

# Rigid Bracings of a Grid

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## ABSTRACT

Rigidity theory is a field of mathematics that can be applied to architectural designs involving specific classes of structures, of which one common example includes scaffoldings. The structure of scaffolding can be braced by beams and rods to withstand the stress of transporting workers and equipment. In reality, there are several factors to consider when bracing such a structure. The type of material used in making the scaffolding and its braces and its ability to withstand stress, and the tightness of the joints between braces and structure are just a few to name. The study of rigidity aims to investigate another factor, namely the methods of bracing such structures and the designs of such bracings, and as a good starting point, grids are perhaps the most similar geometric structure. This research paper investigates this problem using braces that span 2 cells of the grid rather than the standard length brace that spans only 1 cell, and attempts to develop a set of mathematical rules governing the rigidity of such bracings.

## DEFINITIONS

### Frameworks

A  $k$ -dimensional framework  $(V, E, \mathbf{p})$  consists of a graph  $(V, E)$  (known as the *structure graph*) and a function  $\mathbf{p}$  (known as the *embedding function*) from the vertex set into  $k$ -space,  $\mathbf{p}: V \rightarrow \mathbb{R}^k$ , where  $\mathbf{p}(a_i) = \mathbf{p}_i$ . We assume that  $\mathbf{p}_i$  span the  $k$ -space. As this paper deals only with  $m \times n$  grids, which are 2-dimensional frameworks, we shall take the value of  $k$  to be 2 henceforth. In effect, the function  $\mathbf{p}$  “embeds” the grid into the familiar Cartesian plane by assigning each point of the grid  $a_i$ , a position vector  $\mathbf{p}_i$ .

### Motions

An infinitesimal motion of a framework can be defined as a function  $\mathbf{q}: V \rightarrow \mathbb{R}^k$  where each  $\mathbf{q}_i$  is the motion vector assigned to the point  $a_i$ . A brace between 2 points  $a_i$  and  $a_j$  imposes the constraint that the distance between them must remain constant. We then have our first condition on the motion and position vectors of  $a_i$  and  $a_j$ :

$$\mathbf{q}_i \cdot (\mathbf{p}_i - \mathbf{p}_j) = \mathbf{q}_j \cdot (\mathbf{p}_i - \mathbf{p}_j) \quad (1)$$

Thus, the projection of the 2 motion vectors  $\mathbf{q}_i$  and  $\mathbf{q}_j$  onto the brace must be equal. Furthermore, if (1) holds true for all points of the grid, we say that the motion  $\mathbf{q}$  is infinitesimally rigid, otherwise we say that  $\mathbf{q}$  is an infinitesimal deformation.

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Elementary Line Motions Motions in which the points on one line move in one direction along the line, and all other points remain fixed, are defined to be elementary line motions.

Minimal Bracing Set of a Grid By virtue of the linearity of (1), the set  $M$  of all possible infinitesimal motions of a grid forms a vector space whose dimension is equal to the number of degrees of freedom of the grid which can be shown to be  $m + n + 2$  for any  $m \times n$  grid. Subtracting the 3 degrees of external freedom attributed to translations and rotation of the grid, we get  $m + n - 1$  degrees of internal freedom. Since each brace removes at most 1 degree of freedom, we need at least  $m + n - 1$  properly placed braces to rigidify a grid. We also have the following theorem.

Theorem 1 The line motions form a basis for the motion space  $M$  of any  $m \times n$  grid.

Shear Equation (1) allows us to define any motion of the grid in terms of line motions along its horizontal and vertical line segments. These are denoted by shear vectors, which are motions that are only horizontal or vertical, taking the direction of the positive  $x$ - and  $y$ -axes to be positive. For convention, we shall always fix the upper-leftmost point of the grid in our investigations, and denote the magnitude and direction of shear vectors relative to this point by a relative shear number. It can be shown that the orientation of a brace in its cell is irrelevant to our study; the same constraints are imposed on the shear vectors no matter its orientation. Furthermore, the resultant shear of a 2-hall can be shown to be simply the sum of the shears of its constituent 1-halls.

