Undergraduate Research Opportunity Programme in Science

Polyhedra

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Contents

1. Introduction

In this project, we will study inclusions, compounds and stellations of regular polyhedra and deltahedra and zonohedra.

1.1. Basic Terms for Describing Polyhedron

We describe polyhedra by using the terms illustrated in figure 1.1 [1] and figure 1.2 below.

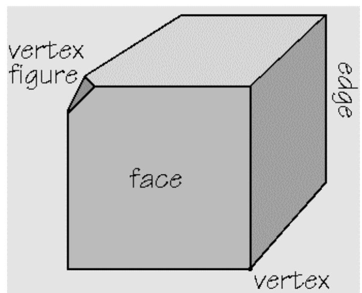


Figure 1.1. Illustrations of basic terms

- A ***face*** is a polygon that bounds a polyhedron.
- An ***edge*** is a line segment where two faces meet.
- A ***vertex*** is a point at which several edges and faces meet.
- A ***vertex figure*** is the polygon which appears if we truncate a polyhedron at a vertex.

Other terms are used to define angles.

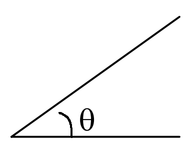


Figure 1.2. Illustrations of basic terms

- ***Dihedral angle*** is the angle between two adjacent faces. The two lines in figure 1.2 represent a plane and the point of intersection of the two lines is the edge where the two planes intersect. θ is the dihedral angle.

- ***Solid angle*** is defined by the amount of surface area of a unit sphere a polyhedron vertex cover. For example, for the cube in figure 1.1, one vertex of the cube covers one eighth of a sphere. The surface area of the portion of the unit sphere it covers is $\pi/2$. Hence, the solid angle of the vertex of the cube is $\pi/2$.

Another common term used is ***valency***. The valency of a vertex is the number of faces or the number of edges each vertex is adjacent to. It is sometimes known as *order*.

2. Polyhedra

2.1. What is a Polyhedron?

A ***polyhedron*** is a surface in the three dimensional space consisting of polygons, with each edge of the polyhedron shared by exactly two polygons [1].

2.2. Convex Polyhedra

In a ***convex*** polyhedron, the line segment joining any two vertices of the polyhedron lies entirely in the interior of the polyhedron. A convex polyhedron has no holes or indentations.

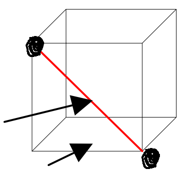


Figure 2.1. Cube

2.2.1. The Platonic Polyhedra

Let P be a polyhedron whose faces are congruent regular polygons. The following statements are equivalent [2]:

- All the vertices of P lie on a sphere.
- All the dihedral angles of P are equal.
- All the vertex figures are regular polygons.
- All the solid angles are congruent.
- All the vertices have the same valency.

The above statements are the characteristics of the Platonic polyhedra. There are only five Platonic polyhedra: the ***tetrahedron***, the ***cube***, the ***octahedron***, the ***dodecahedron*** and the ***icosahedron***.

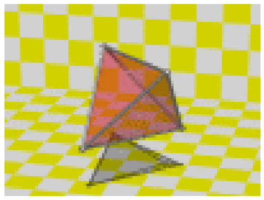
Naming Polyhedra

Polyhedra are named according to the Greek names for their number of faces.

Number	**In Greek**	**Name of polyhedra**
4	Tetra-	Tetrahedra
6	Hexa-	Cube (Hexahedra)
8	Octa-	Octahedra
10	Dodeca-	Dodecahedra
12	Icosa-	Icosahedra

Table 2.1. Naming of the five Platonic polyhedra

Describing Polyhedra



Tetrahedra Cube Octahedra

Dodecahedra Icosahedra

Figure 2.2. Five Platonic polyhedra

	Vertex / V	Edges / E	Faces / F	P = polygon	D = degree
Tetrahedra	4	6	4 squares	3	3
Cube	8	12	6 triangles	4	3
Octahedra	6	12	8 triangles	3	4
Dodecahedra	20	30	12 pentagons	5	3
Icosahedra	12	30	20 triangles	3	5

Table 2.2. Description of the five Platonic polyhedra

The following are formulae that can be used to derive table 2.2:

- $V = \frac{F \bullet P}{D}$ since
  - $F \bullet P$ is the total number of edges generated if all the faces are separated
  - When all the faces are joined together, the degree represents the number of edges that will join to form a vertex. Hence, $V = \frac{F \bullet P}{D}$.
- $E = \frac{F \bullet P}{2}$ since
  - $F \bullet P$ is the total number of edges generated if all the faces are separated
  - When all the faces are joined together, two edges will join together to form a new edge in the new polyhedron. Hence, $E = \frac{F \bullet P}{2}$.

2.3. Duality of Polyhedra

Duality is a property of a polyhedron. For every polyhedron there exists a dual polyhedron. Starting with any regular polyhedron, the dual can be constructed in the following way [1]:

(1) Place a point in the center of each face of the original polyhedron;
(2) Connect each new point with the new points of its neighboring faces;
(3) Erase the original polyhedron.

The dual polyhedron is the polyhedron defined by the edges drawn in step (2).

Looking at one Platonic polyhedron at a time, it is found that:

- A tetrahedron is dual to a tetrahedron;
- A cube is dual to an octahedron and vice versa; and
- A dodecahedron is dual to an icosahedron and vice versa.

The number of faces in one polyhedron is the same as the number of vertices in its dual polyhedron since the each vertices of the dual polyhedron corresponds to a center of the face of the original polyhedron. This can be verified by checking the above dual pairs with table 2.2.

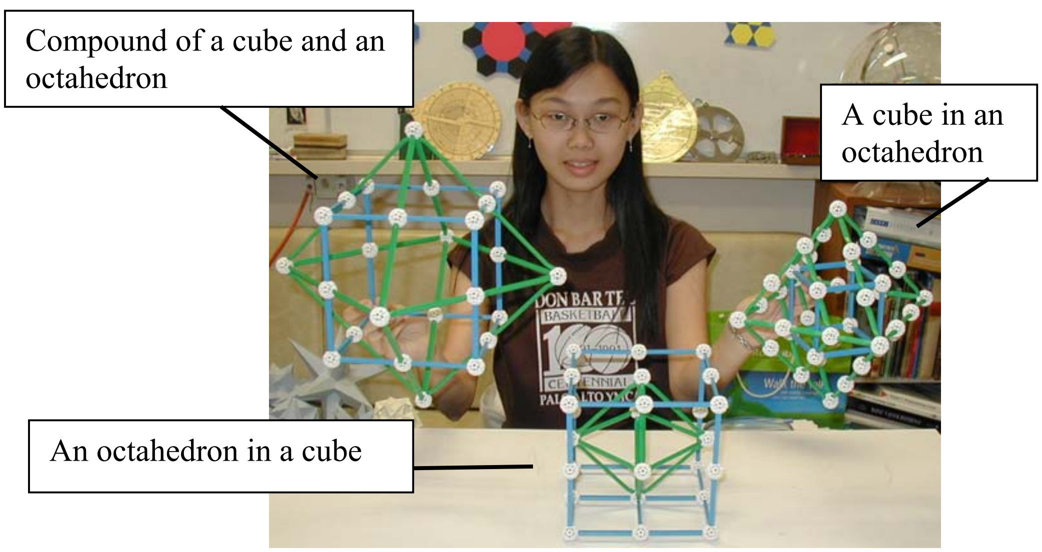


Figure 2.3. The cube is dual to an octahedron

3. Inclusions of Polyhedra

A regular polyhedron is ***included*** in another if their centers are the same and the vertices of the included polyhedron lie on the outer polyhedron. [2]

There are different ways of including a polyhedron in another. Table 3.1 shows some possible ways of including one polyhedron in another, while figure 3.1 shows another way of inscribing a cube in an octahedron (Piero della Francesca) where the vertices of the cube lies on the edge of the octahedron instead of the face as shown in figure 3.1. Of course there are other ways of including other polyhedron in another. However, in this project, we will not discuss them.

