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WEAK* CONVERGENCE IN HIGHER DUALS OF ORLCZ SPACES

DENNY H. LEUNG

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ABSTRACT. It is shown that the spaces \((\Sigma \oplus E)_{l=1}^{\infty}(\Gamma)\) are Grothendieck spaces for a class of Banach lattices \(E\) which includes the Orlicz spaces with weakly sequentially complete duals.

A Banach space is said to be a Grothendieck space if weak* and weak sequential convergence coincide in the dual. The simplest nontrivial example of a Grothendieck space is \(l^\infty\). In [7], the question of when the space \((\Sigma \oplus E)_{l=1}^{\infty}(\Gamma)\) is Grothendieck is treated. In particular, it is shown there that \((\Sigma \oplus L^p)_{l=1}^{\infty}(\Gamma)\) is Grothendieck if \(2 \leq p < \infty\) and \(\Gamma\) is countable. In this paper, we extend this result to a class of Banach lattices which includes the Orlicz spaces with weakly sequentially complete duals. We close these introductory remarks by mentioning that H. P. Lotz [6] has shown recently that the weak \(L^p\) spaces are Grothendieck spaces.

1. Let us start by fixing some notation. Let \(E\) be a (real) Banach lattice, \(\Gamma\) an arbitrary index set, and \(F = (\Sigma \oplus E)_{l=1}^{\infty}(\Gamma)\). For \(x \in F\), we write \(x = (x(\gamma))\), where \(x(\gamma) \in E\) for every \(\gamma \in \Gamma\). If \(x' \in F'\) and \(A \subset \Gamma\), define \(x'_{\chi_A} \in F'\) by \(\langle x, x'_{\chi_A} \rangle = \langle x_{\chi_A}, x' \rangle\) for all \(x \in F\). It is easily seen that the equation \(\mu_{x'}(A) = \|x'_{\chi_A}\|\) defines a finitely additive measure on \(\Gamma\); consequently, we may identify \(\mu_{x'}\) with an element of \(l^\infty(\Gamma)'\).

LEMMA 1. If \((x'_i)\) is a positive weak* null sequence in \(F'\), then \((\mu_{x'_i})\) is relatively weakly compact in \(l^\infty(\Gamma)'\).

PROOF. Let \(\mu_i = \mu_{x'_i}\). If \((\mu_i)\) is not relatively weakly compact, then there exist a partition \((A_i)\) of \(\Gamma\) and \(\varepsilon > 0\) such that \(\mu_i(A_i) > \varepsilon\) for all \(i\). By definition of \(\mu_i\), there is a positive normalized sequence \((x_i) \subset F\) such that \(x_i_{\chi_{A_i}} = 0\) and \(\langle x_i, x'_i \rangle > \varepsilon\) for all \(i\). Let \(x = \sup_i x_i\). Then \(\|x\| = 1\) and \(\langle x, x'_i \rangle > \varepsilon\) for all \(i\), contrary to the fact that \((x'_i)\) is weak* null.

THEOREM 2. Let \(E\) be a Banach lattice with positive cone \(E_+\). Suppose there exist a function \(\tau: E_+ \to [0, \infty]\) and a positive real number \(M\) with the following properties:

1. \(\tau(0) = 0;\)
2. \(\|x\| \leq 1 \Rightarrow \tau(x) \leq M;\)
3. For every disjoint sequence \((x_i)_{i=1}^n \subset E_+, \sum_{i=1}^n \tau(x_i) \leq M\tau(\sum_{i=1}^n x_i);\) and
4. For every sequence \((x_i)_{i=1}^\infty \subset E_+\) with \(\sum_i \tau(x_i) \leq 1\), \(\sup_i x_i\) exists and \(\|\sup_i x_i\| \leq M\).

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Then, for any index set $\Gamma$, every disjoint positive weak* null sequence $(x'_i)$ in $F = (\Sigma \oplus E)_{t \in \Gamma}$ has a weakly Cauchy subsequence.

