Homework Set 4

Hand in independently written solutions by 14th November.

(1) Suppose that the sequences of real-valued functions $(f_k)_{k=1}^{\infty}$ and $(g_k)_{k=1}^{\infty}$ converge uniformly on a set S to functions f and g respectively.

   (a) Show by an example that the sequence of products $(f_k g_k)_{k=1}^{\infty}$ does not have to converge uniformly on S.

   (b) However, if we also assume that there exists $M < \infty$ so that $|f_k(x)|, |g_k(x)| < M$ for all k and all $x \in S$, then $(f_k g_k)_{k=1}^{\infty}$ converges to fg uniformly on S.

(2) Suppose that the power series $\sum_{k=0}^{\infty} b_k x^k$ has radius of convergence ∞. Let $g : \mathbb{R} \to \mathbb{R}$ be its sum. Show that there is a function $f : \mathbb{R} \to \mathbb{R}$ that satisfies $f(x) + f'(x) = g(x)$ for all $x \in \mathbb{R}$ and $f(0) = 0$.
(Look for a power series solution $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Solve for the coefficients a_k formally and then justify the convergence of the resulting series. In my calculations, I find that I need to show that if ε is small, then

$$\left(\frac{1 + \varepsilon + \varepsilon^2 2! + \cdots + \varepsilon^k k!}{k!}\right)^{1/k}$$

is also small for large k. I do it by splitting up the sum into two parts, the first $k/2$ terms and the remaining terms.)

(3) Let

$$S = \{(x, y) \in \mathbb{R}^2 : y^2 \leq x \leq y, 0 \leq y \leq 1\}.$$

Define $f : S \to \mathbb{R}$ by $f(x, y) = xy$. Evaluate $\int_S f \, dA$. Justify all your steps.