Lambda) \otimes V(\Lambda')$ of weight v with $F(x) = y$. We want to find an expression for v. Let v and v' be extreme vectors for Λ and Λ'. Then $w = f(v)$ and $w' = f'(v)$ are extreme vectors for λ and λ'. We can write

$$x = \textstyle\sum c_i E_{-\beta_{i_s}} \cdots E_{-\beta_{i_1}} v \otimes E_{-\gamma_{i_t}} \cdots E_{-\gamma_{i_1}} v',$$

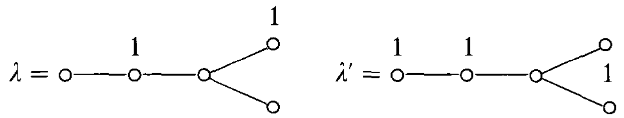


Fig. 1.

where $\beta_{i_1} + \cdots + \beta_{i_s} + \gamma_{i_1} + \cdots + \gamma_{i_t} = \Lambda + \Lambda' - \nu$ for all i. Hence

$$y = F(x) = \sum c_i E_{-\beta_{i_s}} \cdots E_{-\beta_{i_1}} w \otimes E_{-\gamma_{i_t}} \cdots EW_{-\gamma_{i_1}} w',$$

where $\beta_{i_1} + \cdots + \beta_{i_s} + \gamma_{i_1} + \cdots + \gamma_{i_t} = \lambda + \lambda' - \nu$ for all i, so $\nu = \Lambda + \Lambda' - \sum k_j \alpha_j$. If $\Lambda = \Lambda'$ and $\lambda = \lambda'$, then

$$x' = \sum c_i E_{-\gamma_{i_t}} \cdots E_{-\gamma_{i_1}} v' \otimes E_{-\beta_{i_s}} \cdots E_{-\beta_{i_1}} v = \pm x$$

depending on whether $2\Lambda - \sum k_j \alpha_j$ lines in $S^2 \Lambda$ or $\Lambda^2 \Lambda$. Since

$$y' = \sum c_i E_{-\gamma_{i_t}} \cdots E_{-\gamma_{i_1}} w' \otimes E_{-\beta_{i_s}} \cdots E_{-\beta_{i_1}} w$$

it follows that $x' = \pm x$ iff $y' = \pm y$, so $2\Lambda - \sum k_j \alpha_j \in S^2 \Lambda$ iff $2\lambda - \sum k_j \alpha_j \in S^2 \lambda$. □

We will say that a chain $\beta_1, \ldots, \beta_n$ of distinct simple roots links λ and λ' if

$$(\lambda, \beta_1) \neq 0, \quad (\beta_i, \beta_{i+1}) \neq 0 \text{ for } i = 1, \ldots, s-1 \quad \text{and} \quad (\beta_s, \lambda') \neq 0.$$

We will call $\beta_1, \ldots, \beta_n$ a minimal chain if in addition

$$(\lambda, \beta_i) = 0 \text{ for } 2 \leq i \leq s \quad \text{and} \quad (\beta_s, \lambda') = 0 \text{ for } 1 \leq i \leq s-1.$$

Consider the example described in Fig. 1. We see that α_2 and $\alpha_4, \alpha_3, \alpha_5$ are minimal chains while the chain α_1, α_2 links λ and λ' but is not minimal.

Dynkin's second-highest weights rule [2, Theorem 3.1] says the following.

Theorem 2. *If $\beta_1, \ldots, \beta_s$ is a minimal chain linking λ and λ', then $\lambda + \lambda' - \sum_{i=1}^{s} \beta_i$ is the highest weight of a summand in $\lambda \otimes \lambda'$. Furthermore, if μ is the highest weight of a summand in $\lambda \otimes \lambda'$ other than $\lambda + \lambda'$, then there is a minimal chain $\gamma_1, \ldots, \gamma_s$ such that*

$$\mu \leq \lambda + \lambda' - \sum_{i=1}^{s} \gamma_i.$$

Proof. The theorem is proved by constructing an extreme vector of weight $\lambda + \lambda' - \sum_{i=1}^{s} \beta_i$. This extreme vector will then generate an irreducible summand. For details, see [1]. □

If the chain $\beta_1, \ldots, \beta_s$ is not minimal, we have the following theorem.