PROOF. Assume the contrary. We obtain a disjoint positive weak* null sequence $(x'_i)$ which is not weakly sequentially precompact. By Rosenthal's theorem, we may assume that $(x'_i)$ is equivalent to the $l^1$ basis. Since $(x'_i)$ is lattice isomorphic to $l^1$, there exist $\varepsilon > 0$ and a positive sequence $(x_{ij})_{i \geq j} \subset F$ with the following properties:

(a) For every $i$, $(x_{ij})_{1 \leq j \leq i}$ is a pairwise disjoint sequence such that $\|\sum_{j \leq i} x_{ij}\| < 1$; and

(b) $\langle x_{ij}, x'_j \rangle > \varepsilon$ for $1 \leq j \leq i$.

Define $A_{ij} \subseteq \Gamma$ by $A_{ij} = \{\gamma | \tau(x_{ij}(\gamma)) \geq 1/\sqrt{i}\}$. Note that $\|\sum_{j \leq i} x_{ij}\| < 1 \Rightarrow \|\sum_{j \leq i} x_{ij}(\gamma)\| < 1$ for all $\gamma \Rightarrow \tau(\sum_{j \leq i} x_{ij}(\gamma)) \leq M$. Hence $\sum_{j \leq i} \tau(x_{ij}(\gamma)) \leq M^2$ since the $x_{ij}$'s are disjoint. Thus

\[
\bigcap_{j \in B} A_{ij} = \emptyset
\]

for all $B \subseteq \{1, 2, \ldots, i\}$ with card $B > M^2 \sqrt{i}$. Recall the sequence $(\mu_i)$ as defined in the proof of Lemma 1. Fix $i$ and let $C_i = \{j \leq i | \mu_j(A_{ij}) < \varepsilon/2\}$. For $j \in C_i$, we let $z_j = x_{ij} \chi_{A_{ij}}$, then

\[
\langle z_j, x'_j \rangle \geq \langle x_{ij}, x'_j \rangle - \langle x_{ij}, x'_j \chi_{A_{ij}} \rangle \geq \varepsilon - \|x_{ij}\| \mu_j(A_{ij}) \geq \varepsilon/2
\]

while $\tau(z_j(\gamma)) \leq 1/\sqrt{i}$ for all $\gamma$ by definition of $A_{ij}$. If $(\text{card } C_i)_{i=1}^\infty$ is unbounded, there exists an infinite subset $I$ of $\mathbb{N}$ such that for every $i \in I$, there exists $j_i \in C_i$ with the $j_i$'s distinct for different $i$'s. Without loss of generality, we may also assume that $\sum_{i \in I} 1/\sqrt{i} \leq 1$. Choose $z_{j_i}$ as given above. Since

\[
\sum_i \tau(z_{j_i}(\gamma)) \leq \sum_{i \in I} \frac{1}{\sqrt{i}} \leq 1
\]

for all $\gamma$, $z(\gamma) \equiv \sup_i z_{j_i}(\gamma)$ exists for all $\gamma$ and $\|z(\gamma)\| \leq M$ by property (4). Hence $z \equiv (z(\gamma)) \in F$. However,

\[
\langle z, x'_j \rangle \geq \langle z_{j_i}, x'_j \rangle \geq \varepsilon/2
\]

for all $i \in I$, contrary to the fact that $(x'_i)$ is weak* null. Hence $(\text{card } C_i)_{i=1}^\infty$ is bounded by some constant $K < \infty$. Now $\mu_i$ is relatively weakly compact in the AL-space $l^\infty(\Gamma)'$ by Lemma 1, hence there exists $0 \leq \mu \in l^\infty(\Gamma)'$ such that $(\mu_i) \subset [0, \mu] + (\varepsilon/4)U$, where $U$ denotes the unit ball of $l^\infty(\Gamma)'$. Let $D_i = \{j \leq i | \mu_j(A_{ij}) \geq \varepsilon/2\}$ for every $i$. By the above, card $D_i \geq i - K$ for all $i$. Also $\mu(A_{ij}) \geq \varepsilon/2$ for all $j \in D_i$. Using equation $(*)$, we see that