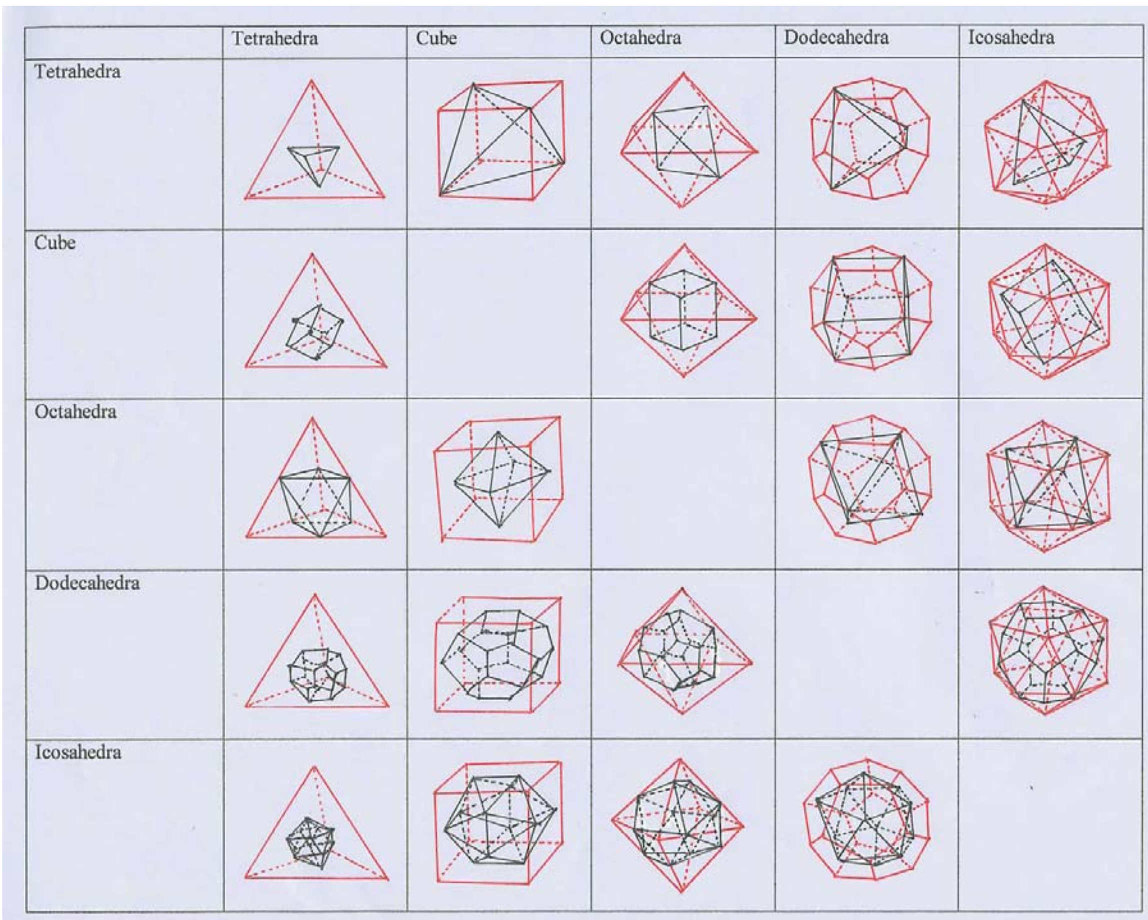


Figure 3.1. Inclusions

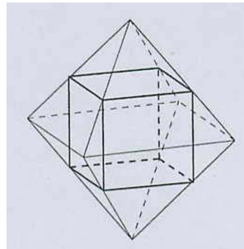


Figure 3.1. Cube in an octahedron (Piero della Francesca)

Types of inclusions	Tetrahedra (T)	Cube (C)	Octahedra (O)	Dodecahedra (D)	Icosahedra (I)
Vertex on vertex	T ⊂ D				
Vertex on vertex and edge on face	T ⊂ C	C ⊂ D			
Vertex on edge	T ⊂ O	C ⊂ O (Piero)	O ⊂ T O ⊂ D O ⊂ I		I ⊂ O
Vertex on face	T ⊂ I T ⊂ T	C ⊂ T C ⊂ O C ⊂ I	O ⊂ C	D ⊂ T D ⊂ I D ⊂ O	I ⊂ D
Edge on face				D ⊂ C	I ⊂ C
Face on face					I ⊂ T

Note: ⊂ stands for inscribed in

Table 3.2. Summary of the type of inclusions

It is noted from table 3.2 that any polyhedron can be included in another polyhedron in some ways.

Inclusions by duality

One polyhedron can be included in another polyhedron by observing the duality of each polyhedron. Inclusions by the duality property can be done by following steps (1) and (2) of the constructing the dual polyhedron in section 2.3.

Because of the duality property, the following inclusions are possible:

- A tetrahedron in a tetrahedron since the tetrahedron is the dual of itself;
- An cube in an octahedron;
- A dodecahedron in an icosahedron.

However, this property only generates some inclusions. Other methods of inclusions include the use of midpoints, golden ratio and double inclusions.

Inclusions by vertex on vertex:

This is the most preferred type of inclusions since the resulting structure will be very compact. The included polyhedron makes use of the existing vertices of the outer polyhedron. Hence, the outer polyhedron must have more or equal number of vertices than the included polyhedron. Some examples of such inclusions are:

- A tetrahedron included in a cube where four vertices of the cube are the vertices of the tetrahedron.
- A cube in a dodecahedron where eight dodecahedron vertices are the vertices of the cube.

Inclusions using edges:

A polyhedron can be included in another polyhedron by placing the vertices of the included polyhedron on the edges of the outer polyhedron and the vertices can cut the edges at either the midpoint of the edge of by golden ratio. 1: $\frac{1+\sqrt{5}}{2} \approx 1.61803...$ is the golden ratio. Examples of inclusions where the vertices of the included polyhedron lie on the midpoint of the outer polyhedron are: an octahedron in a tetrahedron and an octahedron in a dodecahedron. A cube in an octahedron (Piero della Francesca) and the icosahedron in an octahedron are examples of inclusions by the golden ratio method.

Inclusions by double inclusions:

This is a very nice method of including. For example, a tetrahedron can be included in an octahedron by inscribing a cube into the octahedron and then a tetrahedron into the cube. If the cube is ignored, the tetrahedron is gracefully inscribed into the octahedron. However, not all inclusions can be done this way. Take a tetrahedron in an icosahedron for example; an icosahedron can be included nicely in a dodecahedron by duality property. However, the vertices of the tetrahedron in the icosahedron may not lie on the dodecahedron.

Inclusions by midpoints of faces:

The midpoints of the faces of the outer polyhedron are the vertices of the included polyhedron. This method seems similar to inclusion by duality. In fact, the set of inclusions generated by duality property is a subset of the sets of inclusions generated by using midpoint of faces. This is because the duality method restrict that all the midpoints of faces of the outer polyhedron are the vertices of the included polyhedron, while inclusions by using the midpoint of faces method does not. The tetrahedron in an icosahedron is an example of such inclusion. The centers of the four-icosahedron faces are the vertices of the tetrahedron. Another example is a cube in an icosahedron.

