Exercises for §7

(48) In each of the following cases, determine if $f \in \mathcal{R}(g,[0,1])$ using either the definition or the Riemann integrability criterion. Find $\int_a^b f\,dg$ if it exists.
  (a) $f(x) = 1$ if x is rational and $f(x) = 0$ otherwise; $g(x) = x$.
  (b) $f(x) = x$;
$$g(x) = \begin{cases} \frac{-1}{2^n} & \text{if } x \in [\frac{n-1}{n}, \frac{n}{n+1}) \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } x = 1. \end{cases}$$
  (c) $f(x)$ is defined arbitrarily for $x \in [0, 1/3]$ and $f(x) = 0$ for $x \in (1/3, 0]$; the same function g as in part (b).

(49) This exercise is concerned with the converse to part (3) of Proposition 31. Let $a, b \in \mathbb{R}$ with $a < b$ and suppose that g is a real-valued function on $[a, b]$. Assume that $f \in \mathcal{R}(g,[a,c]) \cap \mathcal{R}(g,[c,b])$.
  (a) Show that if either (i) f is continuous at c and g is bounded, or (ii) f is bounded and g is continuous at c, then $f \in \mathcal{R}(g,[a,b])$.
  (b) Give an example to show that in general, f does not have to belong to $\mathcal{R}(g,[a,c])$.

(50) (a) Give an example of a function f so that f is not Riemann integrable on an interval $[a, b]$ but $|f|$ is.
  (b) If g is increasing on $[a, b]$, f is bounded and $f \in \mathcal{R}(g,[a,b])$, show that $|f| \in \mathcal{R}(g;[a,b])$.

(51) Suppose that f and g are real-valued functions on $[a, b]$. If f and g are both increasing and have no common discontinuities, show that $f \in \mathcal{R}(g,[a,b])$. (The result holds if the word "increasing" is replaced by "monotone", meaning either increasing or decreasing. The "monotone" case follows from the "increasing" case by parts (1) and (2) of Proposition 31.)

(52) Suppose that $c < a < b < d$, $g : (c, d) \to \mathbb{R}$ is continuously differentiable on (c, d). If $f : [a, b] \to \mathbb{R}$ is bounded and $fg' \in \mathcal{R}[a, b]$, show that $f \in \mathcal{R}(g,[a,b])$.
(Use the Mean value theorem plus apply Q8 to g'.)