Theorem 3. *If $\beta_1, \ldots, \beta_s$ is a nonminimal chain linking λ and λ', then*

$$m(\lambda + \lambda' - \beta_1 - \cdots - \beta_s, \lambda \otimes \lambda') \geq 1.$$

Proof. To see this, we construct v and v' such that $v \leq^{\text{sub}} \lambda$ and $v' \leq^{\text{sub}} \lambda'$ and $\beta_1, \ldots, \beta_s$ is a minimal chain linking v and v'. This can be done by setting $v_i = 0$ for $\alpha_i \in \{\beta_2, \ldots, \beta_s\}$ and $v_i = \lambda'_i$ otherwise. Then

$$m(\lambda + \lambda' - \beta_1 - \cdots - \beta_s, \lambda \otimes \lambda') \geq m(v + v' - \beta_1 - \cdots - \beta_s, v \otimes v') = 1. \qquad \square$$

In the example described in Fig. 1 the highest summand is $\lambda + \lambda' = [1, 2, 0, 1, 1]$, the second-highest summands are $\lambda + \lambda' - \alpha_2 = [2, 0, 1, 1, 1]$ and $\lambda + \lambda' - \alpha_3 - \alpha_4 - \alpha_5 = [1, 2, 0, 0, 0]$. In addition, we have four nonminimal chains, giving us the following summands $\lambda + \lambda' - \alpha_1 - \alpha_2 = [0, 1, 1, 1, 1]$, $\lambda + \lambda' - \alpha_2 - \alpha_3 - \alpha_5 = [2, 1, 0, 2, 0]$, $\lambda + \lambda' - \alpha_2 - \alpha_3 - \alpha_4 = [2, 1, 0, 0, 2]$ and $\lambda + \lambda' - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 = [0, 2, 0, 0, 2]$.

The following theorem is due to Krämer [4].

Theorem 4. *Assume that* $m(\gamma_i, \lambda^i \otimes \lambda'^i) = p_i \geq 1$ *for* $i = 1, 2$. *Then*

$$m(\gamma_1 + \gamma_2, (\lambda^1 + \lambda^2) \otimes (\lambda'^1 + \lambda'^2)) \geq \max(p_1, p_2).$$

For $\lambda'^i = \lambda^i$ *we get the following corresponding results.*

- *If* $\gamma_i \in S^2 \lambda^i$, *then* $\gamma_1 + \gamma_2 \in S^2(\lambda^1 + \lambda^2)$.
- *If* $\gamma_i \in \Lambda^2 \lambda^i$, *then* $\gamma_1 + \gamma_2 \in S^2(\lambda^1 + \lambda^2)$.
- *If* $\gamma_1 \in S^2 \lambda^1$ *and* $\gamma_2 \in \Lambda^2 \lambda^2$, *then* $\gamma_1 + \gamma_2 \in \Lambda^2(\lambda^1 + \lambda^2)$.

The proof is based on Frobenius' Reciprocity Theorem and follows from results in an earlier paper by Krämer. Krämer's Theorem can be very useful for finding summands, but it only provides lower bounds on the multiplicities of the summands.

4. Finding summands in $\lambda \otimes \lambda = S^2 \lambda \otimes \Lambda^2 \lambda$

In this section we will show how the method of successive subtractions can be used to determine summands in products of the form $\lambda \otimes \lambda = S^2 \lambda \oplus \Lambda^2 \lambda$. The method of successive subtractions allows us in theory to compute any tensor product by going through the following steps. We first compute the weight system for each of the factors, and then determine the weight system of the product. We then find the highest weight in the weight system of the product and subtract the weight system for the representation determined by this highest weight from the weight system of the product. We then find a maximal weight among the remaining weights and subtract the weight system corresponding to the representation determined by this weight. Continuing in this way, we will eventually obtain the whole decomposition. While providing a simple solution from a theoretical point of view, it is clear that the approach outlined above will in general lead to formidable calculations. But by only considering parts of the weight system, we can easily obtain some partial results.