3.1. Nested Platonic Solids

A nested Platonic solid is a framework of all the polyhedra, with a series of inclusion of one polyhedron in another. After looking at all the possible inclusions of the polyhedra, it is very clear that we can form nested Platonic solids starting from any polyhedron.

Since vertex on vertex inclusions and vertex on edge inclusions are the two most preferred choice of inclusion. We shall first restrict the construction of nested Platonic solid to vertex on vertex and vertex on edge inclusions only. It is found that only one nested Platonic solid is possible, i.e. $I \subset O \subset T \subset C \subset D$ which is shown in figure 3.2. In fact, this nested Platonic solid is the most compact of all the other nested Platonic solids. It is also the easiest to build using zome models.

Other inclusions involving faces of polyhedron results in very big models and there is a difficulty in locating the position on the outer polyhedron where the vertex, edge or face of the included polyhedron should lie. Hence, these nested Platonic solids are not preferred.

Figure 3.2. Nested Platonic Solid [12]

4. Compounds of Polyhedra

A ***compound*** is a collection of two or more interpenetrating polyhedra, whose centers coincide. [1]

4.1. Forming Compound By Observing Duality

A compound can be generated using the duality property of a polyhedron. Consider two similar tetrahedra. Letting their centers coincide; we get a compound of two-tetrahedra. This compound is called the stella octangula. Similarly, we can obtain the cube and octahedron compound and the dodecahedron and icosahedron compound.

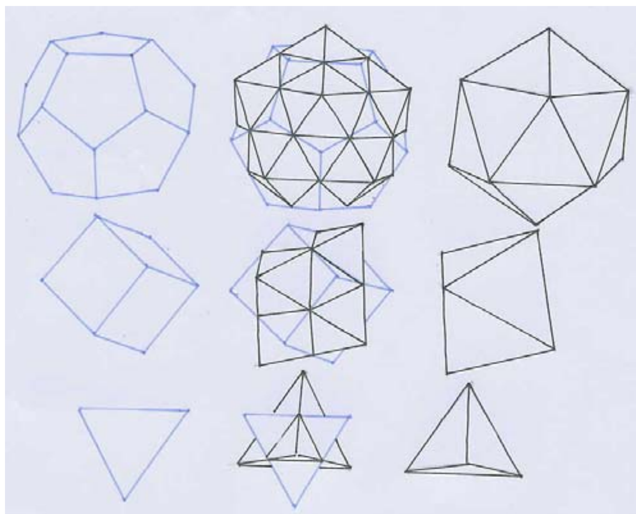


Figure 4.1. The dual pairs and their interpenetrating solids

4.2. Forming Compound By Observing Inclusions

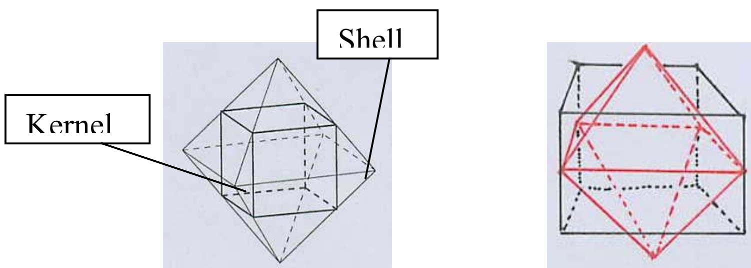
However, it is noted that there are more than one way where we can generate compounds. Take the stella octangula for example. It is observed that we can include a tetrahedron in a cube in two different ways and when we include both tetrahedra in the cube together, we can also get the stella octangula. Hence, it can be concluded that we can also generate compounds by considering the number of times a polyhedron can be included in another.

We can also generate other tetrahedra compound by observing the possible inclusions of tetrahedra. In section 3, we see that a tetrahedron can be included in an octahedron, a dodecahedron and an icosahedron. Note that, a tetrahedron can be included in an octahedron in three ways and in a dodecahedron in five ways. Hence, we obtain a three-tetrahedra compound by including a tetrahedron in an octahedron three times and a five-tetrahedra compound by including a tetrahedron in a dodecahedron five times. Note that the midpoints of the faces of an icosahedron are the vertices of a dodecahedron. And the tetrahedron is included in a dodecahedron by vertex on vertex inclusion and in an icosahedron by vertex on face inclusions. No matter where the tetrahedron lies in the dodecahedron, it also lies on the icosahedron. Hence, the number of ways of including a tetrahedron in an icosahedron is the same as the number of ways of including a tetrahedron in a dodecahedron. We will also get the five-tetrahedra compound by considering the tetrahedron in an icosahedron.

Now, let's look at the cube, a cube can be included in an octahedron in three ways. Hence, we obtain the compound of three-cubes. Note that a cube is included in a dodecahedron with each edge of the cube corresponding to one diagonal of the pentagon of the dodecahedron. There are 5 diagonals in a pentagon; hence there are five ways of including a cube in a dodecahedron. A five-cubes compound is obtained. The number of ways of including a cube in an icosahedron is the same as that of the dodecahedron for the same reason mentioned above for the tetrahedron.

4.3. Forming Compound By Observing Symmetry

Other than considering the number of ways a polyhedron can be included in another by observation, we can also find the number of components of one polyhedron in a compound by considering the rotational symmetries of the included polyhedron (kernel), the circumscribed polyhedron (shell) and the compound of the kernel and the shell (amalgam). Let's consider a cube which can be included in an octahedron (Piero della Francesca).



A cube in an octahedron (Piero) Amalgam

Figure 4.2. The kernel, shell and amalgam

The amalgam usually has a lower degree of symmetry than either the kernel or the shell alone. In such cases as in the cube in an octahedron, the symmetry has been lost. However, the lost symmetry is reinstated in the resulting compounds of three cubes and of three octahedra. This idea of reinstating the destroyed symmetry can be used to generate more compound polyhedra. Reinstating the destroyed symmetry can be reflected in the following formulae:

$$\text{Number of components in the compound of shell polyhedra} = \frac{\text{number of symmetries of the kernel}}{\text{number of symmetries of the amalgam}}$$

$$\text{Number of components in the compound of kernel polyhedra} = \frac{\text{number of symmetries of the shell}}{\text{number of symmetries of the amalgam}}$$

The formulae can be verified using the compound of a cube in an octahedron with 8 rotational symmetries.

Number of components of the compound of octahedron

$$= \frac{\text{number of symmetries of the cube}}{\text{number of symmetries of the amalgam}} = 24 / 8 = 3 \text{ octahedra}$$

Number of components of the compound of cube

$$= \frac{\text{number of symmetries of the octahedron}}{\text{number of symmetries of the amalgam}} = 24 / 8 = 3 \text{ cubes}$$

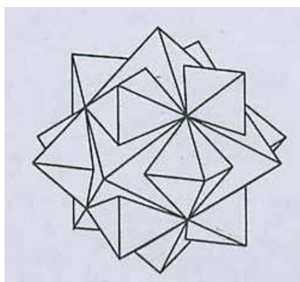


Figure 4.3. Three octahedra.

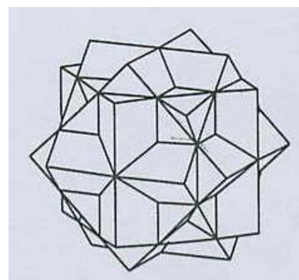


Figure 4.4. Three cubes

Similarly, a five cube compound and two dodecahedra compound can also be formed this way using the compound of a cube in a dodecahedron with 12 rotational symmetries.

