Solutions should be worked out individually. Due 14 April 2009.

(1) Let \( f \) be analytic on \( \text{Ann}(0, R, \infty) \), where \( 0 \leq R < \infty \). Then \( f \) has a Laurent series expansion
\[
f(z) = \sum_{k=0}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^{-k},
\]
valid on \( \text{Ann}(0, R, \infty) \). Assume that \( \lim_{|z| \to \infty} |f(z)| = \infty \), i.e., for all \( M < \infty \), there exists \( r, R \leq r < \infty \), so that \( |f(z)| > M \) for all \( z \) with \( |z| > r \). Show that there exists \( n \in \mathbb{N} \) such that \( a_k = 0 \) for all \( k > n \).
[Consider the function \( f(1/z) \).]

(2) Assume that the integral \( \int_{-\infty}^{\infty} \frac{\log |x|}{x^2 + 1} \, dx \) converges. Find its value. Justify your steps fully.
[Integrate \( \frac{\log(-iz)}{z^2 + 1} \) around a semi-annulus in the upper half plane.]

(3) Let \( h \) be a nonconstant analytic function on an open set \( U \) and let \( z_0 \in U \). Suppose that \( f \) is analytic on \( B'(h(z_0), r) \) for some \( r > 0 \) and that \( f \) has an essential singularity at \( h(z_0) \). Show that \( f \circ h \) has an essential singularity at \( z_0 \).
[Combine the Casorati-Weierstrass Theorem and the Open Mapping Theorem.]

(4) Let \( U \) be an open connected set and let \( \gamma \) be a piecewise smooth closed path so that \( U \cap \{\gamma\} = \emptyset \). Show that the function \( f(z) = n(\gamma, z) \) is constant on \( U \).