Homework Set 3

Hand in independently written solutions by 24th October.

(1) Let U be an open subset of $\mathbb{R}^2$ containing a point (x_0, y_0). Suppose that $f : U \to \mathbb{R}$ has continuous partial derivatives $\partial_1 f$, $\partial_2 f$ on U, and

$$f(x_0, y_0) = 0,\ \partial_2 f(x_0, y_0) \neq 0.$$

Show that there is an open interval I containing x_0 and a differentiable function $g : I \to \mathbb{R}$ so that $g(x_0) = y_0$ and for every $x \in I$,

$$\begin{aligned} f(x, g(x)) &= 0, \\ \partial_2 f(x, g(x)) &\neq 0, \\ g'(x) &= -\frac{\partial_1 f(x, g(x))}{\partial_2 f(x, g(x))}. \end{aligned}$$

(2) Find the minimum value of $xy + 2xz + 2yz$ subject to the conditions $xyz = 4$, $x, y, z > 0$, using the method of Lagrange multipliers. Explain why what you obtain is the minimum value.

(3) Suppose $f, g : [a, b] \to \mathbb{R}$ are functions so that $\int_a^c f\,dg$ exists for all $c \in (a, b)$. If f is bounded on $[a, b]$, g is increasing on $[a, b]$ and continuous at b, show that $\int_a^b f\,dg$ exists.
(Under the assumptions, when c is close to b, both $|U(f, g : P)|$ and $|L(f, g : P)|$ are small for the trivial partition $P = \{c, b\}$ of $[c, b]$. Combine this observation with Proposition 34. Check explicitly that the results you cite are applicable.)