Number of components of the compound of dodecahedra

$$= \frac{\text{number of symmetries of the cube}}{\text{number of symmetries of the amalgam}} = 24 / 12 = 2 \text{ dodecahedra}$$

Number of components of the compound of cube

$$= \frac{\text{number of symmetries of the dodecahedra}}{\text{number of symmetries of the amalgam}} = 60 / 12 = 5 \text{ cubes}$$

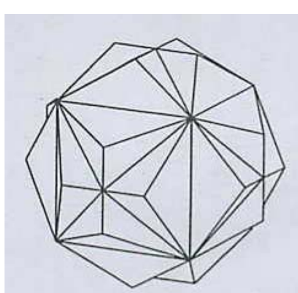


Figure 4.5. Two dodecahedra.

Figure 4.6. Five cubes

Similarly, a four cube compound and four octahedra compound can also be formed this way using the compound of an octahedron in a cube.

Number of components of the compound of cube $= \dfrac{\text{number of symmetries of the octahedron}}{\text{number of symmetries of the amalgam}}$
= 24 / 6
= 4 cubes

Number of components of the compound of octahedra $= \dfrac{\text{number of symmetries of the cube}}{\text{number of symmetries of the amalgam}}$
= 24 / 6
= 4 octahedra

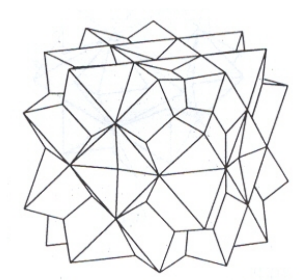


Figure 4.7. Four octahedra.

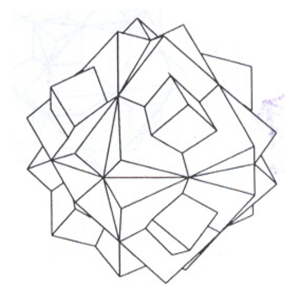


Figure 4.8. Four cubes

4.4. Transitivity [2]

Transitivity can be used to describe polyhedra. A polyhedron is:

- Face-transitive if for any pair of faces, there is a symmetry of the polyhedron which carries the first face onto the other;
- Vertex-transitive if any vertex can be carried onto any other by a symmetry operation;
- Edge-transitive if any edge can be carried to any other by a symmetry operation; and
- Flag-transitive if any one flag triples can be carried onto any other flag by a symmetry operation. Note that a flag triple is a set of face-edge-vertex where the edge is a side of a face and the vertex is an end of the chosen edge.

A polyhedron that is face-transitive, edge transitive and vertex transitive is said to be totally transitive.

Components	Quantity	Vertex – trans	Edge – trans	Face – trans
Tetrahedra	1	✓	✓	✓
	2	✓	✓	✓
	4			
	5	✓	✓	✓
	10	✓	✓	✓
Cubes	1	✓	✓	✓
	3	✓		
	4			✓
	5	✓	✓	✓
Octahedra	1	✓	✓	✓
	3			✓
	4	✓		
	5	✓	✓	✓
Dodecahedra	1	✓	✓	✓
	2			✓
	5			✓
Icosahedra	1	✓	✓	✓

Table 4.1. Transitivity Property of compounds

Note that all the five Platonic polyhedra are totally transitive. The other totally transitive compounds include the two-tetrahedra, five-tetrahedra, ten-tetrahedra, five-cubes and the five-octahedra.

However, there are some interesting observations about the five-tetrahedra compound. The five-tetrahedra compound is built from the dodecahedron where every vertex of the compound is a vertex of a dodecahedron. There are actually two different five-tetrahedra compounds. When we look at the two five-tetrahedra compounds and the ten-tetrahedra compound, it is obvious that there is some relationship between them. Figure 4.7 shows the top view of the two five-tetrahedra compounds. The edges of the five-tetrahedra (right) points outwards towards the right of all vertices while the edges of the five-tetrahedra (left) points to the left. They are actually mirror images of each other. Hence, they are known as the enantiomorphism of each other. The two five-tetrahedra compounds are generated according to the way they intersect each other. Figure 4.8 shows the side view of the two five-tetrahedra compounds. And when these two five-tetrahedra compounds are placed together, vertices to vertices, they form the ten-tetrahedra compound (figure 4.9).

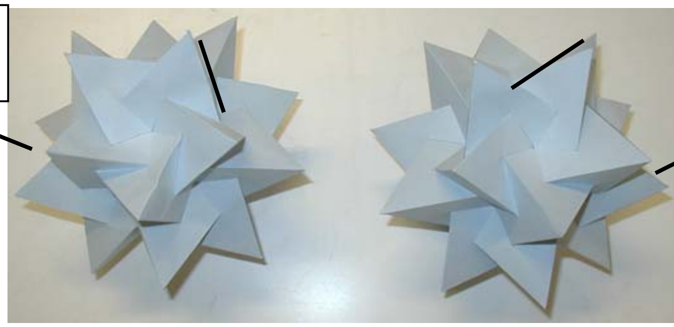


Figure 4.7. Top view of the two five-tetrahedra compounds

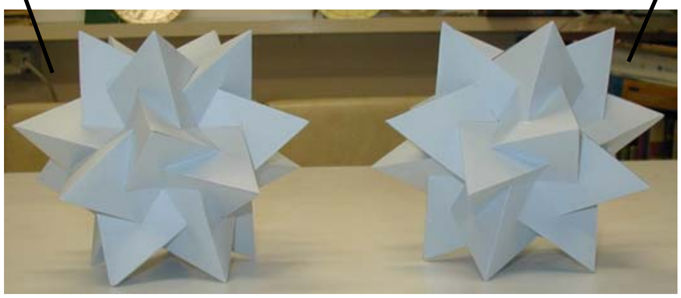


Figure 4.8. Side view of the two five-tetrahedra compounds

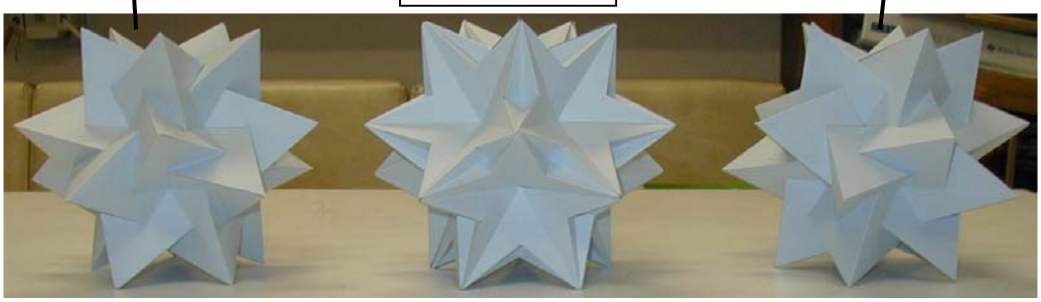


Figure 4.9. Tetrahedra compounds

5. Stellations of Polyhedra

Stellation of polyhedra is a way of creating new polyhedra from existing one through processes similar to creating pentagram from regular pentagon [1]. There are two different types of stellations, namely the edge-stellation and the face-stellation, of polyhedra which will generate different stellated polyhedra.

5.1. Edge-stellations

Non-adjacent edges of the existing polyhedron are extended infinitely until they intersect to form finite regions. The resulting polyhedron of edge-stellation is called an echinus. However, extending edges does not always produce new polyhedron. For example, extending the edges of a tetrahedron, a cube or an octahedron does not produce any new polyhedron since the lines do not intersect apart from at the original vertices. But the dodecahedron and the icosahedron do produce something new. The edge-stellation of dodecahedron and the icosahedron are illustrated in figure 5.1.

