A Mean-Variance Bound For A Three-Piece Linear Function

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Abstract

In this note, we derive a **tight closed form upper bound** on the expected value of a three-piece linear convex function $E[\max(0, X, mX - z)]$ given the mean μ and the variance σ^2 of the random variable X. The bound is an extension of the well-known mean-variance bound for $E[\max(0, X)]$. An application of the bound to price the strangle option in finance is provided.

1 Introduction

Computing upper bounds on the expected value of a convex function $E[f(X)]$ for a random variable X with mean μ and variance σ^2 is a classical problem in probability and optimization. One such commonly studied function is the two-piece linear convex function $f(X) = \max(0, X)$. A simple mean-variance bound in this case is:

$$E[\max(0, X)] \leq \frac{1}{2}\left(\mu + \sqrt{\mu^2 + \sigma^2}\right), \tag{1}$$

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which is obtained from the Cauchy-Schwarz inequality. The two-point distribution that attains the bound is:

$$X = \begin{cases} -\sqrt{\sigma^2+\mu^2} & w.p. \quad \dfrac{1}{2}\left(1-\dfrac{\mu}{\sqrt{\sigma^2+\mu^2}}\right) \\ \sqrt{\sigma^2+\mu^2} & w.p. \quad \dfrac{1}{2}\left(1+\dfrac{\mu}{\sqrt{\sigma^2+\mu^2}}\right). \end{cases} \tag{2}$$

Scarf [6] used this bound for the function $f(X) = \max(0, X - z)$ in a min-max newsvendor model wherein X denotes the random demand for a product and z denotes the order quantity. Likewise, Lo [3] used the bound to obtain an upper bound on a call option price where X denotes the stock price and z denotes the strike price.

We extend this result to find a new closed form upper bound on the expected value of the three-piece linear function:

$$f(X) = \max(0, X, mX - z). \tag{3}$$

The bound is tight and in certain cases shown to be attained by a three point distribution. In the remaining cases, it reduces to the two point distributions as before. We indicate an application of the bound to price a strangle option in finance. We also believe that the bound can be used in newsvendor models with recourse opportunities (cf. Gallego and Moon [1]) and multiple simple recourse problems in stochastic programming (cf. Vlerk [7]), but have not explored it as yet.

2 A New Mean-Variance Bound

We are interested in solving the primal problem:

$$Z \quad = \max_{X\sim(\mu,\sigma^2)} \quad E[\max(0, X, mX - z)], \tag{4}$$

where the maximization is over the set of probability distribution of the random variable X satisfying the given mean and variance requirements. The related dual formulation is:

$$\begin{aligned} Z \quad = \quad & \min \quad y_0 + \mu y_1 + (\mu^2+\sigma^2) y_2 \\ & s.t. \quad g(x) = y_0 + y_1 x + y_2 x^2 \geq \max(0, x, mx - z) \quad \forall x \in \Re, \end{aligned} \tag{5}$$

where y_0, y_1 and y_2 are the dual variables corresponding to the probability-mass, mean and second moment constraints. The dual constraint implies that the quadratic function $g(x)$ is greater than or equal to $f(x) = \max(0, x, mx - z)$ for all x. We assume that $\sigma > 0$. It is then well-known that the two formulations have the same optimal objective value (cf. Isii [2]). Our approach to finding Z is based on solving the primal and dual formulations in closed form. Before proceeding, we make the following assumption.

Assumption 1 *Let $m > 1$ and $z > 0$.*

This ensures that each of the lines in $f(x)$ is maximum in some non-empty interval. All other cases can be easily handled by simple linear transformations of the function. The graphical representation of the functions $f(x)$ and $g(x)$ are provided in Figure 1.

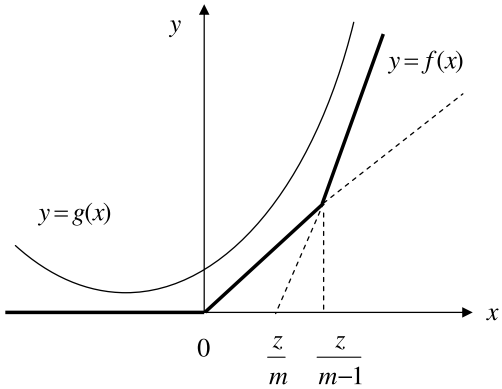

Figure 1: Graphical representation of the functions $f(x)$ and $g(x)$

A classical result due to Rogosinsky [5] states that there there exists an extremal distribution for the problem (4) with at most three support points. However, finding this in closed form is typically not possible (cf. Popescu [4]). We now identify these distributions and the corresponding bounds in closed form for our problem of interest.