#### Associated Bipartite Graphs

Assuming that we label the points of the  $m \times n$  grid  $r_0, \dots, r_m$  row-wise and  $c_0, \dots, c_n$  column-wise, we can associate a bipartite graph  $(V_R, V_C)$  with each grid whose vertices represent its vertical and horizontal segments respectively. We label each vertex in  $V_R$  as either  $r_i r_{i+1}$  or  $r_i r_{i+2}$  and similarly,  $c_j c_{j+1}$  or  $c_j c_{j+2}$  in  $V_C$ . The vertex  $r_i r_{i+1}$  is adjacent to  $c_j c_{j+2}$  if and only if there exists a brace in the cell  $[r_i r_{i+1}, c_j c_{j+2}]$ . We call this brace a *horizontal brace* if the cell is of the form  $[r_i r_{i+1}, c_j c_{j+2}]$  and a *vertical brace* if the cell is of the form  $[r_i r_{i+2}, c_j c_{j+1}]$ . This greatly expedites our study as we can now assign relative shear numbers to the vertices and express the deformations of the grid by means of such number assignments and investigate the relations between these numbers. We have the following theorem relating such a bipartite graph with its grid.

Theorem 2 A grid braced with  $1 \times 2$  braces is infinitesimally rigid if the associated bipartite graph is  $1 \times 2$ -connected.

### **THE BRACING PROBLEM**

Although theorem 2 is analogous to the solution of the bracing problem using short braces, the current problem is much less straightforward. Before our study of the  $2 \times n$  grids, we need to establish the parallelogramic, triangular and cyclic connectivities/conditions that result from our use of  $1 \times 2$  braces. These conditions essentially state that whenever specific structures of braces are found in the grid, there are halls other than those directly braced that are also braced themselves, hence implying the existence of implicit braces within those halls. Hence, in our study, we

need not consider bracings where an actual brace resides in a hall that is already implicitly braced as such a brace is redundant and a minimal set of braces cannot admit redundancy. We now define a few more useful entities before presenting the results.

Partitions and Adjacency If a column-2-hall  $c_i c_{i+2}$  does not contain any horizontal brace,  $c_{i+1}$  is defined to be a partition. 2 partitions A and B are adjacent if there does not exist some partition in that lie between A and B.

Grid Components A grid component is defined to be the collection of halls from a partition to an adjacent partition.

## **INFINITESIMAL RIGIDITY OF $2 \times n$ GRIDS**

After examining many cases of  $2 \times n$  grids (up to the  $2 \times 5$  case), the following results hold true in general for bracings of size  $m + n - 1$  (a minimal set).

Result 1 A  $2 \times n$  grid with no vertical braces is not infinitesimally rigid. There are  $n - 1$  column-2-halls in such a grid. With  $n + 1$  braces, there is necessarily a redundant brace by the cyclic condition, hence such a set of braces cannot rigidify the grid.

Result 2 A  $2 \times n$  grid with 1 vertical brace and  $n$  horizontal braces is infinitesimally rigid. Discounting all obvious cases involving redundant braces by the cyclic condition, we are left with only the bracings where all  $n - 1$  column-2-halls are braced. With 1 horizontal and vertical brace left, application of the parallelogramic and triangular conditions to insert implicit braces quickly yields the result that every column-1-hall and row-1-hall is braced, and thus the associated bipartite graph is  $1 \times 2$ -connected and by theorem 2, the grid is infinitesimally rigid.

Result 3 A grid component which does not contain a vertical brace is not infinitesimally rigid. A non-trivial shear numbering can be assigned to the associated bipartite graph of such a grid component regardless of whether the neighboring grid components are infinitesimally rigid or not, thus the grid cannot be infinitesimally rigid.

Result 4 A grid is infinitesimally rigid if there exists an infinitesimally rigid grid component and the other grid components each have at least 1 vertical. This result is proven in the paper by exhausting all possible configurations of bracings subjected to the given condition.

## **CONCLUSION**

Although only the solution regarding  $2 \times n$  grids is presented here without the capability to state a general solution to the  $m \times n$  case, this is nevertheless a start. Furthermore, it is also easily shown after examining the above solutions that for an  $m \times n$  grid, discounting redundant braces due to the above-mentioned 3 conditions, any bracing set with exactly 1 vertical brace will rigidify the grid.

## REFERENCES

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