Exercises for §6

(43) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = (x^2 + y^2, x^2 - y^2).$$

(a) Find all points (x_0, y_0) where $Df(x_0, y_0)$ is invertible.
(b) For each point (x_0, y_0) found in part (a), find an open set V containing (x_0, y_0) so that f^{-1} exists on $f(V)$. Write a formula for f^{-1} explicitly.
(c) Compute Df^{-1} on $f(V)$ using the formula in (b) and check it against part (3) of the Inverse Function Theorem.

(44) Suppose that for each $i = 1, \ldots, n$, a continuously differentiable function $\gamma_i : (-1, 1) \to \mathbb{R}^n$ is given. Also assume that $\gamma_1(0) = \cdots = \gamma_n(0) = 0$ and that the set of vectors $\{D\gamma_i(0)(1) : 1 \le i \le n\}$ is linearly independent. Show that there is an open set V in $\mathbb{R}^n$ containing 0 so that $f(V)$ is open and $\mathbb{R}^n$ and every $w \in f(V)$ can be expressed uniquely in the form $\gamma_1(x_1) + \cdots + \gamma_n(x_n)$ for some $(x_1, \ldots, x_n) \in V$.
(Inverse Function Theorem.)
(The result can be visualized pictorially as follows. Think of γ_i, $1 \le i \le n$, as curves passing through the point 0. The problem says that n "linearly independent" curves passing through 0 can be used to form a "coordinate system" in a small open set about 0, where every point $b = \gamma_1(x_1) + \cdots + \gamma_n(x_n)$ is given the coordinates $(x_1, \ldots, x_n)$.)

(45) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$f(x, y, z) = (6x + 6y + 4z^3 - 44, -x^2 - y^2 + 8z - 14).$$

Show that there is an open interval I containing 1 and differentiable functions $g : I \to \mathbb{R}$ and $h : I \to R$ so that $g(1) = 1$, $h(1) = 2$ and $f(g(y), y, h(y)) = 0$ for all $y \in Y$. Find $g'(1)$ and $h'(1)$. (Use the Chain Rule.)

Solve the next two exercises using the method of Lagrange multipliers.

(46) Find the maximum and minimum values of $2x + 3y + z$ subject to $x^2 + 2y^2 + 3z^2 = 1$. Justify why the values found are indeed the maximum and minimum values.

(47) Find the minimum distance between a point lying on the ellipse $x^2 + 2y^2 = 1$ and the line $x + y = 4$. Justify your answer.