Homework Set 1

Hand in independently written solutions by 5th September.

(1) If $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$, denote by (a, b) the element $(a_1, \ldots, a_n, b_1, \ldots, b_n)$ of $\mathbb{R}^{n+m}$. Let U be a closed subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ be a bounded function (i.e., $\{f(x) : x \in U\}$ is bounded). Show that f is continuous on U if and only if the set

$$\{(x, f(x)) : x \in U\}$$

is closed in $\mathbb{R}^{n+m}$. Does the result still hold if f is not assumed to be bounded?

(2) Let $T : E \to F$ be a bijective linear operator between finite dimensional normed spaces E and F. Show that there are constants $0 < K_1 < K_2 < \infty$ so that

$$K_1\|x\| \le \|Tx\| \le K_2\|Tx\|$$

for all $x \in E$.

(3) Suppose that $a \in U \subseteq \mathbb{R}^n$ and that $B(a, r) \subseteq U$ for some $r > 0$. Assume that $f : U \to \mathbb{R}$ is *uniformly partially differentiable* at a, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ so that

$$\Big|\frac{f(a + tu) - f(a)}{t} - \partial_u f(a)\Big| \le \varepsilon.$$

for all $u \in \mathbb{R}^n$ with $\|u\| = 1$ and all t with $0 < |t| < \delta$. If, in addition,

$$\partial_u f(a) = \langle(\partial_1 f(a), \ldots, \partial_n f(a)), u\rangle$$

for all $u \in \mathbb{R}^n$ with $\|u\| = 1$. Show that f is differentiable at a.