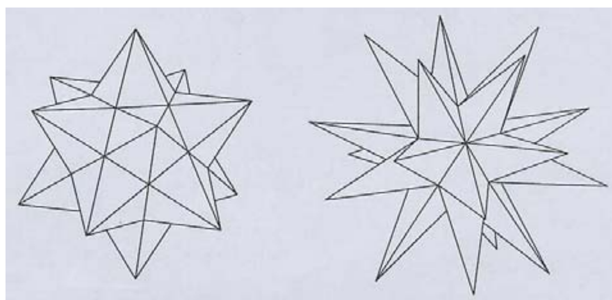


Figure 5.1. Edge-stellated dodecahedron and icosahedron

5.2. Face-stellations

Extending the faces of a given polyhedron infinitely until they intersect to give finite regions is known as face-stellation. A face-stellated polyhedron is called an ostera. Like edge-stellation, face-stellation does not always produce new polyhedron. It is noted that a tetrahedron and a cube does not produce any stellated polyhedron mainly because there are too few edges and faces to be stellated. On the other hand, a series of stellation of the dodecahedron and icosahedron generates many different polyhedra.

Figure 5.2 shows the great dodecahedron and the great icosahedron through a series of stellations. Some further stellations results in the models shown in figure 5.3.

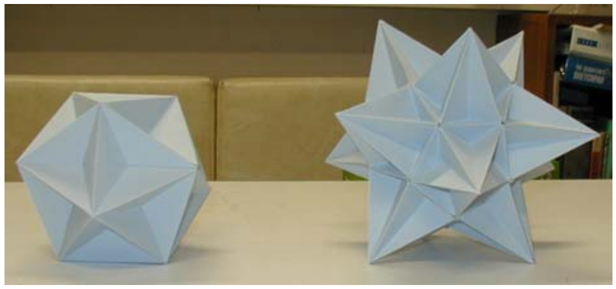


Figure 5.2. The great dodecahedron (left model) and the great icosahedron (right model)

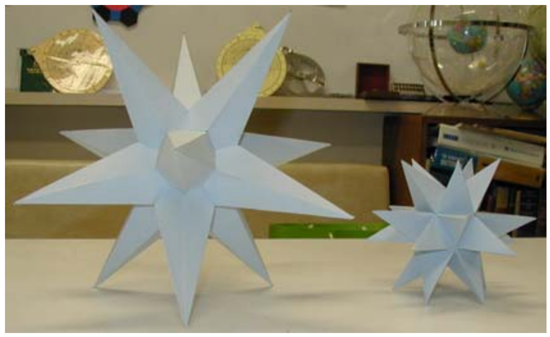


Figure 5.3. Stellations of the icosahedra (left model) and the dodecahedron (right model)

There are many stellations of dodecahedron and icosahedron. However, in this project, we will only study on the stellation of dodecahedron.

5.3. Stellations of the Dodecahedron

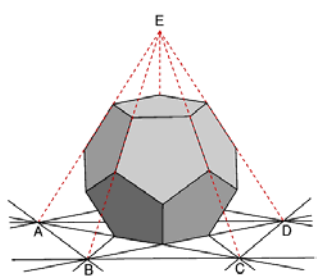


Figure 5.4. Stellated diagram

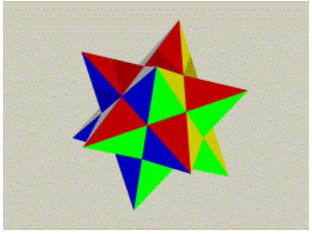
<u>Procedure of stellation of dodecahedron</u>

Stand the dodecahedron on one face and imagine projecting the other faces down on to the plane of that face. Each will meet it in a line. The lines will join at the points such as A, B, C, D.

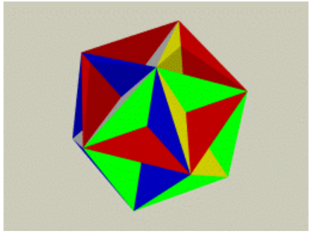
If you project the faces from the plane they meet at E, forming a pentagonal pyramid standing on the face. In this is a way you can form a new polyhedron from the original one. Completing the whole process gives rise to a small stellated dodecahedron.

This process can be seen as building a layer of 12 pyramids on the dodecahedron faces. This is an illustration of a face-stellation.

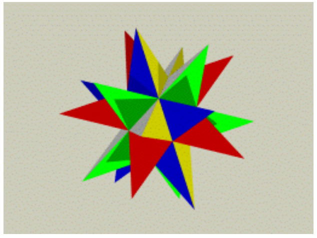
A second layer of 30 wedges that sit between the pyramids form the great dodecahedron, this is the second stellation of the dodecahedron. Adding a final layer of 20 spikes, each of which is an asymmetric triangular dipyramid, which fits into the hollows between the wedges completes the third stellation of the dodecahedron. Figure 5.5 shows the first, second and third stellated dodecahedron.



Small stellated dodecahedron



Great dodecahedron



Great stellated dodecahedron

Figure 5.5. Stellations of dodecahedron

Figure 5.6. Zome models of the first, second and third stellation of the dodecahedron

6. Deltahedra

To talk about deltahedra, we first need to understand pyramids.

6.1. Pyramids

A ***pyramid*** is a polyhedron with a regular n-gon at the base, with equilateral triangles attached to each edge of the n-gon and the extra vertex of the triangles is brought together at a point. This extra vertex formed other then the base vertices is called the apex.

Since the main difference of pyramids is their base, they are named according to the shape of their base:

- Triangular base - tetrahedra (one of the five regular polyhedron)
- Square base - square pyramid
- Pentagonal base - pentagonal pyramid

However, there is no pyramid with hexagonal base since 6 equilateral triangles attached to the sides of a hexagon when brought together at a vertex will lie in a plane. In fact, any n-gon based pyramid with $n \geq 6$ does not exist. This can be further explained by Euler's formula which states that every convex regular polyhedron has a vertex of valency at most five. Suppose there exist a pyramid of n-gon base where $n \geq 6$, the apex of that pyramid will have valency more than 6 which violates the Euler's formula. Hence, there only exist pyramids of n-gon base, where n less than 6.

When two congruent pyramids are placed together base to base, the resulting configuration is called a dipyramid. It can also be called a ***deltahedron***. A deltahedron any convex polyhedron with equilateral triangular faces. Deltahedra is named after the Greek capital letter delta, which is a triangle. We denote Deltahedra of n faces by Dn.

6.2. Convex Deltahedra

The above method, using pyramid, only generate some deltahedra and not all of them. Hence, other ways have to be derived to generate all the possible Deltahedra.

6.2.1. Classification

A mathematical method can be used to classify deltahedra.

Euclid mentioned this in 1915. However, it was not very complete and part of it was wrong.

Euler’s formula holds for all convex polyhedra.
(1) $F + V - E = 2$, where F is the number of faces, V is the number of vertices and E the number of edges.

Now, if we multiply the number of faces by 3, we will count each edge twice because two faces share each edge. Hence,
(2) $3F = 2E$.
From equation (2), since E is an integer, F must be even. Hence, any deltahedron has an even number of faces.

If we substitute $F = \frac{2E}{3}$ in Euler's formula and simplify, we obtain
(3) $3V - E = 6$

Now, a complex deltahedron can have only three essentially different types of vertices. At each vertex, 3, 4, or 5 faces must meet. Let V_3 equal the number of vertices at which 3 faces (also 3 edges) meet. Define V_4 and V_5 similarly. Then
(4) $V_3 + V_4 + V_5 = V$

Also,
(5) $3V_3 + 4V_4 + 5V_5 = 2E$ because each edge joins 2 vertices.

