

RECONSTRUCTION OF VERMEER'S "THE MUSIC LESSON"

An application of projective geometry

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SUMMARY

Johannes Vermeer was an extraordinary Dutch artist who created amazingly photorealistic paintings with lifelike qualities. Several authors have suggested that he might have used a kind of *camera obscura* to obtain such realism.

In one of his paintings, “the Music Lesson”, a reflection of the back wall can be seen, and several authors have attempted to reconstruct the room that this painting was created in by applying principles of projective geometry. Several key measurements of the room are of great interest, namely, position of the camera and the direction it is pointing at; the total length of the room (distance from the front wall to the back wall); and the vertical angle of the mirror hanging overhead. Quite a few authors have attempted to calculate these values but there does not seem to be a consensus on them.



Figure 1 - The Music Lesson



Figure 2 - Detail of the Music Lesson

For this project, I attempted to find these three values for myself, by using basic principles of projective geometry. I assumed that Vermeer did use a *camera obscura* and thus sufficiently accurate data can be obtained from his paintings. In this paper, I have attempted to explain the way I found these values by using as simple mathematics as possible, to make it more accessible for the lay reader to

understand the basics of projective geometry.

1 – BASIC PROJECTIVE GEOMETRY

In Euclidean geometry, we are taught that parallel lines do not ever intersect each other. However, that logical knowledge seems to be contradicted by what we see in our daily lives. Railway tracks seem to converge at a certain point in the distance. Pillars of a building seem to get smaller and smaller as they are further from us, and if we consider the line joining the top of the pillars and the bottom of the pillars, the two lines do seem to converge at a certain point.

We can explain this phenomenon by introducing the axioms defining projective geometry, but instead of approaching projective geometry in that manner, we would use the more intuitive visual approach. This approach might seem less mathematically rigorous, but it is much more accessible to the lay reader. This geometric method is inspired by Salgado's *Manual de Perspectiva Modular*. In that book, he introduces a technique for drawing a 3D perspective diagram. It can be seen as the pen and paper equivalence to 3D modelling software such as 3DsMax and AutoCAD.

The diagram below shows a basic idea of how Salgado's method works.

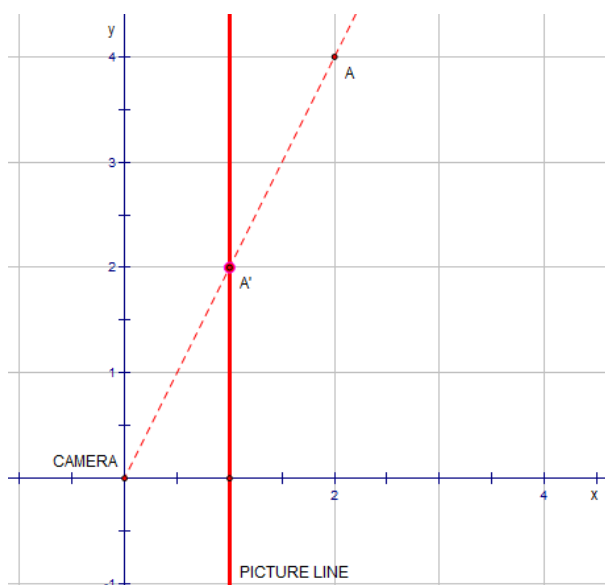


Figure 3 – Image of point A (2, 4) on the picture line

In this diagram, the picture line is parallel to the y-axis at $x = 1$. The camera is located at the origin and the object A is at $(x, y) = (2, 4)$. A' , the image of A on the picture line is located at point $(1, 2)$.

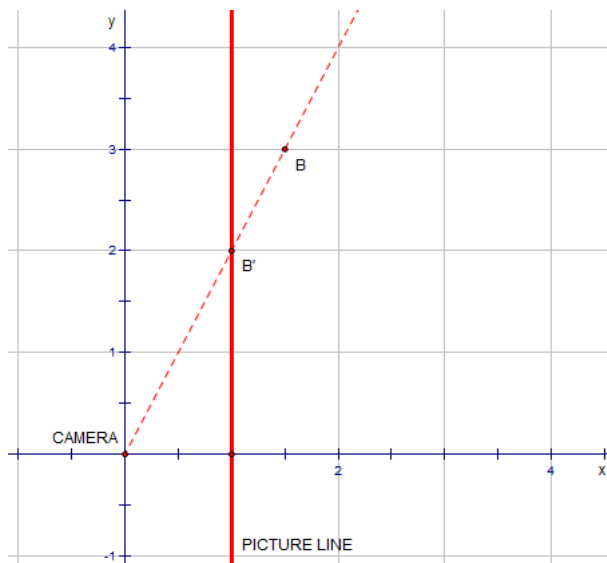


Figure 4 - Image of point B (1.5, 3) on the picture line

In this diagram, the picture line is still parallel to the y-axis at $x = 1$.

The camera is again located at the origin and the object B is at $(x, y) = (1.5, 3)$

We can see that B' , the image of B on the picture line is also located at point $(1, 2)$, the same location as A' .

So we see that two distinct points in the 2D coordinate system are mapped to the same point on the picture line. In fact, we can see that any point on the projected line along the camera, object X and object Y would be mapped to the same image at point $(1, 2)$. To be exact, the point (x, y) in the 2D plane is projected on the point $\left(1, \frac{y}{x}\right)$.

2 – POINTS IN A PROJECTIVE PLANE

We can extend the above method to the projection of 3D coordinates into 2D coordinates on the picture plane by using two diagrams to illustrate the top view and the side view of the system.

Let us set the camera to be at the origin and let the picture plane to be at $z = 1$. Let the object A be located at $(x, y, z) = (6, 4, 2)$. The convention used would be y as parallel to the vertical axis of the picture plane, x parallel to the horizontal axis of the picture plane, and z as normal to the picture plane.

We would illustrate the 3D model with two diagrams, the first one showing the top view of the environment, and the next one showing the side view of the environment.

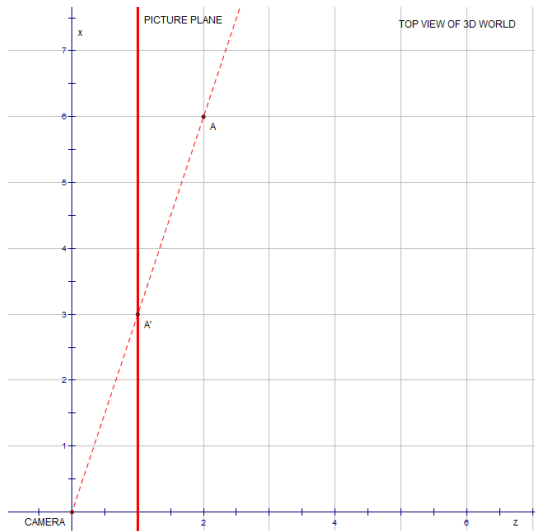


Figure 5 - Image of point A (6, 4, 2) on the picture plane. Top view (x-axis and z-axis)

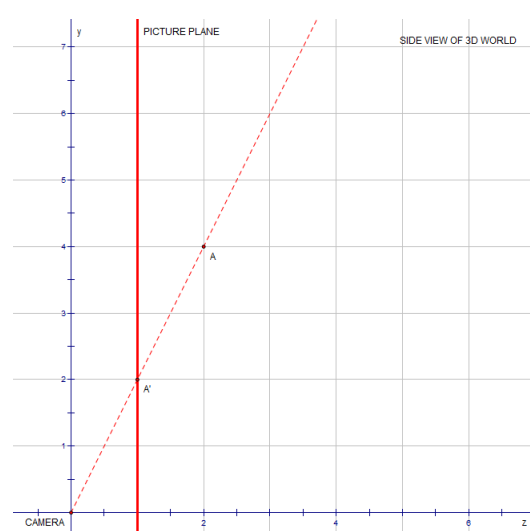


Figure 6 - Image of point A (6, 4, 2) on the picture plane. Side view (y-axis and z-axis)

Thus, the image of A on the picture plane is at the coordinate (3, 2) on the picture plane. As with the 2D case, we can see that any other points that lie on the line formed by the camera and A would also be mapped to the same coordinate on the picture plane.

For example, the point (9, 6, 3) would also be mapped to the coordinate (3, 2) on the picture plane - equivalent to the 3D coordinate of (3, 2, 1).

So we have arrived at the first relationship between the picture plane and the real world.

Let the two dimensional projective space be denoted as RP^2 and the three dimensional Euclidean space be R^3 .

Definition 1.1 (points in projective geometry): A point in RP^2 is a line that passes through the origin of R^3 .

For our specific case where the camera is located at the origin, and the image plane is at $z = 1$, the point (kx, ky, kz) would be mapped to the point $(\frac{kx}{kz}, \frac{ky}{kz})$ on the picture plane. So the point $(\frac{x}{z}, \frac{y}{z})$ on the picture plane corresponds to the line formed by the set of points (kx, ky, kz) on the 3D coordinate system for all values of k.

3 - GENERALIZATION FOR ARBITRARY CAMERA POSITIONS

What about the more general case? What if we move the camera to another location? What if the camera is pointing in another direction? What if the picture plane is located at a nearer or further distance from the camera?

For the general case, we can apply a set of geometric transformations on the coordinate system, such that it becomes the specific case that we have examined.

One possible sequence of transformations is as follows.

The first transformation should be a translation such that the camera is located at the origin, followed by a rotation along the y-axis until the camera direction is located on the y-z plane. The third transformation in the sequence is another rotation along the x-axis until the camera is pointing along the z-axis. The last transformation is a uniform scaling along all three axes, such that the picture plane is on the plane $z = 1$.