Let L denote the root lattice $\{\sum_{i=1}^{n} k_i\alpha_i \mid k_i \geq 0\}$. We want to find a basis for $V(2\lambda - D, \lambda \otimes \lambda)$ where $D \in L$. We can find $D_1, \dots, D_m \in L$ and bases $\{v_i^1, \dots, v_i^{p_i}\}$ for $V(\lambda - D_i, \lambda)$ such that for each $i = 1, \dots, m$ there is an i' with $1 \leq i' \leq m$ such that $\lambda - D_i, \lambda - D_{i'} \in \Delta(\lambda)$ and $D_i + D_{i'} = D$. A basis for $V(2\lambda - D, \lambda \otimes \lambda)$ is then given by

$$\{v_i^k \otimes v_{i'}^l \mid 1 \leq i \leq m, 1 \leq k \leq p_i, 1 \leq l \leq p_{i'}\}.$$

It follows that a basis for $V(2\lambda - D, S^2\lambda)$ is given by

$$\{v_i^k \otimes v_{i'}^l + v_{i'}^l \otimes v_i^k \mid 1 \leq i \leq m \text{ with } i \neq i', 1 \leq k \leq p_i, 1 \leq l \leq p_{i'}\}$$
$$\cup \{v_i^k \otimes v_i^l + v_i^l \otimes v_i^k \mid 1 \leq i \leq m \text{ with } i = i', 1 \leq k < l \leq p_i\}$$
$$\cup \{v_i^k \otimes v_i^k \mid 1 \leq i \leq m \text{ with } i = i', 1 \leq k \leq p_i\},$$

while

$$\{v_i^k \otimes v_{i'}^l - v_{i'}^l \otimes v_i^k \mid 1 \leq m \text{ with } i \neq i', 1 \leq k \leq p_i, 1 \leq l \leq p_{i'}\}$$
$$\cup \{v_i^k \otimes v_i^l - v_i^l \otimes v_i^k \mid 1 \leq i \leq m \text{ with } i = i', 1 \leq k < l \leq p_i\}$$

is a basis for $V(2\lambda - D, \Lambda^2\lambda)$.

In particular, if D is such that $\lambda - D/2$ is not a weight of λ, then

$$\dim V(2\lambda - D, S^2\lambda) = \dim V(2\lambda - D, \Lambda^2\lambda),$$

while if $\lambda - D/2$ is a weight, then

$$\dim V(2\lambda - D, S^2\lambda) = \dim V(2\lambda - D, \Lambda^2\lambda) + \dim V(\lambda - D/2, \lambda).$$

Theorem 5. *Assume that $\lambda_\alpha = q$. Then $m(2\lambda - k\alpha, \lambda \otimes \lambda) = 1$ for $0 \leq k \leq q$ and the summands with k even lie in $S^2\lambda$ while the summands with k odd lie in $\Lambda^2\lambda$.*

Proof. It is easy to see that $\lambda - k\alpha$ is a weight of λ with multiplicity 1 for $0 \leq k \leq q$. In the product we get weights of the form $2\lambda - k\alpha$ with $0 \leq k \leq 2q$, but only the ones with $0 \leq k \leq q$ will be dominant. For $0 \leq k \leq q$ we have $m(2\lambda - k\alpha, \lambda \otimes \lambda) = k + 1$ with $k/2 + 1$ weights in $S^2\lambda$ and $k/2$ weights in $\Lambda^2\lambda$ if k is even, and with $(k + 1)/2$ weights in both $S^2\lambda$ and $\Lambda^2\lambda$ if k is odd. The multiplicities of these dominant weights in the summands are given in Table 1. We write (a, b) to denote a weights in $S^2\lambda$ and b weights in $\Lambda^2\lambda$. The statements about the summands in $S^2\lambda$ and $\Lambda^2\lambda$ can be deduced from Table 1. □

For $k = 1$, this is just Dynkin's rule about the second-highest weights, while for $k = 2$ or $k = 3$ it was proved by Manturov [6].