Hence,
(6) $E = \frac{3V_3 + 4V_4 + 5V_5}{2}$

Substituting from (4) and (6) for V and E in equation (3) and simplifying, we arrive at
(7) $3V_3+2V_4+V_5 = 12$
and every deltahedron must satisfy this equation.

Since the vertices of each polyhedron can only have valency 3, 4 or 5, for easy notation, we shall denote each polyhedron by:
(V_3, V_4, V_5)

Below are the only solutions:
(0,0,12), (0,1,10), (0,2,8), (0,3,6), (0,4,4), (0,5,2), (0,6,0), (1,0,9), (1,1,7), (1,2,5), (1,3,3), (1,4,1), (2,0,6), (2,1,4), (2,2,2), (2,3,0), (3,0,3), (3,1,1), (4,0,0)

However, only some of the above solutions correspond to a convex deltahedron. And there are exactly eight convex deltahedron (Freudenthal and van der Waerden 1947). Any deltahedron starting having $V_3 \neq 3$ must be a tetrahedron or a triangular dipyramid. Hence, any of the above solution with $V_3 \neq 3$ other than the tetrahedron and the triangular dipyramid cannot be the answer. That left us with only 9 deltahedra. The deltahedron that is supposed to be the D_{18} (0,1,10) is also eliminated using the following proof.

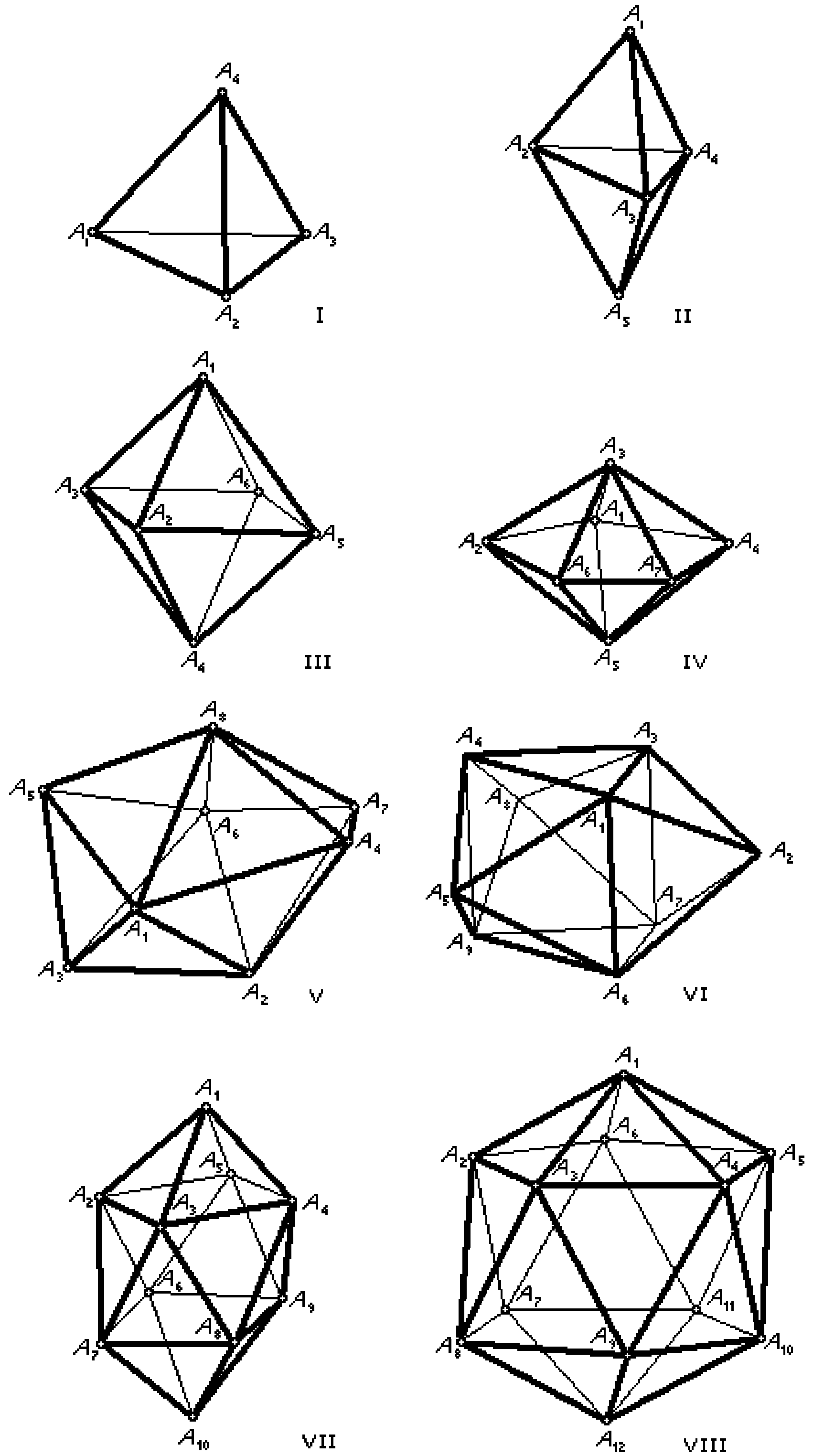


Figure 6.1. Deltahedra [4]

The following list shows all the possible convex deltahedra and figure 6.1 shows all the deltahedra corresponding to the list:
(I) D_4 - tetrahedra (4, 0, 0)
(II) D_6 - triangular dipyramid (2, 3, 0)
(III) D_8 - octahedra (0, 6, 0)
(IV) D_{10} - pentagonal dipyramid (0, 5, 2)
(V) D_{12} - snub disphenoid (0, 4, 4)
(VI) D_{14} - tri-augmented triangular prism (0, 3, 6)
(VII) D_{16} - gyro elongated square dipyramid (0, 2, 8)
(VIII) D_{20} - icosahedra (0, 0, 12)

Now, there is something interesting about the naming of the deltahedra.

Let's look at the dipyramid first. The triangular dipyramid and the pentagonal dipyramid are formed from the construction of deltahedra using pyramids. They are called dipyramid since two pyramids are place together base to base. The octahedron can also be thought of as a square dipyramid.

Disphenoid is a belt of four isosceles triangles between two opposite edges and the tetrahedron is a special kind of the disphenoid. D_{12} is given the name snub disphenoid because it can be seen as tearing the tetrahedra apart into blue and white parts and inserting a belt of red triangles in between as shown in figure 6.2.

Figure 6.2. The snub disphenoid

D_{14} is called the tri-augmented triangular prism because it can be viewed as a having three square pyramids attached to the sides of a triangular prism as illustrated in figure 6.3.

Figure 6.3. Tri-augmented triangular prism

D_{16} is called the gyro elongated square dipyramid. As seen in figure 6.4, there are two square pyramids (the red and the green) facing each other with a belt of yellow triangles in between. Sometimes, the icosahedron can also be called the gyro elongated pentagonal dipyramid. However, there are no gyro elongated triangular dipyramid. This is because, if we make the model, it is observed that there are flat surfaces bounding the polyhedron. Hence, it cannot be considered as a convex polyhedron.