4 – LINES ON THE PROJECTIVE PLANE

What does a line in RP^2 correspond to in R^3 ? A line in RP^2 can be defined just by taking two points on that line in RP^2 which would correspond to two lines in R^3 , both of which passing through the origin of R^3 . In R^3 , these two lines then define a plane.

Definition 1.2 (lines in projective geometry): *A line in RP^2 is a plane that passes through the origin of R^3 .*

Once we have established these two, we then look at how two intersecting lines in R^3 are projected on RP^2 . We will be looking at how the intersection of the two lines is mapped onto the picture plane. For simplicity's sake, we only show one diagram.

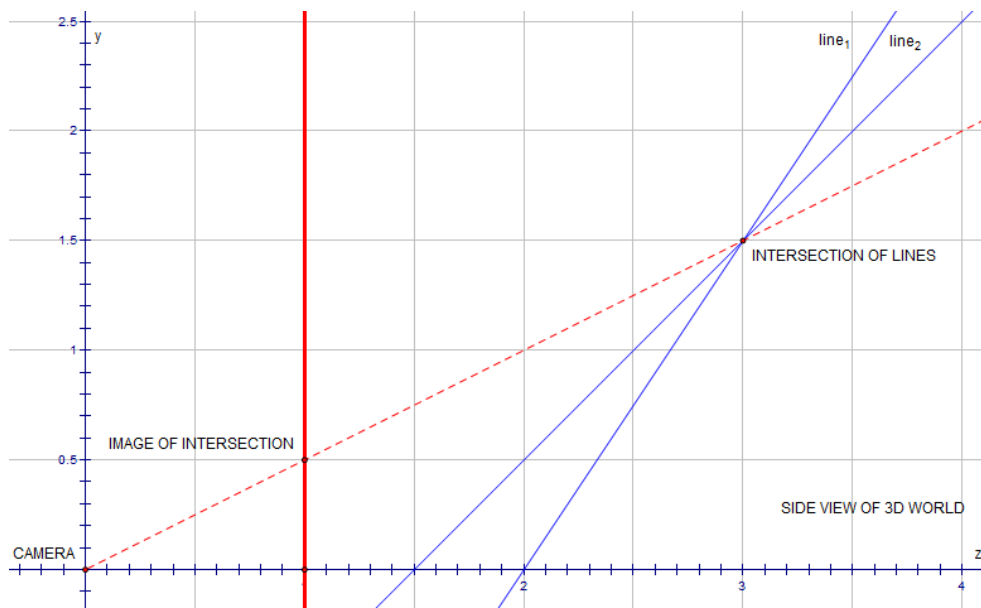


Figure 7 – Projected image on RP^2 of the intersection of two lines in R^3

To find where the intersection is mapped on the picture plane, we draw a visual line passing through the origin and the intersection of the two lines. The intersection of that visual line and the picture plane would be where the intersection would be mapped on the picture plane, so from Figure 7, we get the y-coordinate of the image on the picture plane as 0.5. By repeating the above approach with the top view, we can get the x-coordinate of the image on the picture plane.

5 – PARALLEL LINES IN THE PROJECTIVE PLANE

But what about two parallel lines? We know that railroad tracks converge in a distance, so let us see where two parallel lines converge in the picture plane.

In the following sequence of diagrams, we see how the projected image of the intersection of two lines changes as the gradient of one line changes. As we progress through the sequence, we fix the black line while adjusting the gradient of the blue line to progressively go closer to the gradient of the black line.