We will now show how a weaker version of Theorem 5 can be deduced from Krämer's Theorem. Consider ω_α, the fundamental weight corresponding to α. We know that

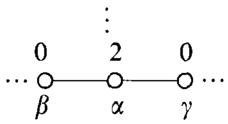
$$\omega_\alpha \otimes \omega_\alpha \supset (2\omega_\alpha)_s \oplus (2\omega_\alpha - \alpha)_a.$$

Table 1

	$\lambda \otimes \lambda$	$2\lambda_s$	$(2\lambda - \alpha)_a$	$(2\lambda - 2\alpha)_s$	$(2\lambda - 3\alpha)_a$	$\cdots$
2λ	(1, 0)	(1, 0)	–	–	–	
$2\lambda - \alpha$	(1, 1)	(1, 0)	(0, 1)	–	–	
$2\lambda - 2\alpha$	(2, 1)	(1, 0)	(0, 1)	(1, 0)	–	
$2\lambda - 3\alpha$	(2, 2)	(1, 0)	(0, 1)	(1, 0)	(0, 1)	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

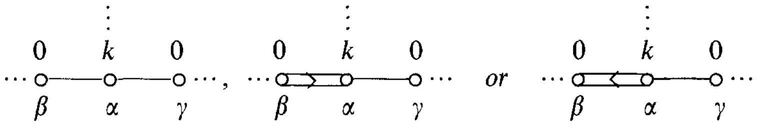
If we take the q-fold tensor power of ω_α, we can deduce from Krämer's Theorem that the multiplicities are at least one.

Manturov [6] also considered the case where the Dynkin diagram contained three simple roots α, β and γ joined by simple bonds with $\lambda_\alpha = 2$ and $\lambda_\beta = \lambda_\gamma = 0$.



The dots indicate arbitrary combinations of simple roots and bonds. Manturov then stated that $2\lambda - 3\alpha - \beta - \gamma \in \Lambda^2\lambda$. This can be generalized as follows.

Theorem 6. *If λ is of the form*



with $k \geq 1$, then

$$\lambda \otimes \lambda \supset (2\lambda)_s \oplus (2\lambda - \alpha)_a \oplus \cdots \oplus (2\lambda - k\alpha)$$
$$\oplus (2\lambda - 2\alpha - \beta - \gamma)_s \oplus (2\lambda - 3\alpha - \beta - \gamma)_a$$
$$\oplus \cdots \oplus (2\lambda - (k+1)\alpha - \beta - \gamma).$$

The s *and* a *summands alternate in each series, the given summands have multiplicity equal to* 1, *and they are the only summands of the form $2\lambda - a\alpha - vb\beta - c\gamma$ with $b, c \leq 1$ that occur in the product.*

If λ is of the form

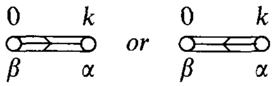


Table 2

λ (1)	–	–	–
$\lambda - \alpha$ (1)	–	–	–
$\lambda - 2\alpha$ (1)	$\lambda - \alpha - \beta$ (1)	$\lambda - \alpha - \gamma$ (1)	–
$\lambda - 3\alpha$ (1)	$\lambda - 2\alpha - \beta$ (1)	$\lambda - 2\alpha - \gamma$ (1)	$\lambda - \alpha - \beta - \gamma$ (1)
$\lambda - 4\alpha$ (1)	$\lambda - 3\alpha - \beta$ (1)	$\lambda - 3\alpha - \gamma$ (1)	$\lambda - 2\alpha - \beta - \gamma$ (2)
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\lambda - k\alpha$ (1)	$\lambda - (k-1)\alpha - \beta$ (1)	$\lambda - (k-1)\alpha - \gamma$ (1)	$\lambda - (k-2)\alpha - \beta - \gamma$ (2)
–	$\lambda - k\alpha - \beta$ (1)	$\lambda - k\alpha - \gamma$ (1)	$\lambda - (k-1)\alpha - \beta - \gamma$ (2)
–	–	–	$\lambda - k\alpha - \beta - \gamma$ (2)
–	–	–	$\lambda - (k+1)\alpha - \beta - \gamma$ (2)

with $k \geq 1$, then

$$\lambda \otimes \lambda \supset (2\lambda)_s \oplus (2\lambda - \alpha)_a \oplus \cdots \oplus (2\lambda - k\alpha)$$
$$\oplus (2\lambda - 2\alpha - \beta)_s \oplus (2\lambda - 3\alpha - \beta)_a$$
$$\oplus \cdots \oplus (2\lambda - (k+1)\alpha - \beta).$$