Theorem 1 *Define:*

$$x_1 = \frac{-z}{m(m-1)}, \; x_2 = \frac{z}{m(m-1)} \text{ and } x_3 = \frac{(2m-1)z}{m(m-1)}.$$

The tight upper bound Z in (4) reduces to the following four cases:

Region 1: Three point distribution

If $\max[(x_2 - \mu)(\mu - x_1), (x_3 - \mu)(\mu - x_2)] \leq \sigma^2 \leq (x_3 - \mu)(\mu - x_1)$, *then*

$$Z = \frac{1}{2}\left[\mu + \frac{m(m-1)(\mu^2 + \sigma^2)}{2z} + \frac{z}{2m(m-1)}\right].$$

Region 2: Two point distributions

(2a) *If* $\sigma^2 \leq (x_2 - \mu)(\mu - x_1)$*, then*

$$Z = \frac{1}{2}\left[\mu + \sqrt{\mu^2 + \sigma^2}\right].$$

(2b) *If* $\sigma^2 \leq (x_3 - \mu)(\mu - x_2)$*, then*

$$Z = \frac{1}{2}\left[(m+1)\mu - z + \sqrt{((m-1)\mu - z)^2 + (m-1)^2\sigma^2}\right].$$

(2c) *If* $\sigma^2 \geq (x_3 - \mu)(\mu - x_1)$*, then*

$$Z = \frac{1}{2}\left[m\mu - z + \sqrt{(m\mu - z)^2 + m^2\sigma^2}\right].$$

Proof:

Our proof is based on constructing a primal and dual feasible solution to (4) and (5) respectively with the same objective value. Using strong duality, we can then claim that this is indeed the tight upper bound.

Region 1: Three point distribution

The dual feasible function $g(x)$ lies above the lines $y = 0$, $y = x$ and $y = mx - z$ respectively. From Figure 1, it is clear that this function can intersect each of these lines at at most one point. Suppose the points are x_1, x_2 and x_3 respectively. Equating the derivative of the function $g(x)$ with the slope of the lines at these points, we get:

$$\begin{aligned} g'(x_1) = 0 \quad &\Rightarrow \quad x_1 = -\frac{y_1}{2y_2}, \\ g'(x_2) = 1 \quad &\Rightarrow \quad x_2 = \frac{1-y_1}{2y_2}, \\ g'(x_3) = m \quad &\Rightarrow \quad x_3 = \frac{m-y_1}{2y_2}. \end{aligned} \tag{6}$$

Similarly, equating the value of the dual function $g(x)$ and the lines at these points, we get:

$$\begin{aligned} g(x_1) &= y_0 + y_1 x_1 + y_2 x_1^2 = 0, \\ g(x_2) &= y_0 + y_1 x_2 + y_2 x_2^2 = x_2, \\ g(x_3) &= y_0 + y_1 x_3 + y_2 x_3^2 = m x_3 - z. \end{aligned} \tag{7}$$

By substituting (6) into (7), the dual variables are obtained as:

$$\begin{aligned} y_0 &= \frac{z}{4(m-1)m}, \\ y_1 &= \frac{1}{2}, \\ y_2 &= \frac{(m-1)m}{4z}. \end{aligned} \tag{8}$$

The objective value for this dual feasible solution is:

$$y_0 + \mu y_1 + (\mu^2 + \sigma^2) y_2 = \frac{1}{2} \left[\mu + \frac{m(m-1)(\mu^2 + \sigma^2)}{2z} + \frac{z}{2m(m-1)} \right].$$

We next construct a primal solution using the three points x_1, x_2 and x_3 found in (6). From (8), we have:

$$\begin{aligned} x_1 &= \frac{-z}{m(m-1)}, \\ x_2 &= \frac{z}{m(m-1)}, \\ x_3 &= \frac{(2m-1)z}{m(m-1)}. \end{aligned} \tag{9}$$

