Solutions should be worked out individually. Due 27 March 2009.

(1) Suppose that $u$ is harmonic on $B(0, 1)$ and continuous on $\overline{B(0, 1)}$. Define $v$ by

$$v(re^{is}) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{r \sin(s - t)}{1 - 2r \cos(s - t) + r^2} \, dt$$

for $0 \leq r < 1$, $s \in \mathbb{R}$. Show that $v$ is a harmonic conjugate of $u$ on $B(0, 1)$, i.e., $u + iv$ is analytic on $B(0, 1)$.

(2) Let $u$ be a harmonic function on $\mathbb{C}$. Suppose that there exists $M < \infty$ such that $u(z) \leq M$ for all $z \in \mathbb{C}$. Show that $u$ is constant on $\mathbb{C}$. [Exponentiate.]

(3) Let $f$ be a nonconstant analytic function on a connected open set $U$. Suppose that $Z$ is a bounded set such that $d(Z, U^c) > 0$ and that $f(z) = 0$ for all $z \in Z$. Show that $Z$ is a finite set. [Here $U^c = \mathbb{C} \setminus U$ and $d(Z, U^c) = \inf\{|z - w| : z \in Z, w \in U^c\}$.

(4) The function $\text{Log}(1 + \sin z)$ has a Taylor series expansion $\sum_{k=0}^{\infty} a_k z^k$ at 0. (There is no need to prove this statement.)

(a) Find the radius of convergence of the series $\sum_{k=0}^{\infty} a_k z^k$. Justify your answer fully.

(b) Find the value of the term $a_4$.

[Obtain a power series representation of $\text{Log}(1 + \sin z)$ at 0 by purely formal manipulations. Why must it be the Taylor series?]