

## Exercises for §4

- (22) (Mean value inequality) Let  $f$  be a differentiable function on an open set  $U$  in  $\mathbb{R}^n$ , with values in  $\mathbb{R}^m$ . Suppose that  $\|Df(x)\| \leq C$  for all  $x \in U$ . If  $U$  contains a *line segment*  $[a, b] = \{(1-t)a + tb : t \in [0, 1]\}$ , show that

$$\|f(b) - f(a)\| \leq C\|b - a\|.$$

- (23) Suppose that  $f : (a, b) \rightarrow \mathbb{R}$  is differentiable on  $(a, b)$ . Then  $f'$  satisfies the intermediate value theorem, i.e., if  $a < c < d < b$  and  $k$  lies between  $f'(c)$  and  $f'(d)$ , then there exists  $\xi \in [c, d]$  where  $f'(\xi) = k$ . This is known as Darboux's Theorem.  
(Consider the function  $g(x) = f(x) - kx$ .)

- (24) Let  $a \in \mathbb{R}$  and  $f : (a, \infty) \rightarrow \mathbb{R}$  be a function. If there is a real number  $L$  so that for any  $\varepsilon > 0$ , there exists  $N \in (a, \infty)$  such that  $|f(x) - L| < \varepsilon$  for all  $x \geq N$ , then we say that

$$\lim_{x \rightarrow \infty} f(x) = L.$$

Let  $f$  be defined and differentiable on  $(0, \infty)$ . If  $\lim_{x \rightarrow \infty} f'(x) = 0$ , show that  $\lim_{x \rightarrow \infty} (f(x+1) - f(x)) = 0$ .

- (25) Let  $f : (a, \infty) \rightarrow \mathbb{R}$  be differentiable on  $(a, \infty)$ . If  $\lim_{x \rightarrow \infty} (f(x) + f'(x)) = 0$ , show that  $\lim_{x \rightarrow \infty} f(x) = 0$ .  
(Contemplate the minima and maxima of the function  $f$  on intervals of the form  $[c, d]$  for large  $c$ .)

- (26) Let  $f : I \rightarrow \mathbb{R}$  be twice differentiable on an open real interval  $I$ . Show that for all  $x \in I$ ,

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

(Find proofs with and without using L'Hôpital's rules.)

- (27) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be such that there is a constant  $C < 1$  with  $|f'(x)| \leq C$  for all  $x \in \mathbb{R}$ . Take any  $x_1 \in \mathbb{R}$  and let  $x_{n+1} = f(x_n)$ ,  $n \in \mathbb{N}$ . Prove that the sequence  $(x_n)_{n=1}^{\infty}$  converges to a point  $x_0$  satisfying  $f(x_0) = x_0$ . Also show that  $x_0$  is the unique solution to the equation  $f(x) = x$ .

- (28) Let  $f$  be a real-valued function defined on an open interval  $(a, b)$  which is differentiable at a point  $c \in (a, b)$ . If  $f'(c) > 0$ , show that there exists  $\varepsilon > 0$  so that  $f(x) > f(c)$  for all  $x \in (c, c + \varepsilon)$  and  $f(x) < f(c)$  for all  $x \in (c - \varepsilon, c)$ .

(29) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & \text{if } (x_1, x_2) \neq (0, 0) \\ 0 & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

Show that

- (a) Both partials of  $f$  exist and are equal to 0 at  $(0, 0)$ ,
- (b)  $f$  is not continuous at  $(0, 0)$ .
- (c) The partials of  $f$  are not continuous at  $(0, 0)$ .

(30) Suppose  $U$  is an open subset of  $\mathbb{R}^n$  and  $1 \leq i, j \leq n$ . If  $f : U \rightarrow \mathbb{R}$  is a function such that  $\partial_k f$ ,  $1 \leq k \leq n$ , exist and are continuous on  $U$ ,  $\partial_j \partial_i f$  exists on  $U$  and is continuous at a point  $a \in U$ . Then  $\partial_i \partial_j f(a)$  exists and is equal to  $\partial_j \partial_i f(a)$ .

(31) Suppose  $a \in \mathbb{R}$ ,  $f, g : (a, \infty) \rightarrow \mathbb{R}$  are differentiable functions on  $(a, \infty)$  so that  $g$  and  $g'$  are never 0. If  $\lim_{x \rightarrow \infty} f(x) = 0 = \lim_{x \rightarrow \infty} g(x)$  and  $\lim_{x \rightarrow \infty} f'(x)/g'(x)$  exists, then  $\lim_{x \rightarrow \infty} f(x)/g(x)$  exists and takes the same value.

(32) Give a “one-step proof” of Taylor’s theorem by considering the functions

$$F(x) = \sum_{k=0}^n \frac{f^{(k)}(x)}{k!} (b-x)^k \quad \text{and} \quad G(x) = (b-x)^{n+1}.$$

(33) Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that  $f$  is infinitely differentiable on  $\mathbb{R}$ . Find the Taylor series expansion of  $f$  at 0.

(You may assume the usual properties of the exponential function, including the value of its derivative.)

(34) Let  $U$  be an open subset of  $\mathbb{R}$  and  $f : U \rightarrow \mathbb{R}$  be a function so that  $f^{(n+1)}(x)$  exists for all  $x \in U$ . Suppose  $a \in U$ ,  $0 = f'(a) = \dots = f^{(n)}(a)$  and  $f^{(n+1)}(a) \neq 0$ . Then

- (a) If  $n$  is even,  $f$  has neither a local maximum nor a local minimum at  $a$ ;
- (b) If  $n$  is odd and  $f^{(n+1)}(a) > 0$ , then  $f$  has a local minimum at  $a$ ;
- (c) If  $n$  is odd and  $f^{(n+1)}(a) < 0$ , then  $f$  has a local maximum at  $a$ .

(Use Exercise 28 and induction.)

- (35) Suppose that  $f, g : (a, b) \rightarrow \mathbb{R}$  are  $n$  times differentiable on  $(a, b)$ . If  $u \in (a, b)$ ,  $f^{(k)}(u) = 0 = g^{(k)}(u)$ ,  $0 \leq k \leq n-1$ ,  $g^{(n)}(x) \neq 0$  for  $x \neq u$ ,  $0 \leq k \leq n$ , and

$$\lim_{x \rightarrow u} \frac{f^{(n)}(x)}{g^{(n)}(x)} = L,$$

show that  $\lim_{x \rightarrow u} \frac{f(x)}{g(x)} = L$ .

- (36) Suppose that  $F : (a, b) \rightarrow \mathbb{R}$  is  $n$  times differentiable on  $(a, b)$ , and that  $F^{(n)}$  is continuous at  $u \in (a, b)$ . Apply the previous exercise with

$$f(x) = F(x) - \sum_{k=0}^n \frac{F^{(k)}(u)}{k!} (x-u)^k$$

and  $g(x) = (x-u)^n$  to show that

$$F(x) = \sum_{k=0}^n \frac{F^{(k)}(u)}{k!} (x-u)^k + \varepsilon(x) |x-u|^n,$$

where  $\varepsilon(x) \rightarrow 0$  as  $x \rightarrow u$ .