(1) Which of the following functions are harmonic on $\mathbb{C}$? Find a harmonic conjugate of each of the harmonic functions.

$$x^2 + y^2; \quad x^2 - y^2; \quad e^{-x} \cos y.$$ 

(2) Here is another proof of the existence of the harmonic conjugate that avoids differentiating under the integral sign.

**Theorem.** Let $U$ be an open subset of $\mathbb{C}$ so that every analytic function on $U$ has an antiderivative on $U$. Then every harmonic function on $U$ is the real part of an analytic function on $U$.

**Step 1.** Let $u$ be harmonic on $U$. Define $f = u_x - iu_y$. Show that $f$ is analytic on $U$.

**Step 2.** Let $F$ be an antiderivative of $f$ on $U$. Write $F = U + iV$. Show that $u + iV$ is analytic on $U$.

[Note that the Theorem applies, in particular, if $U$ is convex or star-shaped.]

(3) (a) If $f$ is analytic on an open set $U$ and $u$ is harmonic on an open set $V \supseteq f(U)$, show that $u \circ f$ is harmonic on $U$.

(b) Find all functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi \circ u$ is harmonic for all harmonic functions $u : \mathbb{C} \to \mathbb{R}$.

(4) Suppose that $u$ is harmonic on $\mathbb{C}$ and that there exists $M \in \mathbb{R}$ with $u(z) \leq M$ for all $z \in \mathbb{C}$. Show that $u$ is constant.

[Consider exp$(f)$, where $u = \text{Re} f$.]

(5) Suppose that $f$ is analytic on $B(0, 1)$ and continuous on $\overline{B(0, 1)}$. Let $u = \text{Re} f$.

(a) Show that for all $z \in B(0, 1)$

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} u(w) \frac{w + z}{w - z} \frac{dw}{w} + i \text{Im} f(0),$$

$$= \frac{1}{\pi i} \int_{\gamma} u(w) \frac{dw}{w - z} \overline{f(0)},$$

where $\gamma(t) = e^{it}, \ t \in [-\pi, \pi]$.

(b) Deduce a formula for $f'(z)$ similar to the second one in part (a).

(6) The Poisson Integral Formula for the upper half plane can be obtained from the Poisson Integral Formula for the unit ball via a conformal (i.e., analytic) transformation.

(a) Show that the map $\varphi(z) = -\frac{z - 1}{z + 1}$ maps $B(0, 1)$ one-one onto the upper half plane $\{w \in \mathbb{C} : \text{Im} w > 0\}$ and the deleted unit circle $\{z : |z| = 1, z \neq -1\}$ one-one onto the $x$-axis.
(b) Suppose that $U$ is harmonic on the open upper half plane $\{w \in \mathbb{C} : \text{Im } w > 0\}$, continuous on the closed upper half plane $\{w \in \mathbb{C} : \text{Im } w \geq 0\}$, and that $\lim_{|w| \to \infty} U(w)$ exists. (Here the limit is taken through $w$ in the closed upper half plane and the value of the limit is meant to be a real number.) Show that for all $a + ib, b > 0$,

$$U(a + ib) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{b u(x) \, dx}{(b - x)^2 + a^2}. $$

[The function $u = U \circ \varphi$ is harmonic on $B(0, 1)$ and, with $u(-1)$ suitably defined, continuous on $\overline{B}(0, 1)$. Apply the Poisson Integral Formula to $u$. (Use the real part of the formula in Q5(a).) Make a change of variable in the integral to change the integration from the unit circle to the $x$-axis.]