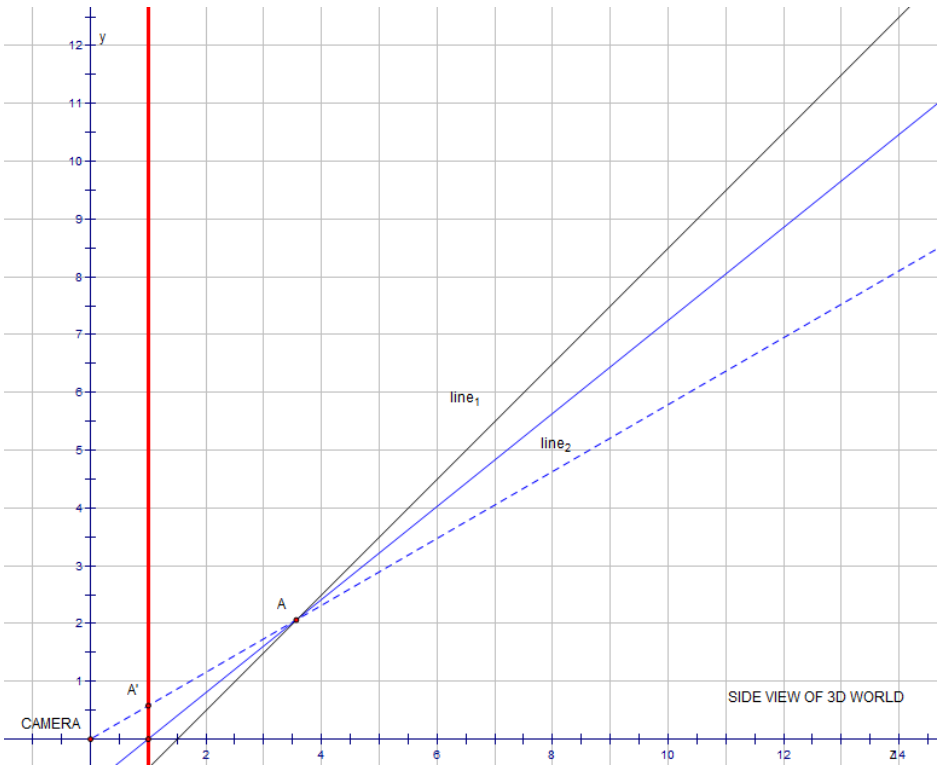


Figure 8a Intersection of parallel lines – Part 1

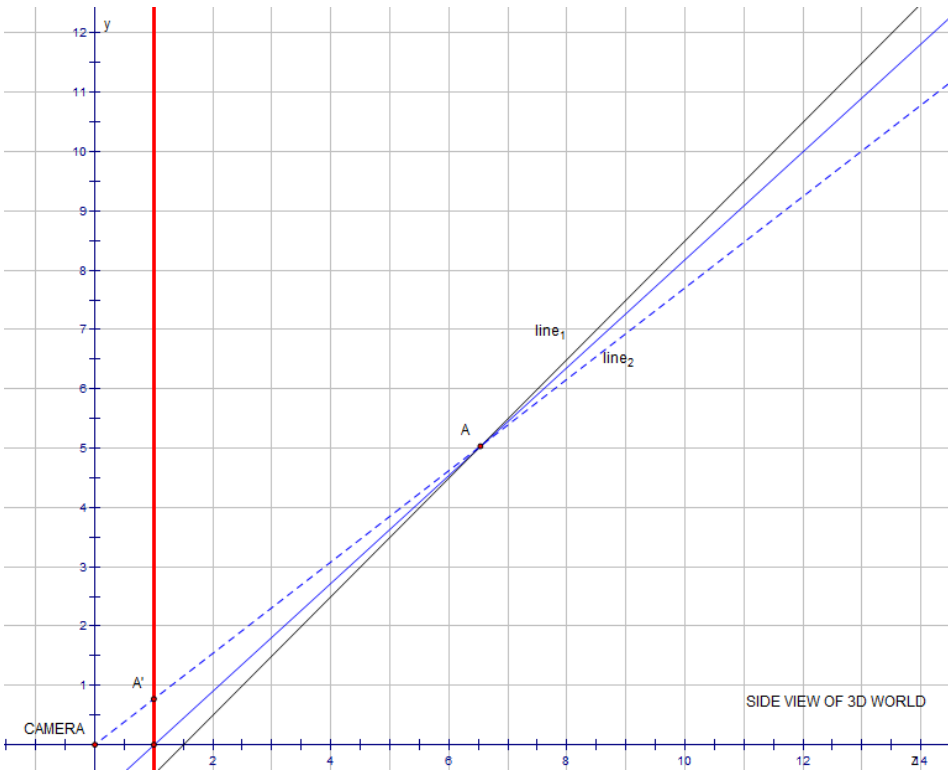


Figure 8b Intersection of parallel lines – Part 2

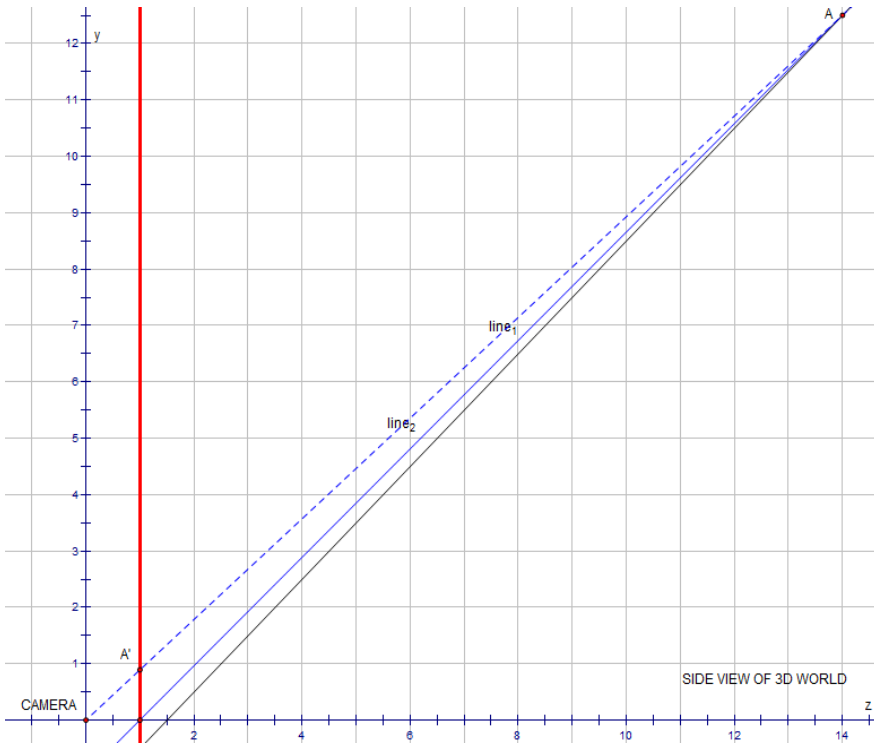


Figure 8c Intersection of parallel lines – Part 3

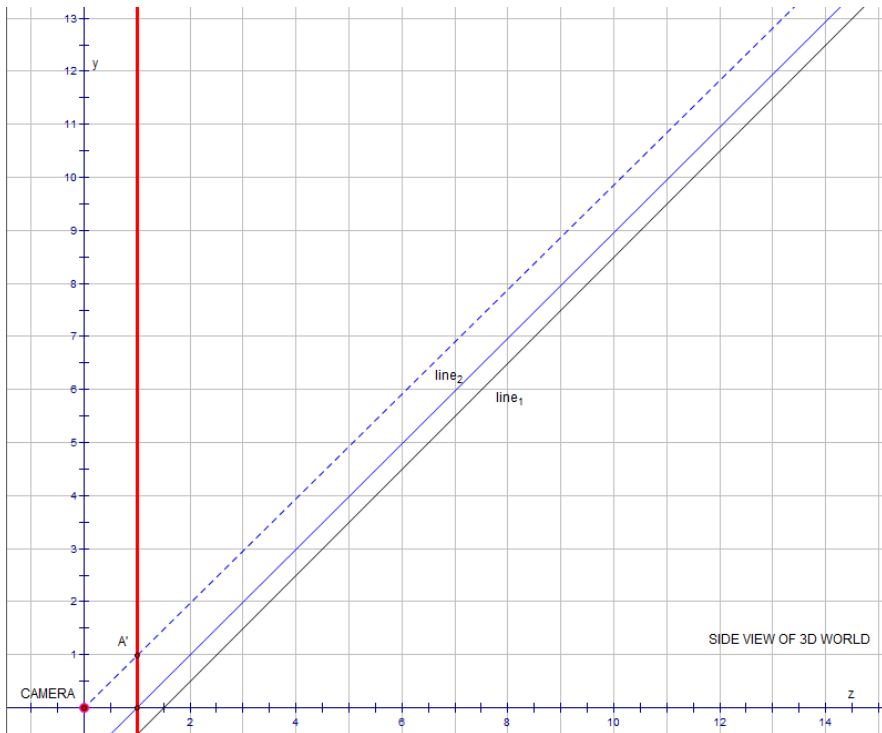


Figure 8d - Intersection of parallel lines – Part 4

As the blue line's gradient increases such that it is almost parallel to the black line, we see that the projection line (dashed) also becomes close to being parallel to the black line.

From this observation, we then deduce that the projection of parallel lines in 3D to the picture plane will intersect at a point. We also see that the image of this intersection depends solely on the gradient of the parallel lines.

If we let the angle formed by the parallel lines and the z-axis on the diagram be θ , and the image plane is at $z = 1$ then the coordinate of the intersection image is at $(z, y) = (1, \tan \theta)$.

Lemma 1.3 (Image of intersection of parallel lines): *The image of parallel lines under projection intersects at the position where another line parallel to them intersects the picture plane.*

6 – FINDING THE CAMERA DISTANCE

We would be using this result to find the distance of Vermeer's camera to the picture plane, by making use of the numerous vanishing points that can be constructed on his painting. In essence, we are doing the opposite of the above; we are using the known locations of the vanishing points to find out the distance of the camera to the picture plane.

We would be using the tiles on the floor of the Music Lesson as a reference, and assuming that they are square in real life. There are several vanishing points that can be obtained from these tiles. If we look at the floor tiles in the painting, it seems that the camera is pointing at the direction of the diagonals of the tiles. There would then be 3 sets of parallel lines that can be obtained from the tiles; two sets are the sides of the tiles, and the third is the set of parallel lines connecting the corners of the tiles travelling into the picture plane.

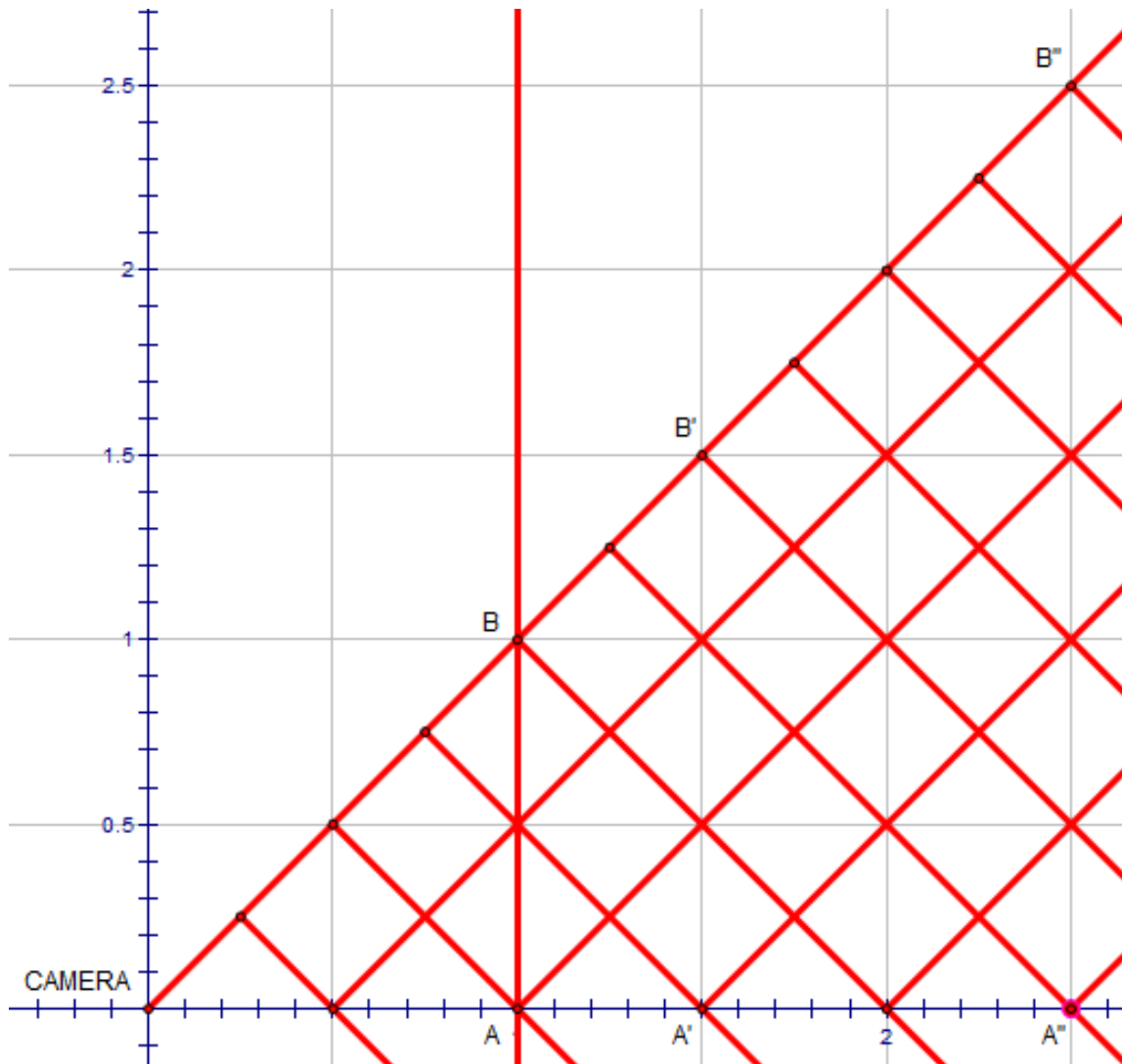


Figure 9 – Top View of tiles

The figure above helps to illustrate how we can use the vanishing points to calculate the distance from the camera to the picture plane. The diagram above assumes that the diagonal of the tiles are parallel to the direction the camera is pointing at. So on the picture plane, B is the vanishing point of lines 45° from the direction of the camera.

For the special value of 45° , we notice an interesting behaviour. If we measure the in-painting distance between the left vanishing point and the central vanishing point and

compare it to how many tiles does it correspond to at that particular z-value, we can get the real-world distance in tiles of that z-value to the camera.

For example, the in-painting distance between the left vanishing point and the central vanishing point, would be equal to the in-painting distance of $A'B'$ and $A''B''$. If we measure that distance, in term of tiles at that z-value, we see that distance is equivalent to 3 tile diagonals for $A'B'$ and 5 tile diagonals for $A''B''$. Notice how the distance of A' from the camera is indeed 3 diagonals, and the distance of A'' from the camera is 5 diagonals. In fact, this would be true for every z-value. Thus, we can find the real-world distance of the camera to any z-value in this manner.

Steadman's book, *Vermeer's Camera*, uses this very method to find the camera distance from the picture plane. He wrote that the left and right vanishing points are equidistant to the central vanishing point, and continued to use the above method to calculate the camera distance.

A fundamental assumption used by this method that the tiles are arranged precisely in the orientation given in the above figure (i.e. 45°). If this assumption were to be true, the central vanishing point would be located exactly in the midpoint of the left and right vanishing point.