The s *and* a *summands alternate in each series, the given summands have multiplicity equal to* 1, *and they are the only summands of the form $2\lambda - a\alpha - b\beta$ and $b \leq 1$ that occur in the product.*

Proof. In order to use the method of successive subtractions, it will be necessary to calculate several weight multiplicities. We will use Freudenthal's formula

$$m(\mu, \lambda) = \frac{2}{(\lambda + \mu + 2\delta, \lambda - \mu)} \sum_{v \in \Delta^+} \sum_{k \geq 1} m(\mu + kv, \lambda)(\mu + kv, v).$$

In our case $\mu = \lambda - a\alpha - b\beta - c\gamma$, so the only roots we need to consider are the ones of the form $a'\alpha + b'\beta + c'\gamma$. But since we know the part of the Cartan matrix involving α, β and γ, we know that the only such roots are

$$F = \{\alpha, \beta, \gamma, \alpha + \beta, \alpha + \gamma, \alpha + \gamma + \beta\}.$$

The important point is that these calculations only depend on a small part of the weight system and the root space. We do not even need to know which algebra $\mathfrak{g}$ is. It is even independent of the type of bonds we have. The last statement is not entirely obvious (in fact, I only realized it after the referee had asked me to extend the theorem to the case where the roots had different lengths). The relevant part of the weight system is listed in Table 2 (with the multiplicities in brackets). The length of the root strings depends only on k. If the roots have different lengths we scale them so that $(\alpha, \alpha) = 2$. When using Freudenthal's formula we can think of k as an unknown. We then get a fraction of the form $(ak + b)/(ck + d)$. But the terms involving k all come from products of the form (λ, α) so a and c are independent of the lengths of the other roots. But since $(ak + b)/(ck + d)$ is equal to a natural number e, we see that e is also

Table 3

$2\lambda - p\alpha$ with $0 \le p \le k$ and p even	$(p/2 + 1, p/2)$
$2\lambda - p\alpha$ with $0 \le p \le k$ and p odd	$((p+1)/2, (p+1)/2)$
$2\lambda - p\alpha - \beta$ with $2 \le p \le k$	(p, p)
$2\lambda - p\alpha - \gamma$ with $2 \le p \le k$	(p, p)
$2\lambda - p\alpha - \beta - \gamma$ with $2 \le p \le k + 1$	$(3p - 2, 3p - 2)$

Table 4

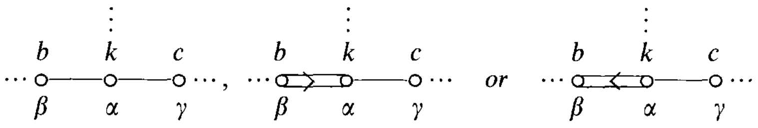
	2λ	$2\lambda - r\alpha$	$2\lambda - r\alpha - \beta - \gamma$
$M - p\alpha$ with $p \ge 0$	1	1	1
$M - \beta$	–	1	–
$M - p\alpha - \beta$ with $p \ge 1$	1	2	–
$M - \beta - \gamma$	–	1	–
$M - \alpha - \beta - \gamma$	1	4	–
$M - p\alpha - \beta - \gamma$ with $p \ge 2$	2	5	–

independent of the lengths of the other roots. Hence we can do all our computations assuming that the roots are of equal length.