Let p_1, p_2 and p_3 denote the probabilities of these three points. To satisfy the probability-mass and mean, variance requirements, we have:

$$\begin{aligned} &p_1 + p_2 + p_3 = 1, \\ &p_1 x_1 + p_2 x_2 + p_3 x_3 = \mu, \\ &p_1 x_1^2 + p_2 x_2^2 + p_3 x_3^2 = \mu^2 + \sigma^2. \end{aligned}$$

Solving for the values of p_i that satisfy these three equations, we get:

$$\begin{aligned} p_1 &= \frac{\sigma^2 + (\mu - x_2)(\mu - x_3)}{(x_2 - x_1)(x_3 - x_1)}, \\ p_2 &= \frac{\sigma^2 + (\mu - x_1)(\mu - x_3)}{(x_1 - x_2)(x_3 - x_2)}, \\ p_3 &= \frac{\sigma^2 + (\mu - x_1)(\mu - x_2)}{(x_1 - x_3)(x_2 - x_3)}. \end{aligned} \tag{10}$$

For the solution to be primal feasible, we need to ensure that the values of p_i are non-negative. From (10), this is ensured if:

$$\max[(x_2 - \mu)(\mu - x_1), (x_3 - \mu)(\mu - x_2)] \leq \sigma^2 \leq (x_3 - \mu)(\mu - x_1).$$

Assuming the above condition is satisfied, the objective function for this primal feasible solution is given as:

$$\begin{aligned} E[f(X)] &= p_1 0 + p_2 x_2 + p_3 (m x_3 - k), \\ &= \frac{\sigma^2 + (\mu - x_1)(\mu - x_3)}{(x_2 - x_1)(x_2 - x_3)} x_2 + \frac{\sigma^2 + (\mu - x_1)(\mu - x_2)}{(x_3 - x_1)(x_3 - x_2)} (m x_3 - k), \\ &= \frac{1}{2} \left[\mu + \frac{m(m-1)(\mu^2 + \sigma^2)}{2z} + \frac{z}{2m(m-1)} \right]. \end{aligned}$$

Both the primal and dual feasible solutions have the same objective value, implying that these are the primal and dual optimal solutions.

Region 2: Two point distributions

The remaining three bounds correspond to different two point distributions. We indicate the proof for the region (2a) only.

Suppose the dual feasible function $g(x)$ touches the lines $y = 0$ and $y = x$ only. Let these points be a and b respectively. In this case, equating the derivatives and the values as before, we get:

$$\begin{aligned} y_0 &= \frac{1}{16y_2}, \\ y_1 &= \frac{1}{2}, \\ a &= \frac{-1}{4y_2}, \\ b &= \frac{1}{4y_2}. \end{aligned}$$

The best dual solution of this form is obtained by minimizing the dual objective $y_0+\mu y_1+(\mu^2+\sigma^2)y_2$ with respect to y_2. This yields:

$$\begin{aligned} y_0 &= \frac{\sqrt{\sigma^2+\mu^2}}{4}, \\ y_1 &= \frac{1}{2}, \\ y_2 &= \frac{1}{4\sqrt{\sigma^2+\mu^2}}. \end{aligned}$$

The corresponding primal solution is

$$X = \begin{cases} -\sqrt{\sigma^2+\mu^2} & w.p. \quad \frac{1}{2}\left(1-\frac{\mu}{\sqrt{\sigma^2+\mu^2}}\right), \\ \sqrt{\sigma^2+\mu^2} & w.p. \quad \frac{1}{2}\left(1+\frac{\mu}{\sqrt{\sigma^2+\mu^2}}\right). \end{cases}$$

The primal and dual objectives are equal to:

$$\frac{1}{2}\left[\mu+\sqrt{\mu^2+\sigma^2}\right].$$

In this case, we still need to guarantee that the dual feasibility condition is satisfied by checking $y_0 + y_1x + y_2x^2 \geq mx - z$ for all $x \in \Re$. Let Δ be the discriminant of the quadratic function

$y_2x^2 + (y_1 - m)x + (y_0 + z)$. Then, we have:

$$\begin{aligned}
\Delta &= (y_1 - m)^2 - 4y_2(y_0 + z) \\
&= m(m-1) - \frac{z}{\sqrt{\sigma^2 + \mu^2}} \\
&\leq m(m-1) - \frac{z}{\sqrt{(x_2 - \mu)(\mu - x_1) + \mu^2}} && (\text{if } \sigma^2 \leq (x_2 - \mu)(\mu - x_1)) \\
&= m(m-1) - \frac{m(m-1)}{z} z && \left(\text{as } x_1 + x_2 = 0 \text{ and } x_1 x_2 = -\frac{z^2}{m^2(m-1)^2}\right) \\
&= 0.
\end{aligned}$$