Another author, Fred Dubery, also use the same method for the same purpose, but in his book, his measurements show that the distance of the left vanishing point to the central vanishing point is NOT equal to the distance of the central vanishing point to the right vanishing point. Unfortunately, instead of using another method to calculate the camera

distance to account for this length discrepancy, Dubery averages the two values obtained and assumes that this would correspond to the camera distance.

7 – SLIGHT DETOUR, FINDING THE VANISHING POINTS

Faced with two contradicting statements from two different authors, we verify the claims ourselves by obtaining the measurements needed directly off the painting to decide on which claim to believe.

For the first attempt in finding the vanishing points, we use GIMP, a freeware drawing software to extend the lines of the tiles.

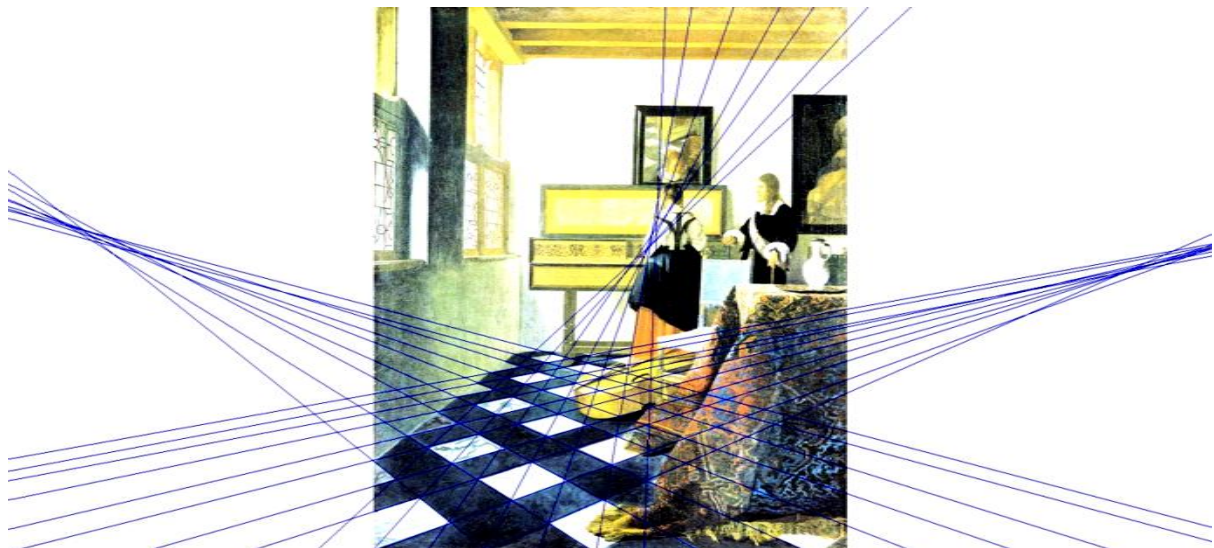


Figure 10 – Manually obtained vanishing points

We can see that there is not a single obvious vanishing point for each set of parallel lines. However, even with this rough result, we can see that Vermeer's drawing is rendered with highly accurate perspective.

Although the result obtained seems to be quite accurate, it is not sufficiently precise for our purposes as errors might have been introduced when drawing the lines. To minimize these

errors, we use MATLAB to calculate the best fit line given manually entered coordinates. We use the MATLAB function *polyfit()* to calculate the best fit line from the coordinates obtained. From the equations of the lines obtained, we calculate the vanishing point by using MATLAB's *mldivide()* (or simply using the operator '\') to calculate the least squares solution to the system of equations, which would be the coordinates of the vanishing point for that set of lines.

We obtained a set of 6 line equations for each of the left, central and right vanishing points respectively. The vanishing points obtained by MATLAB and the lines used to obtain them are displayed below superimposed on the painting.

The coordinates of the vanishing points obtained from MATLAB are as follows. The left vanishing point is at (212.8, 1621.0), the central vanishing point is at (1781.5, 1638.4) and the right vanishing point is at (3167.6, 1702.4)

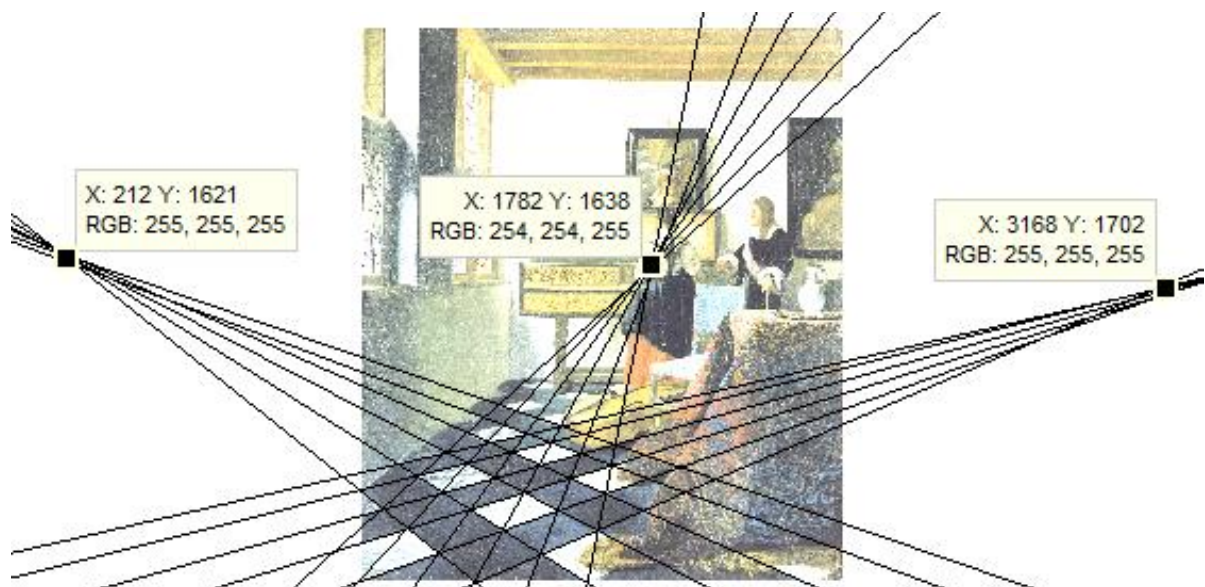


Figure 11 – Vanishing points obtained by MATLAB

From the three coordinates, we calculate the best-fit line.

RESULT 1 – The equation for the vanishing line of the floor is $y = 0.0272x + 1607.2$

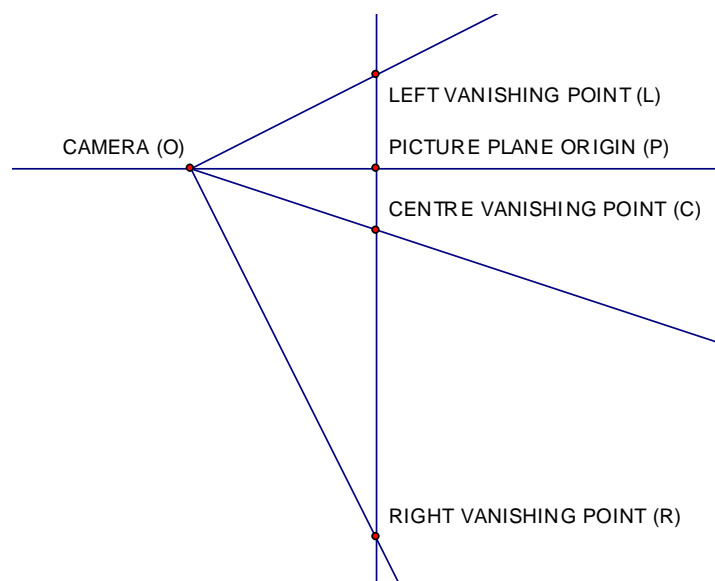
RESULT 2 - The projections of the three vanishing points on this line are (300.8, 1615.4), (1871.2, 1658.1) and (3260.0, 1695.9). Subsequently, we will use these values for the left, central and right vanishing points respectively.

The distance between the left and the central vanishing point is 1571.0 pixels, while the distance between the right and the central vanishing point is 1389.3 pixels. As there is quite a significant difference between the two distances, we can deduce that the central vanishing point is not where the camera is pointing at.

8 – BACK TO FINDING THE CAMERA DISTANCE

Now that we have verified that Dubery was right, how then would we find the distance of the camera from the image plane?

The previous method can be reworked slightly to account for this misalignment. Assuming that the camera is level to the ground, the locations of the three vanishing points above correspond to the locations given in the diagram below.



$$\angle LOC = \angle COR = 45^\circ$$

Letting $\angle POC = \theta$ and the camera distance to be x , we get the following equations.

$$x(\tan \theta + \tan(45^\circ - \theta)) = 1571.0$$

$$x(\tan(45^\circ + \theta) - \tan \theta) = 1389.3$$

Figure 12 – Diagram of angled tiles

Dividing the two equations, we get

$$\frac{(\tan \theta + \tan(45^\circ - \theta))}{(\tan(45^\circ + \theta) - \tan \theta)} = \frac{1571.0}{1389.3}$$

Solving the equation, we get $\theta = -4.96^\circ$ and $x = 1499$ pixels

RESULT 3 – The camera is located 1499 pixels away from the image plane.

This slight misalignment of the center to the direction of the camera suggests that Vermeer, unlike Leonardo da Vinci, did not use a geometric construction method to begin his painting. If Leonardo had done this painting instead, he would have aligned the tile diagonal perfectly with the camera's direction, to ensure that the central vanishing point is located exactly at the midpoint of the left and the right vanishing points.