Multiplying out, we see that the relevant dominant weights for $\lambda \otimes \lambda$ are as listed in Table 3, where we have written (a, b) to denote a weights in $S^2\lambda$ and b weights in $\Lambda^2\lambda$. We must now calculate the multiplicities of these weights in the summands $2\lambda - r\alpha$ for $0 \le r \le k$ and $2\lambda - r\alpha - \beta - \gamma$ for $2 \le r \le k + 2$. They are given in Table 4, where M denotes the possible summands, and the left-most column represents the weights of the summands in the top row. Comparing this table with the multiplicities of $\lambda \otimes \lambda$ given in Table 3, we deduce Theorem 6. (The proof for the G_2 case is similar.) □

Theorem 6 can be combined with the method of subordination to give the following corollary.

Corollary 7. *If λ is of the form*



with $k \ge 1$, then

$$\lambda \otimes \lambda \supset (2\lambda)_s \oplus (2\lambda - \alpha)_a \oplus \cdots \oplus (2\lambda - k\alpha)$$
$$\oplus (2\lambda - 2\alpha - \beta - \gamma)_s \oplus (2\lambda - 3\alpha - \beta - \gamma)_a$$
$$\oplus \cdots \oplus (2\lambda - (k+1)\alpha - \beta - \gamma).$$

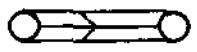
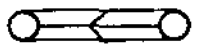
where the s *and* a *summands alternate in each series.*

Table 5

	$\lambda \otimes \lambda$	$2\lambda_s$	$(2\lambda - \alpha)_a$
2λ	(1, 0)	(1, 0)	–
$2\lambda - \alpha$	(1, 1)	(1, 0)	(0, 1)
$2\lambda - \alpha - \beta$	(1, 1)	(1, 0)	(0, 1)

If λ is of the form

b k b k

 *or* 

β α β α

with $k \geq 1$, then

$$\lambda \otimes \lambda \supset (2\lambda)_s \oplus (2\lambda - \alpha)_a \oplus \cdots \oplus (2\lambda - k\alpha)$$
$$\oplus (2\lambda - 2\alpha - \beta)_s \oplus (2\lambda - 3\alpha - \beta)_a$$
$$\oplus \cdots \oplus (2\lambda - (k+1)\alpha - \beta),$$

where the s *and* a *summands alternate in each series.* □

Notice that in this case the multiplicities may be greater than 1, and there may be other summands of the form $2\lambda - a\alpha - b\beta - c\gamma$ with $b, c \leq 1$ occurring in the product.

It is easy to prove Corollary 7 directly using Krämer's Theorem. We first prove the result for $k = 2$ and $b = c = 0$ using the above method. Then we use subordination to handle $b > 0$ or $c > 0$. Then we set $\lambda^1 = \lambda - \omega_\alpha$ and $\lambda^2 = \omega_\alpha$ and add summands from $\lambda^1 \otimes \lambda^1$ to $2\omega_\alpha$ and $2\omega_\alpha - \alpha$ from $\omega_\alpha \otimes \omega_\alpha$. The G_2 case is handled similarly. This proves Corollary 7, but Krämer's Theorem does not give the more precise information contained in Theorem 6.

We will finally determine the multiplicities of summands of the form $2\lambda - \alpha - \beta$, where α and β are not necessarily linked.

Theorem 8. *Let α and β be two roots that are not necessarily linked. Then $2\lambda - \alpha - \beta$ is a summand of $\lambda \otimes \lambda$ if and only if λ_α and λ_β are both nonzero, and in that case*

$$m(2\lambda - \alpha - \beta, S^2\lambda) = 1,$$
$$m(2\lambda - \alpha - \beta, \Lambda^2\lambda) = \begin{cases} 1 & \text{if } (\alpha, \beta) \neq 0, \\ 0 & \text{if } (\alpha, \beta) = 0. \end{cases}$$

Proof. In order for $2\lambda - \alpha - \beta$ to be a summand, either both λ_α and λ_β must be nonzero, or $\lambda_\alpha \neq 0$ and $\lambda_\beta = 0$ but $(\alpha, \beta) \neq 0$. But in the latter case we get the multiplicities listed in Table 5, where we write (a, b) for a weights in $S^2\lambda$ and b weights