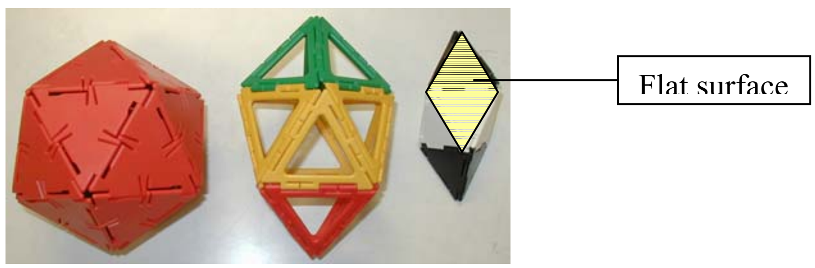


Figure 6.4. The gyro elongated pentagonal dipyramid, the gyro elongated square dipyramid and the gyro elongated triangular dipyramid

6.2.2. Geometric Realization

After looking through the list of deltahedra, we can realize that there is a pattern in the construction of deltahedra. If we observe the list carefully, it can be concluded that each deltahedra can be derived from the others by some transformations:

- By decreasing the number of vertices of valency 3 by two, there is an increase in the number of vertices of valency 4 by 3 and
- By decreasing the number of vertices if valency 4 by one, there will be an increase in the number of vertices of valency 5 by two.

Why not 18

In the list of deltahedra, there is a whole range of deltahedra from four vertices in a tetrahedron to 20 vertices in an icosahedron. However, it is noted that there is no D_{18}. This observation, which is not clearly explained by classification, can be explained clearly when we go through the process of constructing the deltahedra by geometric realization.

The process

For this method, we can construct deltahedra systematically by starting with the smallest deltahedra, the tetrahedra.

The whole process is only possible when there occur two vertices of minimum degree along a "line" with one vertex in between. The "line" must consist of two edges. And the main purpose is to split the polyhedra along these two edges and add two more triangular faces, one to each of these two edge.

Let's start from the tetrahedra (4, 0, 0).
Since all the vertices of the tetrahedra is of valency 3 and there are only six edges. Removing any two edge completes the process and triangular dipyramid (2, 3, 0) is obtained.

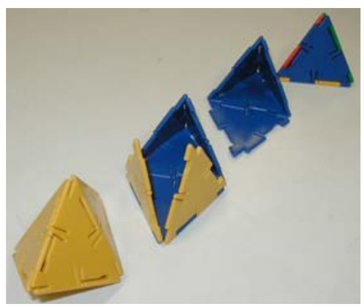


Figure 6.5. From a tetrahedron to a triangular dipyramid

Octahedron (0, 6, 0) is obtained by removing any two edges joining the two vertices of valency 3 and inserting two more faces to the triangular dipyramid.

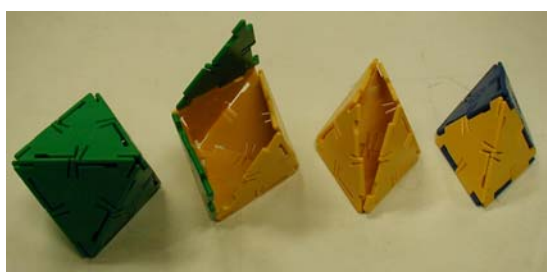


Figure 6.6. From a triangular dipyramid to an octahedron

The process can be continued to generate pentagonal dipyramid, snub disphenoid, tri-augmented triangular prism and gyro-elongated square dipyramid.

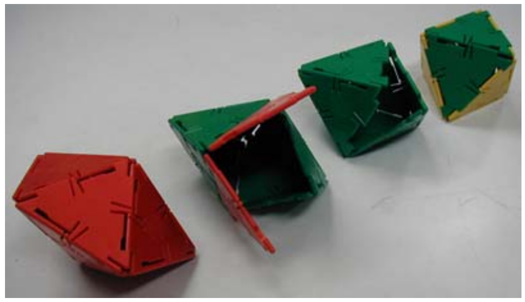


Figure 6.7. From an octahedron to a pentagonal dipyramid

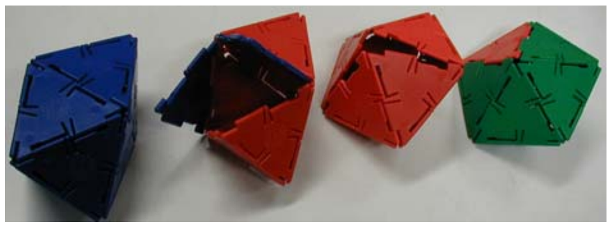


Figure 6.8. From a pentagonal dipyramid to a snub disphenoid

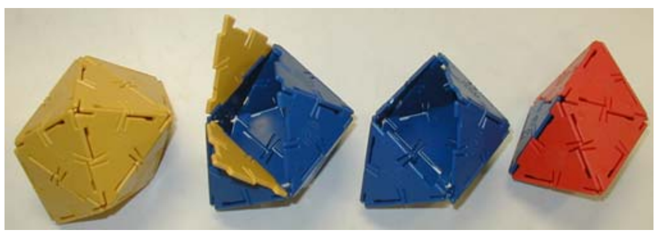


Figure 6.9. From a snub disphenoid to a tri-augmented triangular prism

Figure 6.10. From a tri-augmented triangular prism to a gyro-elongated square dipyramid

However, we will encounter a problem when trying to carry out the same process on the gyro-elongated square dipyramid. It is noted that no two consecutive edges or a "line" connects the only two vertices of valency 4. These two vertices are joined by three edges. By detaching these three edges, we obtained the icosahedra instead of a polyhedron with eighteen faces.

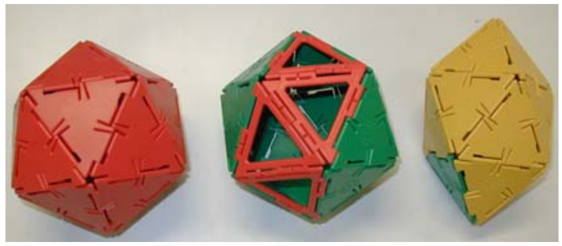


Figure 6.11. From gyro-elongated square dipyramid to an icosahedron

In order to have a D_{18}, the only way is to detach two edges and adding two more triangular faces to the polyhedron. However, by detaching any two edges will give rise to a flat surface where one of the vertices will have valency 6. Hence, D_{18} does not exist.

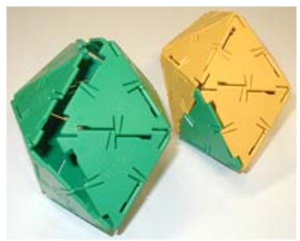


Figure 6.12. No D_{18}

6.2.3. What About Other Polygons?

After looking at the convex deltahedra generated by only triangles, we wonder if it is possible to generate other convex polyhedra with regular polygons of one type, i.e. polyhedra made up of only squares, pentagons or hexagons.

Let's consider a square. Six square faces form a cube, with each vertices of valency 3. If there exist a vertex of valency 2, it is not a polyhedron. And four or more squares meeting at a point produces either a flat surface or a concave polyhedron.

12 pentagonal faces form a dodecahedron with each vertices of valency 3. No vertices should have valency 2 since it produces a flat surface and four pentagons meeting at a vertex does not give a convex polyhedron.

Hexagonal faces do not generate any convex regular polyhedra. Hence, other than the triangular faces, no other n-gon produces a range of convex polyhedron with only one type of regular faces.

7. Zonohedron

A ***zonohedron*** is a convex polyhedron bounded by zonogons. Zonogons are polygons whose sides are parallel and equal. Parallelograms and rhombi are examples of zonogons. [1]

We can create new zonohedra by a systematic process starting with any given polyhedron. Some of these processes are truncation, stellation, dualization, compounding and zonohedrification. [8]. Zonohedrification is a process to generate zonohedra.