So the camera distance is 1499 pixels from the picture plane. The length of the diagonal of the tiles at the furthest wall in the painting is about 105 pixels. So, the camera is located about 14.3 tiles away from the front wall.

RESULT 4 – The camera is located about 14.3 tiles away from the image plane.

From the best fit line, we find the coordinate for the picture plane origin/camera centre.

From the values of θ and x obtained earlier, we get the following result

RESULT 5 – The camera centre is located at (2001.3, 1661.2).

From the camera centre, we can calculate the visual angle of the painting. Since the painting is off centre, we need to measure the visual angle in two steps. The left border of the painting is at $x = 1000$, while the right border is at $x = 2296$. So the horizontal visual angle

can be calculated as $\tan^{-1}\left(\frac{2001.3-1000}{1499}\right) + \tan^{-1}\left(\frac{2296-2001.3}{1499}\right) = 44.9^\circ$. The vertical visual angle can be calculated the same way by assuming that the x and the y axis are equally scaled.

RESULT 6 – The horizontal visual angle is 44.9° .

This off-centeredness suggests that Vermeer cropped the painting, perhaps to focus on a certain detail of the painting. If we look at where the camera centre corresponds to in the painting, we see that it is pointing at a somewhat uninteresting location. Vermeer understandably crops the painting such that the viewer's attention is focused on the girl instead.



Figure 13 – Location of camera centre.

After figuring out the location of the camera, we proceed to calculating the length of the room. We are able to do this due to the presence of a mirror hanging above the girl. In that mirror, we see the reflection of the back wall, which provides a way to estimate the total length of the room. In the next section, we are going to explore how to obtain those measurements, beginning from the angle that the mirror is hanging from the wall.

9 - FINDING THE MIRROR ANGLE

The question of finding the position of the back wall has been explored by Salgado and Steadman in two different approaches. Steadman approached it by reconstructing the actual room itself, placing the camera in the position that he calculated, placing mannequins in place of the two characters in the painting, and lastly adjusting the mirror until the reflection of the mannequin resembles the reflection of the girl in the painting.

This approach, however, led to some inconsistencies between the painting and Steadman's reconstruction. The first one, which Steadman himself acknowledged in his book, is the discrepancy in the angle of the double shadow cast by the mirror on the wall. The double shadow cast by the mirror in Steadman's reconstruction are considerably wider than the one in the original painting.

The second discrepancy is the angle that the mirror hangs, which is quite physically awkward to achieve in real life. For the mirror to hang in that manner, it would have had to be artificially supported in some manner.

The third discrepancy is the rate that the reflection of the tiles in the mirror recedes visually in the reconstruction does not quite match the one in the painting.

In finding the mirror angle, I approached it using the reflection of the tiles in the mirror as the starting point, instead of focusing on the girl. This decision was based on the fact that the tiles would provide a more precise geometric data compared to the girl. As with the approach used to find the camera distance in the first part, we would also be using the vanishing points of the tiles to obtain the mirror angle.

So instead of finding the mirror angle directly, we would use the vanishing points of the tile reflections to find the angle of inclination of the floor reflection. There are insufficient points available on the mirror reflection to convincingly calculate a central vanishing point, but the left and right vanishing points converge quite accurately. So therefore we would only be using two vanishing points to calculate the vanishing line. The left mirror vanishing point is located at (1490.6, 1261.6) and the right mirror point is located at (2258.1, 1189.7).

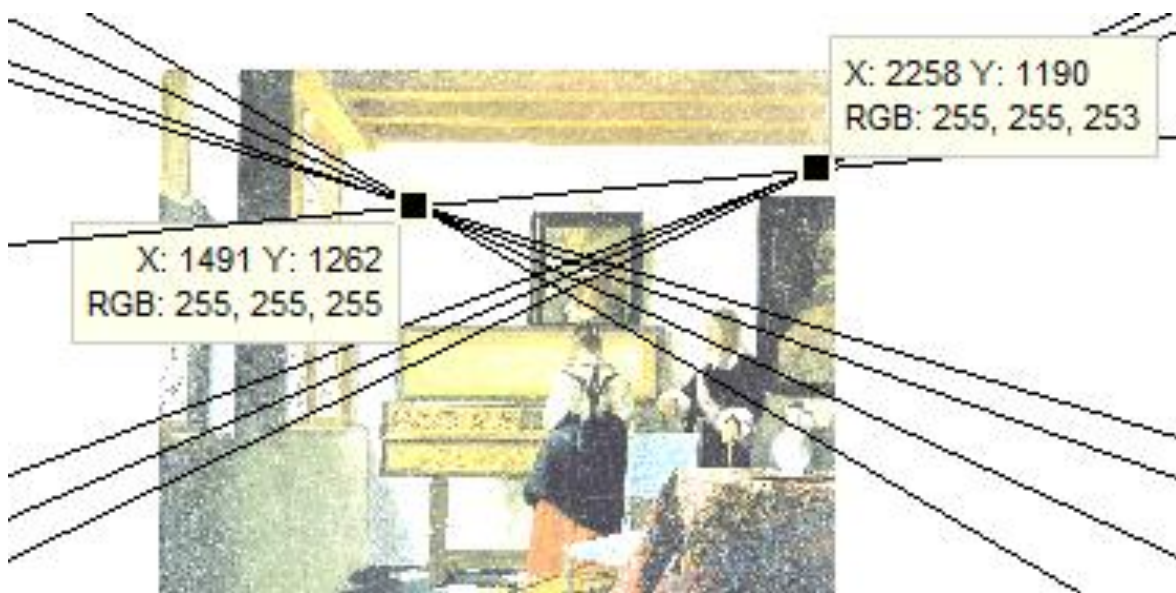


Figure 14 – Mirror vanishing points obtained by MATLAB

Surprisingly, we note that the mirror vanishing line is not horizontal. This means that the mirror is tilted to the right. We can obtain the plane equation of the plane parallel to the floor reflection which passes through the origin, which would just be the plane in R^3 that gets mapped to the vanishing line in RP^2 .

So the plane of the floor reflection is parallel to the plane defined by the following three points, (2001.3, 1661.2, 0), (1490.6, 1261.6, 1499) and (2258.1, 1189.7, 1499), i.e. the camera centre at $z = 0$, and the mirror vanishing points at $z = 1499$ (the camera distance found earlier). We calculate the normal by using cross product.

RESULT 7 – The normal of the plane of the floor reflection is $\begin{pmatrix} 1.07 \\ 11.51 \\ 3.43 \end{pmatrix}$.

Similarly for the plane of the floor, we use the left and right vanishing points of the floor, (300.8, 1615.4, 1499) and (3260.0, 1695.9, 1499), as well as the camera centre, (2001.3, 1661.2, 0), to calculate the normal of the floor.

RESULT 8 – The normal of the plane of the floor is $\begin{pmatrix} 0.12 \\ -4.43 \\ 0 \end{pmatrix}$.

The angle between the two normal = $\cos^{-1} \left(\frac{\begin{pmatrix} 1.07 \\ 11.51 \\ 3.43 \end{pmatrix} \cdot \begin{pmatrix} 0.12 \\ -4.43 \\ 0 \end{pmatrix}}{\begin{vmatrix} 1.07 \\ 11.51 \\ 3.43 \end{vmatrix} \begin{vmatrix} 0.12 \\ -4.43 \\ 0 \end{vmatrix}} \right) = 162.08^\circ$.

RESULT 9 – The angle between the floor and its reflection is 162.08°

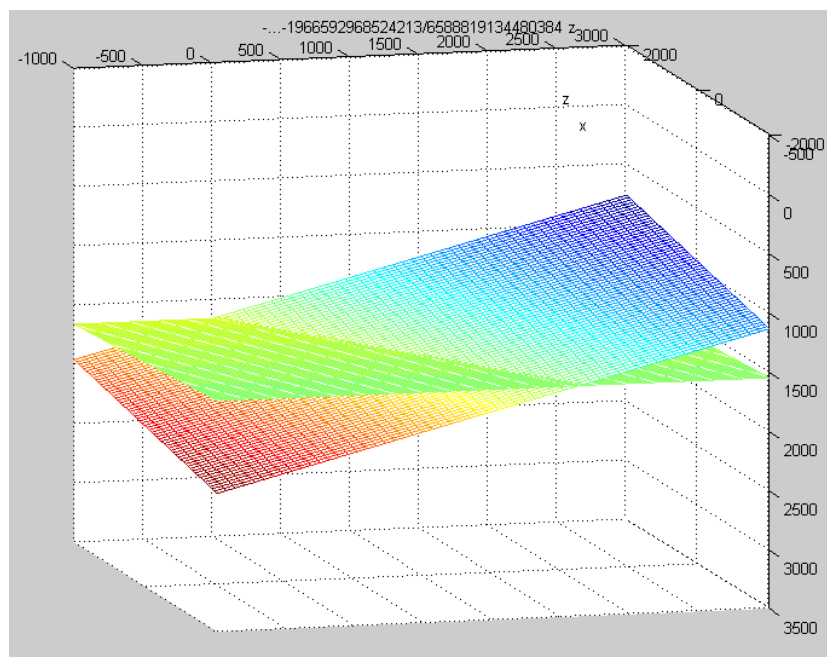


Figure 15 – MATLAB plot of planes parallel to floor plane and reflection plane

The angle between the floor and the mirror is half of the angle between the floor and its reflection.

