Undergraduate Research Opportunity Programme in Science

RIGID BRACINGS OF A GRID

Abstract

LAI YONG CHIENG MARVIN and A/P TAY TIONG SENG

Department of Mathematics

National University of Singapore

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Introduction

Rigidity theory is a body of mathematics developed to aid in designing structures for example scaffolding which is constructed by bolting together rods and beams. This project examines the application of graph theory to rigid bracings of a 2 x n grid by analyzing the number of 1 x 2 braces needed to make a grid rigid, and trying to form ideas and theories as to how a grid can be made rigid.

Definitions:

1. Infinitesimal Rigidity

A k-dimensional framework $(V,E,\mathbf{p})$ consists of a graph (V,E) (known as the *structure graph*) and a function **p** (known as the *embedding function*) from the vertex set into *k*-space, $\mathbf{p}:V \to \Re^k$, where $\mathbf{p}(a_i)=\mathbf{p}_i$. Define an infinitesimal motion of the framework to be the function $\mathbf{q}:V \to \Re^k$ whereby $\mathbf{q}_i$ is a motion vector assigned to the point $\mathbf{p}_i$ (and thus, the point a_i).

The motion **q** is called an **infinitesimal rigid motion** if $(\mathbf{p}_i - \mathbf{p}_j).(\mathbf{q}_i - \mathbf{q}_j)=0$ for all a_i,a_j e V. (Infinitesimal rigid motions are just the rotations and translations of the framework). A framework $(V,E,\mathbf{p})$ is said to be **infinitesimally rigid** if all of its infinitesimal motions are infinitesimal rigid motions.

2. Bipartite Representation

The bipartite of a grid illustrates how the braces are formed in the grid. The vertices of the bipartite graph are divided into 2 sets, the horizontal segments c_ic_j and vertical segments r_ir_j. 2 vertices are adjacent if and only if there is a 1 x 2 brace joining a column to a row. The vertices coloured white represent segments of 1 unit length and the black coloured vertices represent segments of 2 units length.

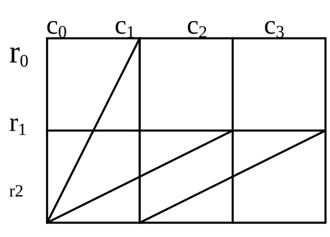

Fig 2.1
Grid representation

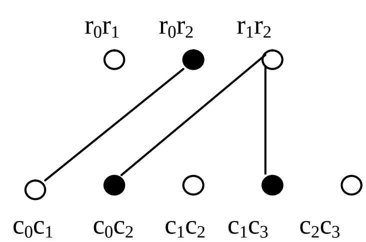

Fig 2.2
Bipartite representation

3. Shear Number

This is a number we assign to each vertex in the bipartite graph, representing the magnitude of the motion vector of each vertical or horizontal segment of the grid. The constraints of these numbers are such:

(i) shear number of a black vertex is equal to the sum of the shear numbers of the 2 white vertices that it is between.

(ii) 2 adjacent vertices have the same shear number.

The 2 constraints serve useful in determining if a grid with 1 x 2 bracings is infinitesimally rigid. A grid is infinitesimally rigid if and only if **all** the vertices of its bipartite graph have shear number zero. This brings us to our theorem.

Theorem: A grid braced with 1 x 2 braces is infinitesimally rigid if the associated bipartite graph is 1 x 2 connected.

Clearly if a bipartite graph is 1 x 2 connected, for any 2 vertices there exists a path joining them. If we were to assign shear number to the vertices, all the vertices would have to have shear number zero.
So we see that if we can prove that a certain bracing of a grid renders its bipartite graph to be 1 x 2 connected, we can conclude that such a bracing is infinitesimally rigid.

Some results:

(i) for any m x n graph, at least m+n-1 braces are needed to ensure infinitesimal rigidity

(ii) triangular connectivity

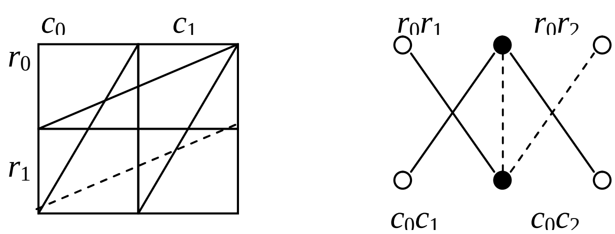

Figure 3: A 2×2 grid with 1×2 braces exhibiting triangular connectivity and its associated bipartite graph

(iii) Parallelogramic connectivity

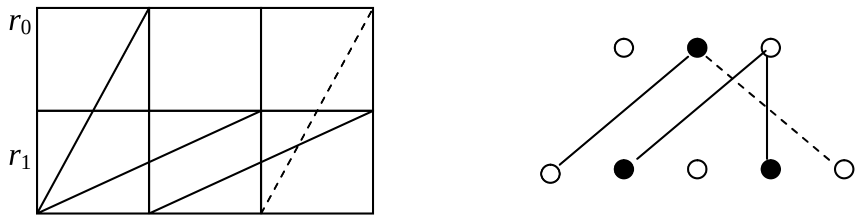

Figure4: A 2×3 grid with 1×2 braces exhibiting Parallelogramic connectivity and its associated bipartite graph.

(iv) Cyclic connectivity

For any 2 x n grid, we start with m+n-1 braces. Using the results, we can filter out any instances of (ii), (iii) and (iv). All these cases would not be infinitesimal as at least one of the braces is implicit, thereby leaving less than m+n-1 braces for the grid.

Conclusion

For our project, we managed to generalize the results for a 2 x n grid as follows.

- A $2 \times n$ grid with no vertical braces is not infinitesimally rigid.
- A $2\times n$ grid with 1 vertical brace and n horizontal braces is infinitesimally rigid
- A grid component which does not contain a vertical brace is not infinitesimally rigid.
- A graph is infinitesimally rigid if there exists an infinitesimally rigid grid component and the other grid components each have at least 1 vertical.

References

1. Ethan D. Bolker and Henry Crapo, "*Bracing Rectangular Frameworks, I*", SIAM Journal of Applied Mathematics Volume 36, Issue 3, 1979.
2. Jack E. Graver, *Graduate Studies in Mathematics Volume 2: Combinatorial Rigidity*, American Mathematical Society, 1991.
3. Jack E. Graver, *Counting on Frameworks*, The Mathematical Association of America, 2001.
4. Tan Weiyu Colin and Tay Tiong Seng, *On the Bracings of $m \times n$ Grids*, SRP Report 2001.