Table 6

	$\lambda \otimes \lambda$	$2\lambda_s$	$(2\lambda - \alpha)_a$	$(2\lambda - \beta)_a$	left over
2λ	(1, 0)	(1, 0)	–	–	0
$2\lambda - \alpha$	(1, 1)	(1, 0)	(0, 1)	–	0
$2\lambda - \beta$	(1, 1)	(1, 0)	–	(0, 1)	0
$2\lambda - \alpha - \beta$	(2, 2)	(1, 0)	(0, 1)	(0, 1)	(1, 0)

Table 7

	$\lambda \otimes \lambda$	$2\lambda_s$	$(2\lambda - \alpha)_a$	$(2\lambda - \beta)_a$	left over
$2\lambda - \alpha - \beta$	(3, 3)	(2, 0)	(0, 1)	(0, 1)	(1, 1)

in $\Lambda^2\lambda$. From Table 5 we deduce that $2\lambda - \alpha - \beta$ will not be a summand unless both λ_α and λ_β are nonzero.

There are now two cases to consider. If $(\alpha, \beta) = 0$, then $\alpha + \beta$ is not a root, and we get the multiplicities listed in Table 6. If follows that $2\lambda - \alpha - \beta \in S^2\lambda$.

If $(\alpha, \beta) \neq 0$, then $\alpha + \beta$ is a root. This makes a difference when calculating $m(\lambda - \alpha - \beta)$ and $m(2\lambda - \alpha - \beta)$. Hence only the last row will be different, as shown in Table 7. It follows that $2\lambda - \alpha - \beta$ lies in both $S^2\lambda$ and $\Lambda^2\lambda$. □

We will finally mention the following result by Wang and Ziller [7, Theorem 2.8] which is proved in a similar way.

Theorem 9. *If* $\beta_1, \dots, \beta_2$ *with* $s \geq 2$ *is a chain linking* λ *with itself and* $\lambda_{\beta_i} = 0$ *for* $2 \leq i \leq s-1$, *then* $m(2\lambda - \beta_1 - \cdots - \beta_s, S^2\lambda) = m(2\lambda - \beta_1 - \cdots - \beta_s, \Lambda^2\lambda) = 1$. □

If $s = 2$, this corresponds to the case $(\alpha, \beta) \neq 0$ above. For $s > 2$ it elaborates on the case $\lambda = \lambda'$ in Theorem 3.

Acknowledgment

Part of this work was done while I was doing part of my military service at the Norwegian Defense Research Establishment (NDRE). I would like to thank Wu-yi Hsiang, University of California at Berkeley; Eldar Straume, University of Tromsø; Terje Wahl, NDRE; McKenzie Wang, McMaster University and Wolfgang Ziller, University of Pennsylvania.

References

[1] R.N. Cahn, Semi-simple Lie Algebras and their Representations, Frontiers in Physics, Vol. 59 (Benjamin/Cummings, Reading, MA, 1984).
[2] E.B. Dynkin, The maximal subgroups of the classical groups, Trudy Moskov. Mat. Obse. 1 (1952) 39–166; English translation: Amer. Math. Soc. Transl. '(2) 6 (1957) 245–378.
[3] M. Krämer, Über untergruppen kompakter Liegruppen als Isotropiegruppen bei linearen Aktionen, Math. Z. 147 (1976) 207–224.
[4] M. Krämer, Some tips on the decomposition of tensor products of compact connected Lie groups, Rep. Math. Phys. 13 (1978) 295–304.
[5] M. Krämer, Eine Klassifikation bestimter Untergruppen kompakter zusammenhängender Leigruppen, Comm. Algebra 3 (1978) 691–737.
[6] O.V. Manturov, Riemannian spaces with orthogonal and symplectic groups of motions and an irreducible group of rotations, Dokl. Akad. Nauk SSSR 141 (1961) 1034–1037.
[7] M.Y. Wang and W. Ziller, On normal homogeneous Einstein manifolds, Ann. Sci. École Norm. Sup. 18 (1985) 563–633.
[8] J.A. Wolf, The geometry and structure of isotropy irreducible homogeneous spaces, Acta Math. 120 (1968) 59–148.