RESULT 10 – The angle between the floor and the mirror is 81.04°

To find the normal of the mirror, we normalize the normals of the reflection and the floor and find the sum.

$$\text{normal} = \frac{\begin{pmatrix} 1.07 \\ 11.51 \\ 3.43 \end{pmatrix}}{\begin{vmatrix} 1.07 \\ 11.51 \\ 3.43 \end{vmatrix}} + \frac{\begin{pmatrix} 0.12 \\ -4.43 \\ 0 \end{pmatrix}}{\begin{vmatrix} 0.12 \\ -4.43 \\ 0 \end{vmatrix}} = \begin{pmatrix} 0.1158 \\ -0.0451 \\ 0.2845 \end{pmatrix}$$

RESULT 11 – The normal of the plane of the mirror is $\begin{pmatrix} 0.1158 \\ -0.0451 \\ 0.2845 \end{pmatrix}$.

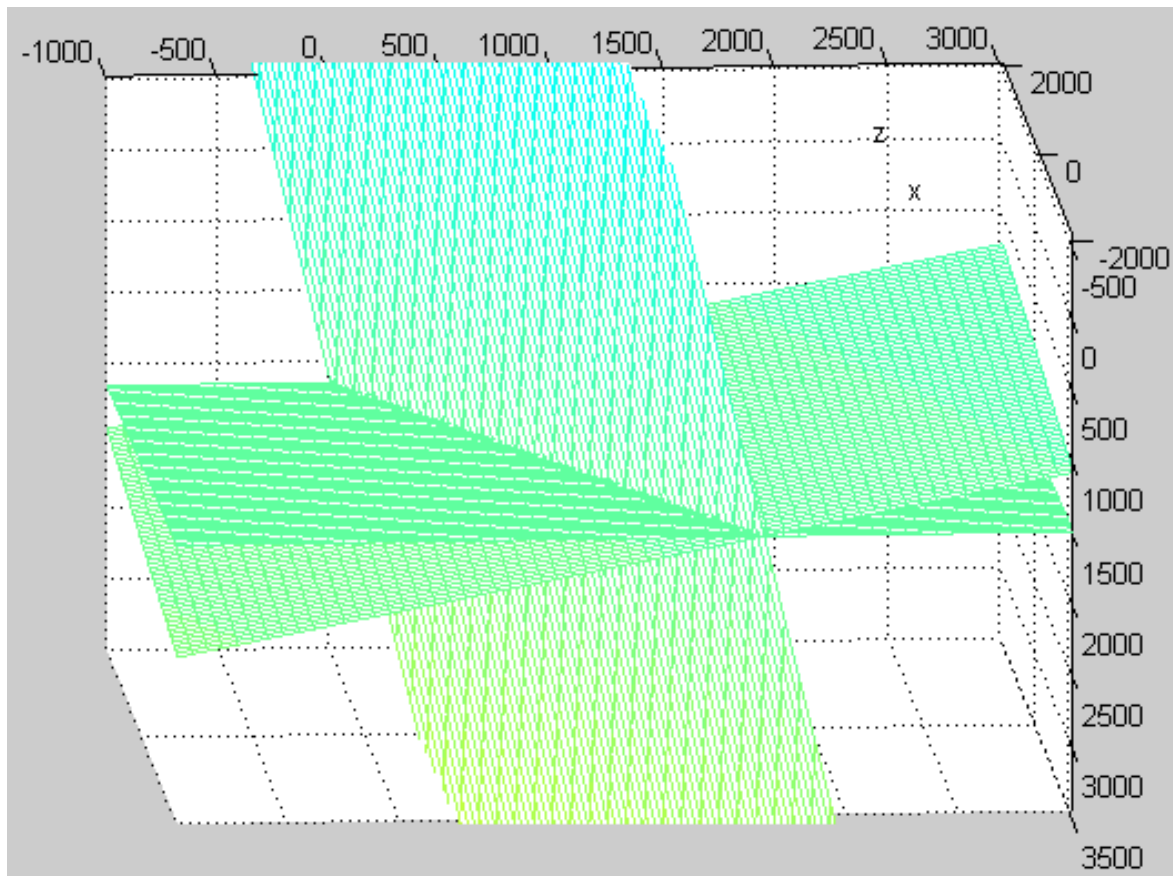


Figure 16 –MATLAB plot of planes parallel to floor plane, mirror plane and reflection plane

10 – PUTTING THEM ALL TOGETHER

Once we have found the normals of the planes needed, we proceed to find the actual plane equations for all three of them. For the floor and mirror planes, we actually have points with known coordinates (correct up to a scaling factor) that we can substitute into the incomplete plane equations. For the reflection plane, we find out the intersection line of the floor and mirror planes and use that line to calculate the equation of the reflection plane.

We know that the equation of the floor plane is $\begin{pmatrix} 0.12 \\ -4.43 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c$.

We let the intersection of the front wall and the left wall be where our image plane is, so at that location, we define $z = 1499$ pixels.

We take a coordinate from the intersection line of the front wall and the floor, (1500, 1943,

1499), to substitute inside. We obtain $c = \begin{pmatrix} 0.12 \\ -4.43 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1500 \\ 1943 \\ 1499 \end{pmatrix} = -8427.5$.

RESULT 12 – The plane equation of the floor plane is $\begin{pmatrix} 0.12 \\ -4.43 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -8427.5$.

We know that the side wall is parallel to the diagonal of the tiles, which is at an angle of 4.96° to the left of the camera direction. Then, assuming that the angle between the side wall and the front wall is 90° ,

We know that the equation of the front wall should be $\begin{pmatrix} \sin -4.96^\circ \\ 0 \\ \cos -4.96^\circ \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c$.

As we defined the z -value of the intersection of the front wall and the side wall to be at $z = 1499$ pixels, we can substitute that reference point to find the equation of the front wall.

We then obtain $c = \begin{pmatrix} \sin -4.96^\circ \\ 0 \\ \cos -4.96^\circ \end{pmatrix} \cdot \begin{pmatrix} 1500 \\ 1943 \\ 1499 \end{pmatrix} = 1363.7$.

So the equation of the front wall is $\begin{pmatrix} \sin -4.96^\circ \\ 0 \\ \cos -4.96^\circ \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1363.7$

For the plane of the mirror, we are going to find out first, where the mirror intersects the front wall, by using finding out the intersection of the edge of the mirror and the shadow formed by the mirror on the front wall. We obtain the coordinate (1918, 1719).



We can calculate the equation of the line formed by this point and the camera centre by using the equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2001.3 \\ 1661.2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2001.3 - 1918 \\ 1661.2 - 1719 \\ 0 - 1499 \end{pmatrix} = \begin{pmatrix} 2001.3 + 83.3t \\ 1661.2 - 57.8t \\ -1499t \end{pmatrix}$$

Equating the two formulas, we can calculate the coordinate of that reference point.

$$\begin{pmatrix} \sin -4.96^\circ \\ 0 \\ \cos -4.96^\circ \end{pmatrix} \cdot \begin{pmatrix} 2001.3 + 83.3t \\ 1661.2 - 57.8t \\ -1499t \end{pmatrix} = 1363.7$$

We then obtain, $t = -1.0241$

So the coordinate of the intersection of the mirror and the front wall is (1916.0, 1720.4, 1535.0).

Equivalently, the equation of the mirror plane is $\begin{pmatrix} 0.1158 \\ -0.0451 \\ 0.2845 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c$. Substituting the

recently obtained reference coordinate (1916.0, 1720.4, 1535.0) in the equation, we

$$\text{obtain } c = \begin{pmatrix} 0.1158 \\ -0.0451 \\ 0.2845 \end{pmatrix} \cdot \begin{pmatrix} 1916.0 \\ 1720.4 \\ 1535.0 \end{pmatrix} = 580.99.$$

RESULT 13 – The plane equation of the mirror plane is $\begin{pmatrix} 0.1158 \\ -0.0451 \\ 0.2845 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 580.99$.

The mirror plane, the floor plane and the reflection plane have a common intersection line, so a point of intersection of the floor and mirror planes is also on the reflection plane.

To find a point on the line, we let $y = 0$ on the floor plane and the mirror plane equations.

We obtain the point (-70229.17, 0, 30578.07) as a solution.

The equation of the reflection plane is $\begin{pmatrix} 1.07 \\ 11.51 \\ 3.43 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c$. Substituting the previously

$$\text{obtained point in the equation, we obtain } c = \begin{pmatrix} 1.07 \\ 11.51 \\ 3.43 \end{pmatrix} \cdot \begin{pmatrix} -70229.17 \\ 0 \\ 30578.07 \end{pmatrix} = 29737.57.$$

RESULT 14 – The plane equation of the reflection plane is $\begin{pmatrix} 1.07 \\ 11.51 \\ 3.43 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 29737.57$.

After obtaining the equations of all three planes, we proceed to locating the back wall. To find the location of the back wall, we will start from a point on the reflection of the back wall in the mirror (chosen arbitrarily).



Figure 17 – Coordinate of back wall reflection point X' on the mirror.

So we know that the point lies on the line defined by the camera centre (2001.3, 1661.2, 0), and (1755, 1317, 1499). Let us call this line our back wall projection line. We obtain the

equation as
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2001.3 \\ 1661.2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2001.3 - 1755 \\ 1661.2 - 1317 \\ 0 - 1499 \end{pmatrix}.$$

RESULT 15 – The line equation is
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2001.3 \\ 1661.2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 246.3 \\ 344.2 \\ -1499 \end{pmatrix}.$$

From this line equation, we find the coordinate of the reflection of this point, which is the intersection of the line just obtained and the reflection plane (result 14). Solving for t , we

get $t = -9.2501$. So the coordinate of the reflection of this point is $(-277.01, -1522.69, 13865.93)$.

