Undergraduate Research Opportunities Programme in Science
Perspective in Mathematics and Art

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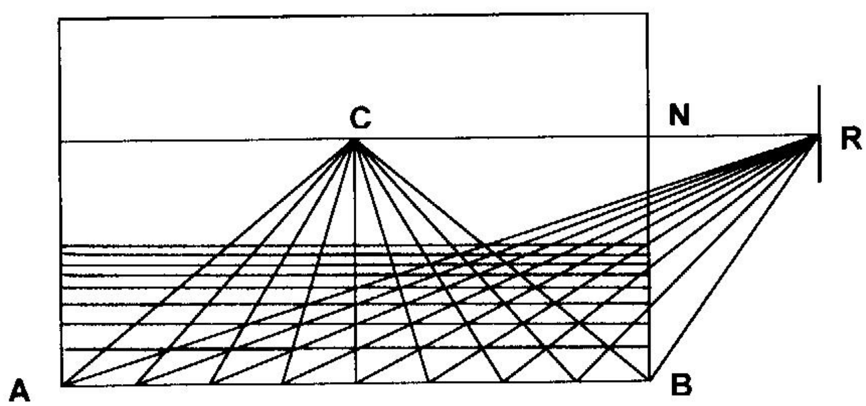
ABSTRACT

The philosophers of the Renaissance considered mathematical investigation integral to some of their theories. Such a combination of mathematics with natural philosophy was known as the "mixed sciences", and this included the study of optics. Ignoring problems of physics and focusing on geometry, the theory of perspective describes how to project a three-dimensional object onto a two-dimensional surface. Specifically, mathematically correct floors are called *pavimenti*. We shall examine two of the main methods of constructing *pavimenti*: Alberti's construction and the distance point construction. Furthermore, we shall look at a method of approximating the size of the moon in paintings. Related to this is the famous moon illusion, the result of a physiological effect known as *oculomotor micropsia*. Finally, there is a brief overview of optical illusions.

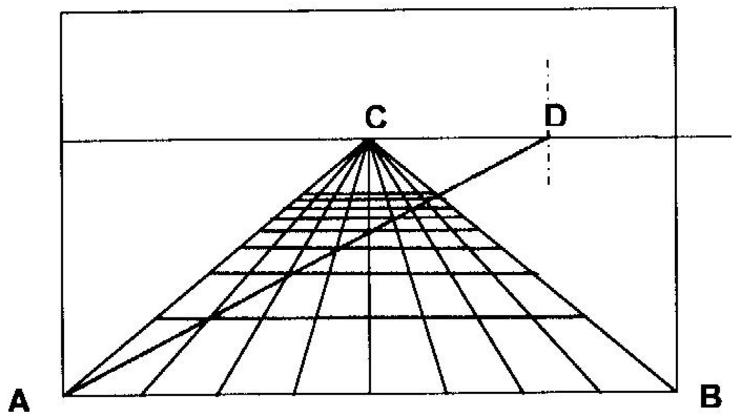
ALBERTI'S CONSTRUCTION



Alberti's Construction

Shown above is Alberti's construction. The line *AB* is first divided equally, and each of the division points are joined to the central vanishing point *C*. The division points are also joined to the right vanishing point R. The lines converging at *C* are known as the orthogonals. The lines parallel to *AB* are known as the transversals. The distance *NR* is the viewing distance, which was how far the artist was from the picture. This is then how far the viewer should stand from the picture.

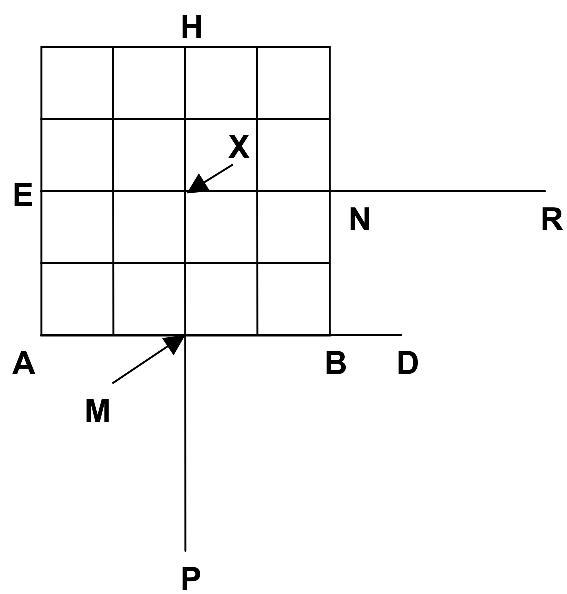
DISTANCE POINT CONSTRUCTION



The distance point construction.