Since Δ is less than or equals to 0 and $y_2 > 0$, the dual feasibility condition is satisfied. Thus the two-point distribution is feasible and the optimal solution in this case. ■

Figure 2 provides a graphical representation of the different cases in Theorem 1 in the mean-variance space. We can interpret the result in Theorem 1 as follows: Suppose we fix the mean of the

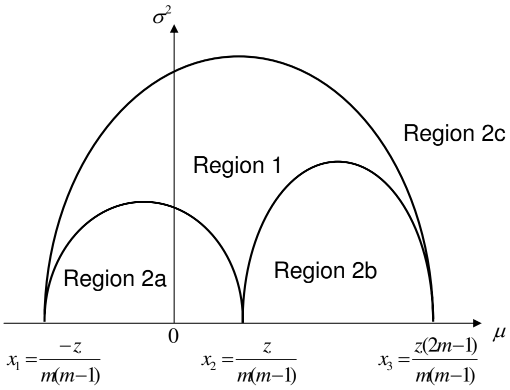

Figure 2: Characterization of the different regions in Theorem 1

random variable μ in the range $[x_1, x_2]$. As we increase the variance σ^2 of the random variable, the extremal distribution moves from region 2a (two point) to region 1 (three point) to region 2c (two point). These can be interpreted as regions of low variance, medium variance and high variance respectively for the particular mean. The characterization of region 1 with the extremal three point distribution is new. This occurs due to the three-piece structure of the objective function. The support points for the three point distribution in region 1 in fact remain unchanged. It is also easy to verify that the bound in region 1 is also an upper bound for the remaining three regions (though not necessarily tight).

3 An Application In Finance

We indicate an application of the bound to price a strangle option in finance. Suppose X denotes the random price of a financial asset at a future time $T > 0$. Consider an investor who at time 0 buys a call and a put option on this asset, both expiring at the same maturity T. Let K_1 and K_2 be the strike prices of this call and put option respectively. In options terminology, with $K_1 > K_2$ this is known as a **strangle**. Such a strangle option is valuable to the investor when the asset price is expected to be volatile but the exact direction of the price movement is unknown. The payoff of the strangle is plotted in Figure 3 and given as

$$f(X) = \max(K_2 - X, 0, X - K_1). \tag{11}$$

The three-piece payoff structure makes it suitable to use our bounds for this option.

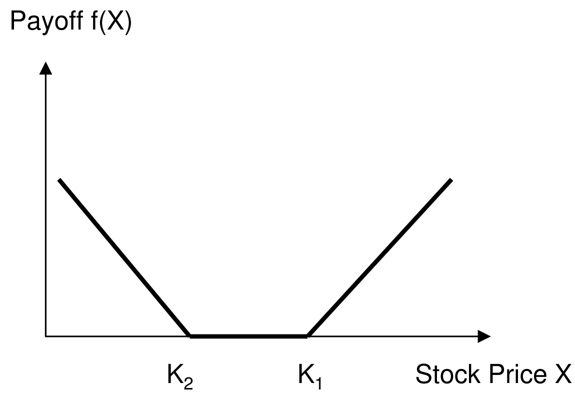

Figure 3: Payoff of Strangle.

Suppose we know the mean and the variance of the asset price under the risk-neutral distribution. The exact distribution is however unknown. A simple upper bound on the expected payoff of the strangle is obtained by using (1):

$$\begin{aligned} E[f(X)] &\leq E[K_2 - X]^+ + E[X - K_1]^+, \\ &\leq \frac{1}{2}\left[K_2 - K_1 + \sqrt{(\mu - K_1)^2 + \sigma^2} + \sqrt{(\mu - K_2)^2 + \sigma^2}\right]. \end{aligned} \tag{12}$$

Such option prices bounds are termed as semi-parametric bounds (cf. Lo [3]). However this is not the tightest possible upper bound on the strangle price since the two-point extremal distributions for the call and the put options are different. We obtain a tighter estimate on the price of the strangle option in this setting.