We know that the line formed by an object and its reflection is perpendicular to the mirror plane. So the line formed by the image point, X' , $(-277.01, -1522.69, 13865.93)$ and its source X is parallel to the normal of the mirror plane. Then the equation of this line is

RESULT 16 – The line equation of X and X' is
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -277.01 \\ -1522.69 \\ 13865.93 \end{pmatrix} + t \begin{pmatrix} 0.1158 \\ -0.0451 \\ 0.2845 \end{pmatrix}.$$

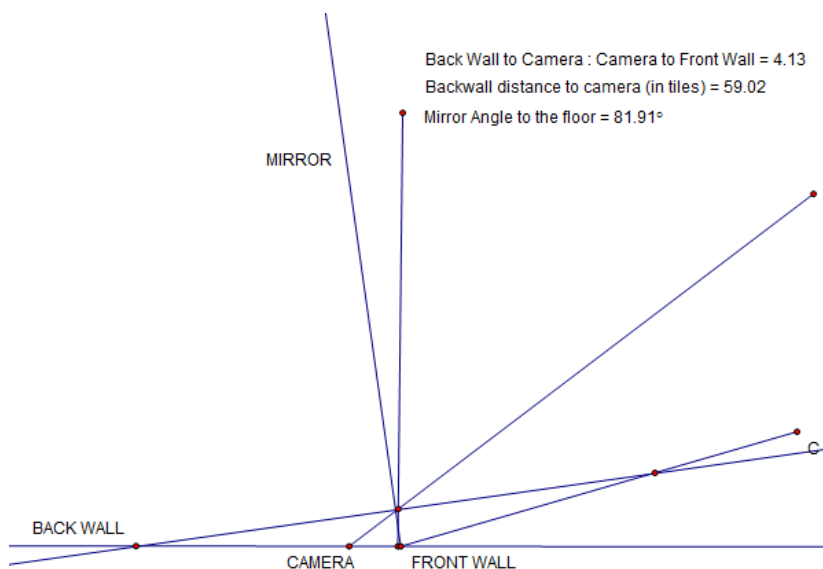
As X is the intersection of the line just obtained and the floor plane, we just equate result 16 and result 12 to obtain $t = -71895.8446$. Substituting this in the equation, we obtain the coordinates of X as $(-8602.5, 1719.8, -6588.4)$.

This value suggests that the back wall is located about 60 tiles behind the camera, which is an obviously unacceptable answer.

What could have caused this large error?

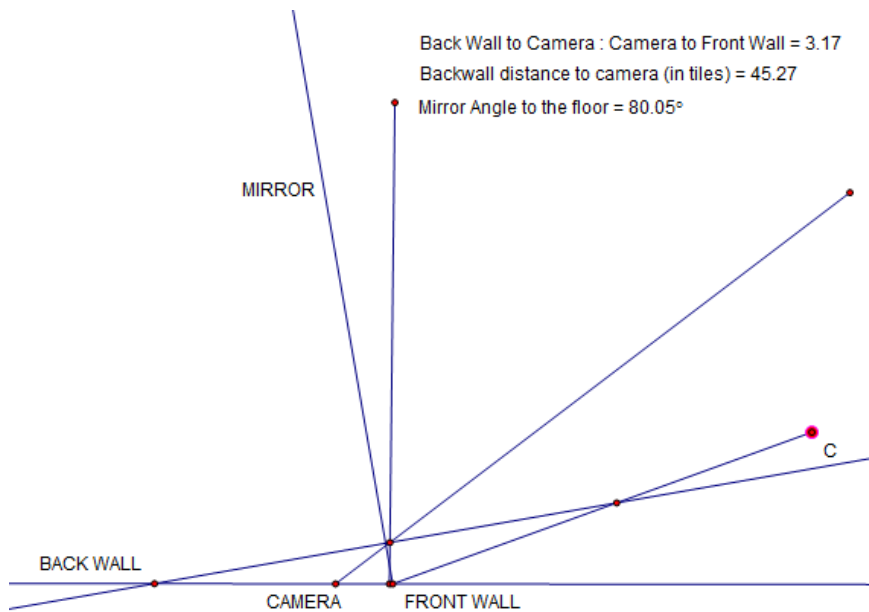
11 – ERROR ANALYSIS

Let us examine the effect of a small change in the mirror angle to the predicted distance of the back wall.



In this diagram, we see that an angle of 81.91° roughly gives a back wall distance of around 60 tiles.

Figure 18a – Effect of changing mirror angle (Part 1)



While in this diagram, we see that an angle of 80.05° gives a back wall distance of around 45 tiles.

A change of 2° in the mirror angle has caused

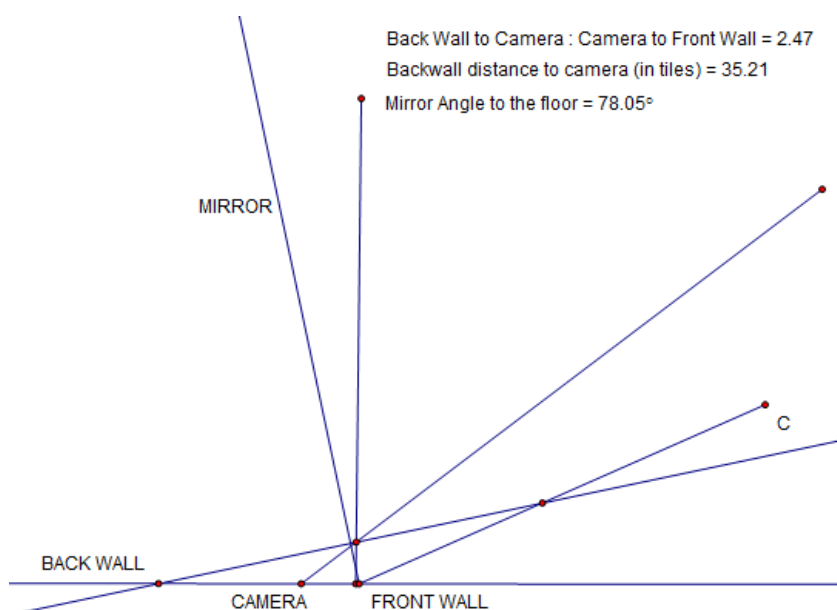
a 15 tile reduction in the predicted distance of the back wall.

Figure 18b – Effect of changing mirror angle (Part 2)

While the change in the

back wall distance is obvious, the projected location of the reflection does not change by much.

Let us examine the effect of a further 2° change in the angle



We see that a total change of 4° change in the mirror angle results in a 25 tile total reduction in the total predicted distance between the back wall and the camera.

Figure 18c – Effect of changing mirror angle (Part 3)

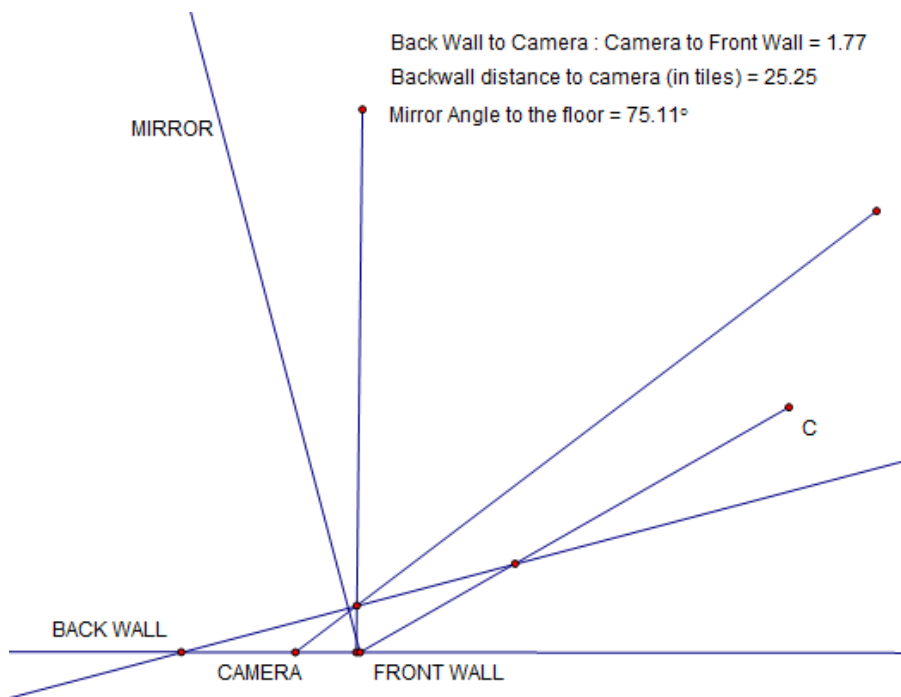


Figure 18d – Effect of changing mirror angle (Part 4)

For a mirror angle of 75°, we see that the predicted distance to the back wall from the camera is around 25 tiles. However, a vertical angle of 15° is quite an awkward angle for a mirror to hang.

What could have contributed to this error in the mirror angle calculated? It is suspected that the measurements obtained for the vanishing lines of the reflection of the tiles are inaccurate, due to the small number of points available for vanishing line calculations. This error could be reduced by using a larger digital image of the painting for better accuracy.

For now, we can only say that the total distance between the back wall and the front wall ranges between 40 to 80 tiles, but this range is too large for any conclusive result to be taken.

However, even in the light of this, the validity of the method outlined in this project is not compromised. The steps taken can be repeated with a better source image to get a more accurate result for the mirror angle and the distance of the back wall.

In the next part, we will compare our obtained results with previously obtained results from the reference materials used in the research for this project.

12 – CONCLUSION and RESULT COMPARISON

	This paper	Steadman	Dubery	Salgado
Horizontal Visual Angle	44.9°	47° (measured from top view plan)	45°	45°
Mirror Angle (to the vertical)	8.1°	17° (measured from side view plan)	NONE	18°
Camera Distance	14.3 tiles	Around 14 tiles	Around 12 tiles (calculated)	14.4 tiles
Distance from back wall to front wall	40 – 80 tiles (inconclusive)	16 tiles	NONE	16.34 tiles

The obvious differences between this paper’s and the previous authors’ results are the mirror angles and the distance from the back wall to the front wall. There is some evidence on the painting that support the hypothesis that the mirror angle is much smaller. For instance, the double shadow on the right of the mirror is much too wide in the Steadman reconstruction model.