The cube, the rhombic dodecahedron, and the rhombic triacontahedron are three important zonohedra. We shall discuss more of them in the next section.

7.1. Rhombic Polyhedra

Rhombic polyhedra are made up rhombic faces. The set of rhombic polyhedron is a subset of the zonohedra. Considering any rhombus, opposite angles are equal, two being acute, two being obtuse. We can generate different polyhedra by changing the combination of obtuse and acute angles at a vertex.

Let's look at the two smallest rhombic polyhedra, the prolate and the oblate. The prolate and the oblate are two polyhedra formed by changing the combination of obtuse and acute angles at a vertex. The acute angles of three rhombi can be fitted together to form a solid angle, and two such sets can be joined to form a prolate while two sets of two obtuse angles and an acute angle brought together form an oblate.

We can also obtain the prolate and the oblate by stretching and skewing a cube and the extent to which we stretch and skew a cube determines the different angles of the rhombi obtained. The more we stretch a cube, the larger the obtuse angle of the rhombus we will get, the acute angle also gets smaller. And hence, in figure 7.1, there are two sets of prolate and oblate, the yellow set having a bigger acute angle and smaller obtuse angle, while the red set have a smaller acute angle and a bigger obtuse angle. The red set of rhombi have diagonals in the golden ratio, while the yellow set have diagonals of ratio $1:\sqrt{2}$. The two sets of prolate and oblate in figure 7.1 form the four rhombohedra with six faces.

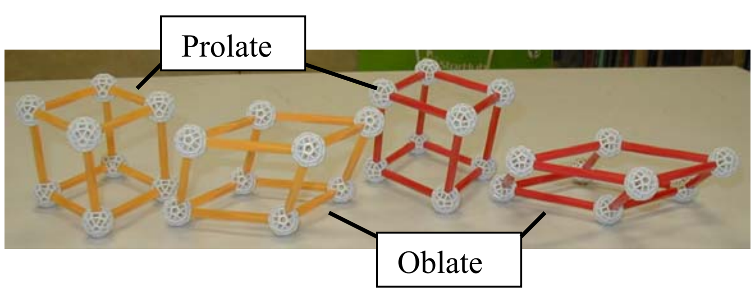


Figure 7.1. Rhombohedra

Other than the four rhombohedra, there are other bigger rhombic polyhedra shown in figure 7.2. They each have 30, 20 and 12 rhombic faces.

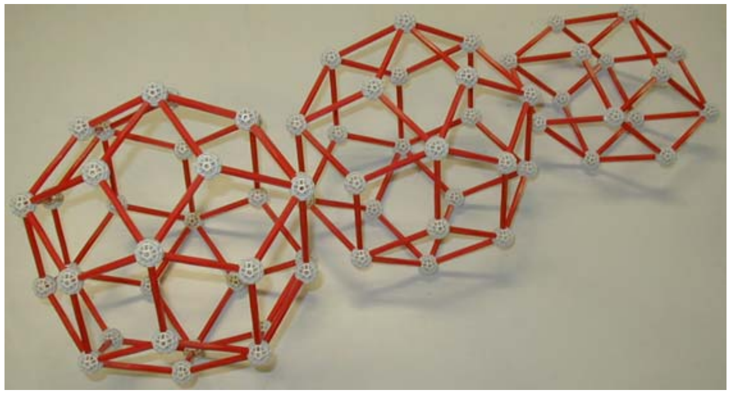


Figure 7.2. Rhombic polyhedra

Now, the 12 and 20 faces rhombic polyhedra can actually be generated by the rhombic triacontahedron with 30 faces, by collapsing 'belts' of rhombi or by collapsing a zone of faces. The process is shown in figure 7.3.

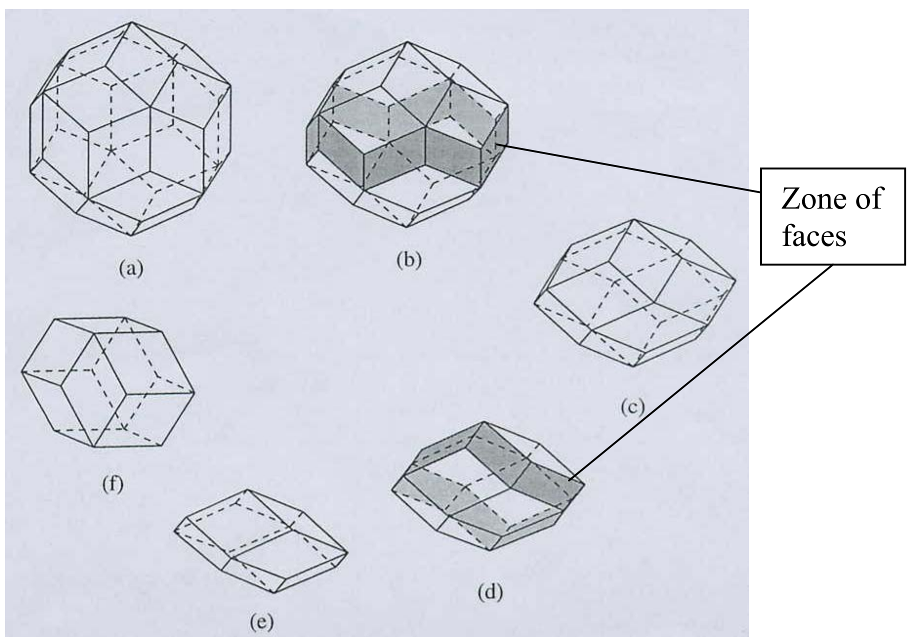
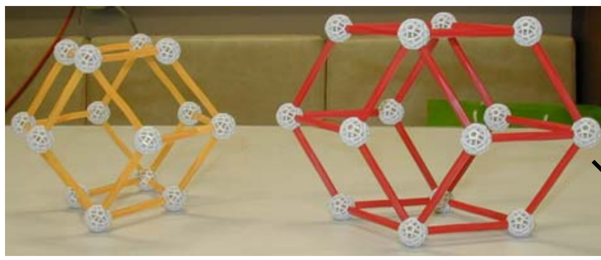


Figure 7.3. Generating rhombic polyhedra

Other than collapsing ‘belts’ of rhombi, we can generate different polyhedra by changing the combination of obtuse and acute angles at a vertex as mentioned earlier. The last polyhedron (figure 7.3(f)) is generated from the previous polyhedron (figure 7.3(e)) by changing the acute angle at the vertex to obtuse angle. These two rhombic polyhedra are known as the rhombic dodecahedra, each with 12 faces. In (figure 7.3(e)), there are rings of four rhombi where the acute angles are pointing at each other. As for the second type of rhombic dodecahedron (figure 7.3(f)), the ring of four rhombuses is formed in a way where the vertex at the obtuse angle points at the vertex of acute angle. This is illustrated in figure 7.4 and figure 7.5.



Rotated 90^0

Figure 7.4. Rhombic dodecahedra

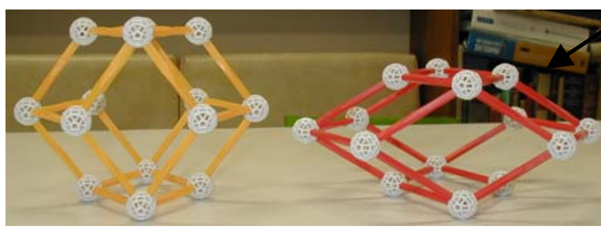


Figure 7.5. Rhombic dodecahedra

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