\[
\sum_{j \in D_i} \mu(A_{ij}) \leq M^2 \sqrt{i} \mu(\Gamma)
\]

for all $i$ and hence $\mu(\Gamma) \geq (\varepsilon/4M^2 \sqrt{i}) \text{card } D_i \geq (\varepsilon/4M^2 \sqrt{i})(i - K)$ for all $i$. This contradiction proves the theorem.
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THEOREM 3. Let $E$ be a countably order complete Banach lattice which satisfies a nontrivial upper estimate. If there is a function $\tau$ on $E$ as in Theorem 2, then $F = (\Sigma \oplus E)_{l^\infty(\Gamma)}$ is a Grothendieck space.

PROOF. Because of the upper estimate condition on $E$, $F'$ is weakly sequentially complete. By [2], it suffices to show that any disjoint positive weak* null sequence in $F'$ is weakly null. But this follows from Theorem 2 and the weak sequential completeness of $F'$.

REMARK. Some condition in addition to the countable order completeness and the upper estimate has to be imposed on $E$ in order for the conclusion of Theorem 3 to hold. In [3], a sequence of finite dimensional lattices $(E_n)$ which satisfy a uniform upper $p$-estimate is constructed such that $F = (\Sigma \oplus E_n)_{l^\infty(\Gamma)}$ is not Grothendieck. Hence $E = (\Sigma \oplus E)_{l^p}$ satisfies an upper $p$-estimate and is obviously order complete while $(\Sigma \oplus E)_{l^\infty(\Gamma)}$ is not Grothendieck.

COROLLARY 4. Under the hypotheses of Theorem 3, all the even duals of $E$ are Grothendieck spaces.

PROOF. By [1, Proposition 1.20], $E''$ is isomorphic to a complemented subspace of some ultraproduct $E_{E''}$; hence $E''$ is a quotient space of some $(\Sigma \oplus E)_{l^\infty(\Gamma)}$. Simple induction now shows that all even duals of $E$ are quotients of (different) $(\Sigma \oplus E)_{l^\infty(\Gamma)}$. But quotients of Grothendieck spaces are themselves Grothendieck.

2. We now apply the results in §1 to Orlicz spaces.

DEFINITION 5. An Orlicz function $\varphi$ is a continuous nondecreasing and convex function defined for $t \geq 0$ such that $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$.

DEFINITION 6. Let $(\Omega, \Sigma, \mu)$ be a measure space and let $\varphi$ be an Orlicz function, the space $L^\omega(\Omega, \Sigma, \mu)$ is the Banach space consisting of all measurable functions $f$ such that $\int \varphi(|f(x)|/\rho) \, d\mu(x) < \infty$ for some $\rho > 0$ with the norm

$$\|f\| = \inf \left\{ \rho > 0 \mid \int \varphi(|f(x)|/\rho) \, d\mu(x) \leq 1 \right\}.$$

For details concerning Orlicz spaces, we refer the reader to [4, 5]. Here, we only wish to point out that (1) every Orlicz space is obviously order complete, and (2) if an Orlicz space $L^\omega$ has a weakly sequentially complete dual, then it satisfies a nontrivial upper estimate. Now, if we define $\tau: (L^\omega)_+ \to [0, \infty]$ by $\tau(f) = \int \varphi(f(x)) \, d\mu(x)$, then it is easily seen that $\tau$ satisfies the conditions in Theorem 2. Hence, by Theorem 3, we get

THEOREM 6. If $L^\omega$ has a weakly sequentially complete dual, then $(\Sigma \oplus L^\omega)_{l^\infty(\Gamma)}$ is Grothendieck for every index set $\Gamma$. Consequently, all even duals of $L^\omega$ are Grothendieck.

REMARK. For $1 \leq p < \infty$, if we let $\varphi(t) = t^p$, then $L^\omega = L^p$. Thus the results of Theorem 6 apply in particular to $L^p$ for $1 < p < \infty$.

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