(1) An open subset $U$ of $\mathbb{C}$ is said to be *star-shaped* if there exists $z_0 \in U$ such that the interval $[z_0, z]$ lies in $U$ for all $z \in U$. Show that if $f$ is analytic on a star-shaped set $U$, then $f$ has an antiderivative on $U$.

(2) For each pair $(f, U)$ given below, determine if $f$ has an antiderivative on $U$ and justify your answer.

(a) $f(z) = \frac{1}{z^2 + 1}$, $U = \mathbb{C} \setminus \{iy : y \in \mathbb{R}, y \leq 1\}$.
(b) $f(z) = \cot z$,
$$U = \{z : \text{Im } z < 0\} \cup \{z : 0 < |\text{Arg}(z + 2\pi i) - \frac{\pi}{2}| < \frac{\pi}{4}\}.$$
(c) $f(z) = \frac{1}{z^2 - 3z + 2}$, $U = \{z : 1 < |z| < 2\}$.

(3) Compute the following integrals.

(a) $\int_{\gamma} \frac{z^2 + 2}{z(z-2)} \, dz$; $\gamma(t) = e^{it}$, $t \in [-\pi, \pi]$.
(b) $\int_{\gamma} \frac{1}{z^2 + 1} \, dz$; $\gamma(t) = 1 + i + 2e^{it}$, $t \in [-\pi, \pi]$.
(c) $\int_{\gamma} \frac{\tan z}{(\pi - 2t)^2} \, dz$; $\gamma(t) = e^{it}$, $t \in [-\pi, \pi]$.

(4) Let $f$ be an entire function. Suppose that there are a natural number $n$ and positive real constants $C$ and $R$ so that $|f(z)| \leq C|z|^n$ for all $z$ with $|z| \geq R$. Show that $f$ is a polynomial of degree at most $n$.

[Show that the appropriate derivative of $f$ is zero. See the proof of Liouville’s Theorem.]

(5) Let $f$ be a complex function that is analytic on an open ball $B(0, r)$ and continuous on the set $B = \{z : |z| \leq r\}$.

(a) Show that for all $\varepsilon > 0$, there exists $\delta > 0$ so that whenever $r - \delta < s \leq r$ and $t \in [-\pi, \pi]$, we have $|f(se^{it}) - f(re^{it})| < \varepsilon$.

[This is a consequence of the uniform continuity of $f$. Prove it by contradiction using the Bolzano-Weierstrass Theorem.]

(b) Show that $\int_{\gamma} f(z) \, dz = 0$, where $\gamma(t) = re^{it}$, $t \in [-\pi, \pi]$.

(6) The purpose of this exercise is to give a proof of the following version of the *Maximum Modulus Principle*:

Suppose that $f$ is analytic on $B(z_0, r)$ and that $r > R > 0$. Let $M = \sup\{|f(z)| : |z - z_0| = R\}$. Then $|f(z)| \leq M$ for all $z \in B(z_0, R)$.

(a) For any $a \in B(0, 1)$, define $m_a(z) = \frac{z-a}{1-\overline{a}z}$. Show that $m_a \circ m_{-a}(z) = m_{-a} \circ m_a(z) = z$ whenever $|z| \leq 1$ and that $|m_a(z)| = 1$ whenever $|z| = 1$. Deduce that $m_a$ maps the set $\{|z| : |z| = 1\}$ one-one onto itself, and maps the set $B(0, 1)$ one-one onto itself.
(b) If $f$ is analytic on $B(0, r)$ for some $r > 1$, show that $|f(0)| \leq \sup\{|f(z)| : |z| = 1\}$. By composing $f$ with the appropriate function from part (a), deduce that

$$|f(a)| \leq \sup\{|f(z)| : |z| = 1\} \text{ whenever } |a| < 1.$$ 

[The first assertion follows from the Cauchy Integral Formula.]

(c) Prove the Maximum Modulus Principle as stated above.