The distance point construction produces the same viewing distance, but via a different approach. The line *AB* is also divided equally. The distance point *D* is determined. A line is then drawn from *A* to *D*, and the intersection of this line with the orthogonals will yield the reference points for the transversals. The distance *CD* is the viewing distance.

GEOMETRICAL PROOF OF ALBERTI'S AND THE DISTANCE POINT CONSTRUCTION



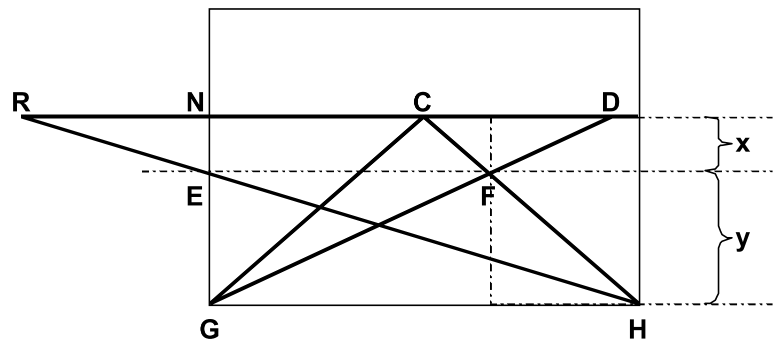
Floor plan of the two constructions.

For Alberti's construction, it is the distance from the viewer to the last transversal which is of great importance. This is intimately related to the viewing distance. We can picture the line *ER* as the line *HP* rotated 90° anticlockwise about the point *X*. Hence, we show that *MP* is equal to *NR*.

For the distance point construction, we can picture the line *MD* as the line *MP* rotated 90° anticlockwise about the point *M*. *D* is the distance point. If one was to stand at *D* instead of *P*, one can easily see that the distances *MP* and *MD* are equal. It follows that *MD* is also the viewing distance.

We have shown that *MD* and *NR* are the correct viewing distances. Hence, *MD* and *NR* must be the same length. It follows that Alberti's and the distance point construction are equivalent.

EQUIVALENCE OF ALBERTI'S AND THE DISTANCE POINT CONSTRUCTION



The viewing distances for both constructions superimposed.

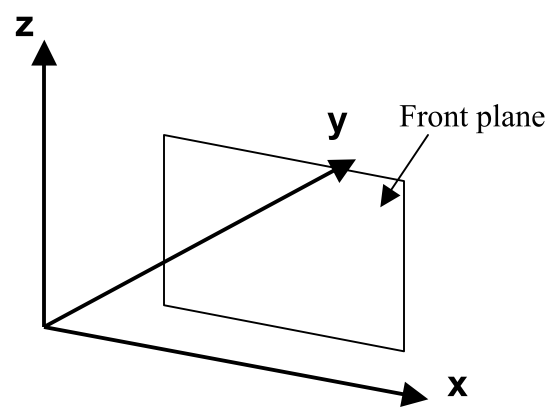
The left portion of the figure shows Alberti's construction, with the *NR* being the viewing distance. The right portion has the viewing distance as *CD*, a result of the distance point construction.

Since the lines *RD* and *GH* are parallel, triangles *CDF* and *GFH* are similar. Hence, $CD/GH = x / y$. In addition, triangles *RNE* and *EHG* are also similar. This means that $NR/GH = NE/EG$. However, $NE = x$ and $EG = y$. It follows that $CD/GH = NR/GH$. Hence, *CD* is the same length as *NR*.

MULTIPLE VANISHING POINTS

It is possible for objects in perspective to have more than one vanishing point. When an objective has two vanishing points, it is said to appear in two-point perspective. Similarly, three vanishing points correspond to three-point perspective.

When we view an object, we can usually make out which is the "front" of it. We shall call the surface of this "front" the front plane. The surface which the object is projected onto is called the projection plane (or the picture plane). Both the observer and the object rest on the ground plane.



Schematic diagram of the front plane, projection plane and the ground plane.

In the figure, the projection plane is taken to be the *xz*-plane. The ground plane is the *xy*-plane. The front plane is represented by the rectangle. Hence, the front plane is always parallel to the projection plane, and perpendicular to the ground plane. When this occurs, only one vanishing point is obtained when the object is viewed in perspective. This is called one-point perspective.

When the front plane is rotated about the *z*-axis, it is no longer parallel to the projection plane. However, it is still perpendicular to the ground plane. When the front plane is viewed in perspective, two vanishing points will be observed.

The front plane can be further rotated such that it is no longer perpendicular to the ground plane. It is also not parallel to the projection plane. The combination of these two factors will yield three vanishing points, when the front plane is view in perspective.

The rules governing two- and three-point perspective can be generalized to more complicated objects.

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