The direct implication of a smaller mirror angle is that the distance from the back wall to the front wall is much larger than calculated by Steadman and Salgado. This would actually provide a possible explanation for the argument that Vermeer used a *camera obscura*, as the bigger space available would give Vermeer ample room for a full-sized *camera obscura*.

APPENDIX A

Coordinate values used to calculate vanishing points of floor.

Left1

x 1232 1222 1212 1201 1187 1177 1168 1158 1148 1127 1052 1042 1033 1024 1013 1005

1138

y 2429 2421 2413 2405 2393 2385 2377 2369 2362 2345 2288 2280 2272 2264 2255 2248

2354

Left2

x 1594 1590 1586 1582 1577 1572 1565 1560 1553 1547 1541 1535 1528 1521 1511 1505

1499 1493 1487 1380 1374 1367 1360 1354 1348 1341 1334 1327 1319 1312 1305 1298

1293 1215 1202 1187 1155 1171

y 2484 2481 2479 2476 2473 2470 2467 2463 2458 2455 2450 2446 2442 2437 2431 2426

2421 2417 2413 2348 2344 2341 2337 2332 2329 2325 2320 2315 2311 2306 2302 2298

2294 2248 2240 2230 2209 2219

Left3

x 1651 1635 1521 1513 1504 1495 1481 1473 1461 1453 1444 1435 1346 1355 1338 1328

1319 1309 1297 1288 1219 1206 1192 1179

y 2361 2351 2291 2288 2282 2278 2271 2267 2260 2257 2251 2248 2203 2208 2198 2192

2188 2183 2177 2172 2138 2130 2122 2114

Left4

x 2173 2165 2151 2137 2120 2111 1768 1653 1646 1638 1627 1611 1604 1594 1583 1566

1473 1465 1456 1445 1435 1425 1417 1407 1335 1325 1314 1305 1295 1286

y 2482 2478 2471 2465 2457 2453 2300 2250 2248 2244 2240 2233 2230 2225 2220 2212

2170 2167 2164 2159 2154 2150 2146 2142 2112 2109 2102 2099 2094 2089

Left5

x 1530 1520 1511 1502 1493 1485 1477 1406 1398 1387 1378

y 2070 2066 2063 2059 2056 2053 2050 2028 2024 2020 2016

Left6

x 1678 1670 1583 1575 1562 1552 1543 1532 1522 1514 1434 1423 1410 1399 1390 1382

1319 1308 1297 1287

y 2178 2175 2143 2141 2133 2131 2128 2123 2119 2117 2088 2083 2079 2075 2071 2068

2045 2041 2035 2032

Right1

x 1453 1472 1495 1511 1529 1543 1562 1581 1599 1618 1640 1657 1680 1701 1727 1750

y 2429 2420 2409 2401 2394 2387 2379 2371 2364 2357 2348 2339 2328 2320 2310 2301

Right2

x 1025 1054 1079 1115 1142 1178 1202 1255 1272 1296 1316 1342 1356 1373 1399 1426

1465 1491

1515 1540 1620 1640 1663 1680 1695 1708 1725

y 2478 2468 2458 2445 2437 2424 2414 2394 2387 2379 2372 2362 2357 2352 2344 2334

2320 2312 2303 2292 2266 2259 2251 2246 2241 2233 2227

Right3

x 1568 1576 1583 1590 1598 1606 1615 1629 1638 1648 1657 1664 1342 1347 1353 1362

1370 1378 1385 1394 1403 1409 1417 1424 1431 1336 1326 1317 1310 1302 1295 1119

1109 1098 1089 1079 1067 1055 1043 1029 1018 1011 1003

y 2208 2205 2203 2200 2197 2195 2192 2189 2185 2183 2180 2177 2277 2276 2274 2272

2269 2266 2263 2260 2258 2255 2253 2250 2248 2278 2282 2284 2287 2290 2292 2345

2349 2352 2355 2358 2362 2366 2369 2374 2377 2378 2382

Right4

x 1503 1520 1533 1548 1563 1576 1490 1016 1026 1040 1051 1229 1237 1257 1275 1292

1308 1331 1348 1003

y 2170 2165 2159 2155 2151 2146 2172 2303 2302 2297 2293 2247 2244 2237 2234 2229

2223 2217 2211 2308

Right5

x 1426 1438 1451 1465 1476 1490 1415 1182 1200 1216 1231 1249 1264 1277 1166

y 2135 2132 2129 2128 2124 2120 2139 2197 2192 2189 2185 2181 2176 2174 2203

Right6

x 1356 1367 1386 1401 1419 1432 1125 1142 1162 1179 1198 1219 1108

y 2113 2110 2108 2102 2097 2093 2165 2161 2157 2154 2148 2143 2170

Center1

x 1537 1510 1482 1434 1383

y 2071 2112 2176 2244 2349

Center2

x 1289 1359 1402 1441 1472 1498 1525 1213

y 2289 2205 2138 2088 2047 2011 1984 2409

Center3

x 1220 1287 1340 1381 1419 1448 1467 1124

y 2246 2167 2114 2064 2032 1994 1968 2340

Center4

x 1590 1563 1530 1482

y 2143 2208 2289 2412

Center5

x 1669 1659 1635

y 2172 2253 2347

Center6

x 1153 1225 1277 1326 1374 1395 1436 1059

y 2200 2138 2085 2047 2013 1980 1958 2292

Right Mirror 1

1841 1837 1770 1766 1761 1757 1754 1750 1746 1742

1393 1395 1427 1429 1431 1433 1435 1437 1439 1441

Right Mirror 2

1857 1852 1848 1843 1809 1805 1800 1796 1791 1786 1782

1358 1361 1363 1365 1378 1380 1382 1385 1388 1389 1392

Right Mirror 3

1784 1778 1774 1771 1766 1762 1758 1754

1371 1373 1374 1376 1377 1379 1381 1382

Left Mirror 1

1744 1748 1752 1756 1761 1766 1770 1773

1408 1411 1413 1415 1418 1420 1423 1426

Left Mirror 2

1741 1745 1749 1752 1783 1787 1792 1797

1376 1379 1381 1383 1396 1398 1400 1403

Left Mirror 3

1766 1770 1774 1778 1782 1786 1812 1817 1820 1825 1828 1832 1836 1840 1843

1360 1361 1363 1365 1367 1369 1377 1378 1380 1381 1382 1384 1385 1387 1388

Left Mirror 4

1744 1749 1753 1757 1763 1767 1844 1848 1851

1336 1338 1340 1342 1343 1346 1367 1369 1370

$$y = -0.4819x + 2280.1$$

$$y = -0.4319x + 2160.6$$

$$y = -0.3776x + 2044.4$$

$$y = 0.5811x + 394.6968$$

$$y = 0.4575x + 580.5057$$

$$y = 0.3545x + 734.4667$$

$$y = 0.3046x + 806.1164$$

APPENDIX A2

x = array of x co-ordinates of points used to find best fit line

y = array of y co-ordinates of points used to find best fit line

p = polyfit(x,y,1); // keep result to find vanishing point

APPENDIX B1

```
I = imread('C:/project.jpg'); //picture file is project.jpg

imshow(I);

lineplot2([0.7900 1455.3]); //values obtained from polyfit(x,y,1) using the sets of data
points

lineplot2([0.6249 1486.8]);

lineplot2([0.5154 1508.1]);

lineplot2([0.4414 1521.5]);

lineplot2([0.3440 1542.9]);

lineplot2([0.3717 1554.3]);

lineplot2([-0.4294 3050.9]);

lineplot2([-0.3555 2841.1]);

lineplot2([-0.3089 2691.4]);

lineplot2([-0.2799 2589.0]);

lineplot2([-0.2517 2495.0]);

lineplot2([-0.2301 2424.3]);

lineplot2([-1.7911 4821.9]);

lineplot2([-1.3627 4054.0]);

lineplot2([-1.0805 3558.9]);

lineplot2([-2.4923 6104.3]);

lineplot2([-4.9214 10399]);

lineplot2([-0.8899 3228.8]);

bLeft = [1455.3; 1486.8; 1508.1; 1521.5; 1542.9;1554.3];

ALeft = [1 -0.7900;1 -0.6249;1 -0.5154;1 -0.4414;1 -0.3440;1 -0.3717];
```

```

VLeft = ALeft\bLeft;           //least square solution for left vanishing point

bRight = [3050.9; 2841.1; 2691.4; 2589.0; 2495.0; 2424.3];

ARight = [1 0.4294; 1 0.3555; 1 0.3089; 1 0.2799; 1 0.2517; 1 0.2301];

VRight = ARight\bRight;       //least square solution for right vanishing point

bCenter = [4821.9; 4054.0; 3558.9; 6104.3; 10399; 3228.8];

ACenter = [1 1.7911; 1 1.3627; 1 1.0805; 1 2.4923; 1 4.9214; 1 0.8899];

VCenter = ACenter\bCenter;    //least square solution for central vanishing point

VLeftX = VLeft(2)             //x coordinate of left vanishing point
VLeftY = VLeft(1)            //y coordinate of left vanishing point

VCenterX = VCenter(2)        //x coordinate of central vanishing point
VCenterY = VCenter(1)       //y coordinate of central vanishing point

VRightX = VRight(2)         //x coordinate of right vanishing point
VRightY = VRight(1)        //y coordinate of right vanishing point

```

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