Proposition 1 *The tight upper bound on* $E\left[\max(K_2 - X, 0, X - K_1)\right]$ *with* $K_1 > K_2$ *and* $X \sim (\mu, \sigma^2)$ *is:*

$$\begin{cases} \frac{1}{2}\left[K_2 - \mu + \sqrt{(\mu - K_2)^2 + \sigma^2}\right] & \textit{if } 4\sigma^2 \leq (K_1 + K_2 - 2\mu)(2\mu - 3K_2 + K_1), \\ \frac{1}{2}\left[\mu - K_1 + \sqrt{(\mu - K_1)^2 + \sigma^2}\right] & \textit{if } 4\sigma^2 \leq (3K_1 - K_2 - 2\mu)(2\mu - K_1 - K_2), \\ \frac{1}{2}\left[K_2 - K_1 + \sqrt{(2\mu - K_1 - K_2)^2 + 4\sigma^2}\right] & \textit{if } 4\sigma^2 \geq (3K_1 - K_2 - 2\mu)(2\mu - 3K_2 + K_1), \\ \dfrac{4\sigma^2 + (K_1 + K_2 - 2\mu)^2}{8(K_1 - K_2)} & \textit{otherwise.} \end{cases} \tag{13}$$

It is possible to strengthen the bound slightly for the strangle using the additional information that the stock price X is always nonnegative (see Lo [3]). For the numerical example we consider next, this information is however not useful in tightening the bounds.

Numerical Example: We consider a single asset example taken from Lo [3] with a current stock price of $S_0 = \$40$. Call and put options are trading on this stock with a time to maturity of $T = 1/52$ year (or one week). The annual risk free interest rate is $r = 6\%$ with an annual compound standard deviation of s. We consider two different cases, $s = 0.2$ (small) and $s = 0.8$ (large). Assuming a lognormal distribution for the asset price, the mean and variance of the terminal stock price under the risk-neutral distribution (cf. [3]) is given as

$$(\mu, \sigma^2) = \left(S_0 e^{rT}, S_0^2 e^{2rT}(e^{s^2 T} - 1)\right).$$

We compare the mean-variance bounds for the strangle price:

$$e^{-rT} E[\max(K_2 - X, 0, X - K_1)]$$

from (12) and (13) with the closed form Black-Scholes price. The strike prices K_1 and K_2 are varied between 30 to 50 with $K_1 \geq K_2$. The results are provided in Table 1 and Figure 4. From the table, it is clear that the improvement using the new bound (13) is larger as the variance increases. For

$s = 0.8$, the best improvement over bound (12) is obtained for a strangle with strike price $K_1 = 45$ and $K_2 = 35$. In this case, the tight bound is 0.9919 with the three point distribution:

$$X = \begin{cases} 30 & \text{w.p. } 0.0970, \\ 40 & \text{w.p. } 0.8014, \\ 50 & \text{w.p. } 0.1016. \end{cases} \tag{14}$$

Under this extremal distribution, the strangle is in-the money at $X = 30$ and 50 while out-of-the money at $X = 40$. On the contrary, using the simple extension of Lo's bound in this case provides a weaker upper bound of 1.6959.

		$s = 0.2$			$s = 0.8$		
K_1	K_2	Black-Scholes	Bound (13)	Bound (12)	Black-Scholes	Bound (13)	Bound (12)
30	30	10.0346	10.0958	10.0958	10.0457	10.9776	10.977
35	30	5.0404	5.1216	5.1313	5.2718	6.2566	6.3539
40	30	0.4658	0.5783	0.6089	1.7971	2.2488	2.7203
45	30	0.0000	0.0614	0.0920	0.3603	0.8538	1.3253
50	30	0.0000	0.0309	0.0614	0.0475	0.4960	0.9470
35	35	5.0404	5.1611	5.1611	5.4921	6.7245	6.7245
40	35	0.4658	0.5783	0.6387	2.0175	2.6295	3.0909
45	35	0.0000	0.0617	0.1218	0.5807	0.9919	1.6959
50	35	0.0000	0.0603	0.0912	0.2678	0.8421	1.3176
40	40	0.8854	1.1106	1.1106	3.5370	4.4515	4.4515
45	40	0.4196	0.5322	0.5937	2.1002	2.5844	3.0565
50	40	0.4196	0.5322	0.5631	1.7874	2.2027	2.6782
45	45	4.9481	5.0710	5.0710	5.6576	6.6557	6.6557
50	45	4.9481	5.0303	5.0404	5.3448	6.1773	6.2774
50	50	9.9423	10.0041	10.0041	10.0262	10.8933	10.8933

Table 1: Price and bounds for the strangle option.

