Exercises for §12

(65) Evaluate $\int_0^1 \int_{1-y}^1 \ln(1+x^2)\,dx\,dy$. Justify your steps.

(66) By considering "elliptic coordinates" (r,θ), where $x = ar\cos\theta, y = br\sin\theta$ for some $a, b > 0$, evaluate the integral

$$\int_0^1 \int_0^{\frac{1}{3}\sqrt{1-x^2}} \sqrt{x^2+9y^2}\,dy\,dx.$$

Justify your steps.

(67) Justify the following manipulations. Let $I = \lim_{a\to\infty} \int_{-a}^a e^{-x^2}\,dx$. Then

$$\begin{aligned} I^2 &= \lim_{a\to\infty} \int_{-a}^a \int_{-a}^a e^{-x^2-y^2}\,dx\,dy \\ &= \lim_{r\to\infty} \int_{D_r} e^{-x^2-y^2}\,dA = \pi, \end{aligned}$$

where $D_r = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le r^2\}$.

The next series of exercises show what can go wrong if we relax some conditions in Fubini's theorem.

(68) Let $R = [0,1]\times[0,1]$ and define $f : R \to \mathbb{R}$ by $f(x,0) = 1$ if x is rational, $f(x,y) = 0$ otherwise. Show that f is Riemann integrable on R but the function f_0 defined by $f_0(x) = f(x,0)$ is not Riemann integrable on $[0,1]$.

(69) Let $R = [0,1]\times[0,1]$ and define $f : R \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 2^{2n} & \text{if } \frac{1}{2^{n+1}} < x, y < \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ -2^{2n+1} & \text{if } \frac{1}{2^{n+2}} < x < \frac{1}{2^{n+1}} \text{ and} \\ & \quad \frac{1}{2^{n+1}} < y < \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ both exist but take different values.

(70) Let $R = [0,1]\times[0,1]$ and define $f : R \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 2^{n} & \text{if } \frac{1}{2^{n+1}} \le x, y < \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ -2^{n+1} & \text{if } \frac{1}{2^{n+2}} \le x < \frac{1}{2^{n+1}} \text{ and} \\ & \quad \frac{1}{2^{n+1}} \le y < \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ both exist and are equal, but that f is not integrable on $\mathbb{R}$.