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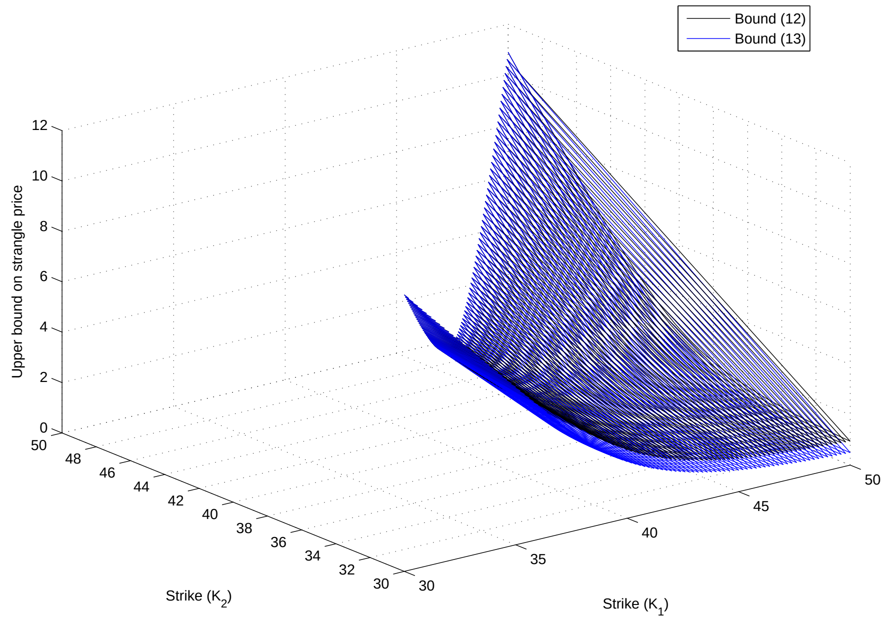

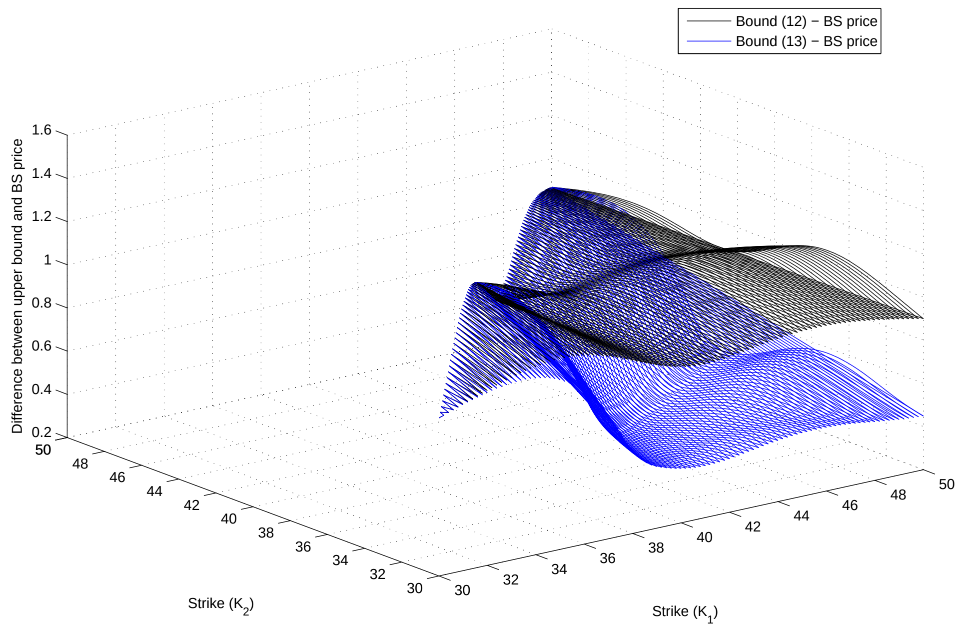

Figure 4: Upper bound and difference with BS price for strangle for $s = 0.8$.

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