Gale's Vingt-et-en

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ABSTRACT

David Gale is a professor emeritus of mathematics at *The University of California, Berkeley*. From 1991 to 1997, he served as an associate editor of *The Mathematical Intelligencer*[3] in charge of the "Mathematical Entertainments" column. Among his of specialization are game theory, geometry and combinatorics. Other famous problems of his include the "*Bridg-It"* and "*Subset Take-Away*". He won the prestigious *Lester R. Ford Award*[4] in 1980 for "*The game of Hex and the Brouwer fixed-point theorem* (*Amer. Math. Monthly* 86 (1979), 818-827)"

"Gale's Vingt-et-en" is a famous unsolved problem in combinatorics. It is named after the French mathematician who invented it, David Gale[1] (vingt-et-en means 21 in French). Using two Qbasic programs, the two variables in the game were varied and the results plotted on tables.

Gale's VINGT-ET-EN

Cards numbered 1 through 10 are laid on the table. P1 chooses a card. Then P2 chooses cards until his total of chosen cards exceeds the card chosen by P1. Then P1 chooses until his cumulative total exceeds that of P2, etc. The first player to get 21 wins. Who is it? The rule can be interpreted to mean either "21 exactly" or "21 or more". Jeffery Magnoli, a student of Julian West, thought the second interpretation was more interesting, and found a first-player win in six-card onze(eleven) and in eight-card dix-sept(seven-teen).

In this project, the winning limit was interpreted as $\geq m$, that is, in Gale's Vingt-et-en, $m=21$ and hence the winning limit is "21 or more". Our analysis was also based on the assumption that both players play optimally.

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TERMINOLOGY USED

In this project, the number of playing cards is denoted by ***n*** and the winning limits by ***m***. For example n=10, m=21 is exactly the case for Gale's vingt-et-en.

RESULTS

For ***n***=10, ***m***=21, the unique winning card is 1. For $n \geq 9$, P2 will always win regardless of what card P1 chooses to start with when the winning limit is set as $m = n + 7$. For $n \geq 14$, P2 will always win regardless of what card P1 chooses to start with when the winning limit is set as $m = n + 17$. For $n \geq 13$, P2 will always win regardless of what card P1 chooses to start with when the winning limit is set as $m = n + 20$. Other winning limits for P2 appear quite randomly and do not follow a fixed pattern. We also found that if ***m*** is the minimum limit that results in a draw, then the limit $m+ k$ always results in a draw for all positive integers k.

The cases for ***k*** from 1-8 were analyzed and for ***k*** from 1-6, the winning cards were found to be unique and they appear to be quite random. For a fixed ***k*** though, the winning card was found to be identical as ***n*** varies. For ***k***= 7, P1 will still be able to win for small ***n***. For sufficiently large ***n***, P2 will win regardless of which card P1 chooses to start with. For ***k***= 8, the winning card for ***n***= 8 is different from the winning card for $n > 8$. This is contrary to what was observed for ***k*** from 1-6. Using the Qbasic program to extrapolate, we see more examples of such phenomena as ***n*** increases. However, we also note that the winning card is unique, at least up to the data we have considered so far.

SOME PROPOSED CONJECTURES

Uniqueness of Winning Card

So far, we have not come across any case with non-unique winning card. There is reason to believe that the winning card is unique.

The Draw Formula

The data tells us that the minimum limit ***m*** which results in a draw is slightly greater than half of the initial total value of the cards on the table ($=\frac{n(n+1)}{2}$). It is postulated that m can be calculated from the formula below, called the ***Draw Formula***:

$$m = \left\lfloor \frac{n(n+1)}{4} + \frac{1}{2} \right\rfloor + 1$$

To prove that the postulated ***Draw Formula*** is indeed correct is equivalent to proving this:

(n, m) is a draw $\Rightarrow$ ($n+4$, $m+2n+5$) is a draw

If the ***Draw Formula*** for n cards is equal to m, then the ***Draw Formula*** for n+4 cards is m+2n+5.

SOME INTERESTING PHENOMENA TO NOTE

For some n, together with the minimum draw limit, m, P1 can choose to play the card labeled n as his starting card and still be able to force a draw. For other n, there is a maximum value he can choose his starting card to be. For small n, it is easy to see why one cannot choose a higher valued card to start with. We observe that the maximum value of the starting card appears to be quite random and not directly dependent on n or m. Also, $(h-m)_i - (h-m)_{i-1}$ appear to follow a step function, where h= total value of cards, m= minimum draw limit.

SUGGESTIONS FOR FURTHER RESEARCH

Larger data size

The computers used to generate the data for analysis were just personal computers and hence took a long time to churn out data, which explains why the data size is so small. Subject to the availability of the resources, the computer program can be set up to run on a server and left there to generate data by itself thus increasing the data size.

Verify and if possible prove the postulates

Verify and if possible, prove the postulates by considering more data.

Account for the random winning limits

Try to account for the random winning limits for P2 (n =9, m = 22 is one example). Also, we note that for the limit $m = n + k$, $1 \le k \le 6$, the winning card for P1 is the same for all n but for k = 7, P2 will actually win for a minimum n. For $k \ge 8$, we see that the winning card is different for different n but tends to settle down to a fixed winning card as n gets large. Having a larger data size might allow us to see some patterns not visible on our data set.

Establish if the winning card is unique

The computer program only searches for a winning card then it stops. We can modify it to search for all winning cards (if any) and try to establish if the winning card for each (n, m) is unique.

Try to establish a best winning strategy

Based on our data, we postulate that P1 should avoid choosing to start with the higher valued card in order not to lose. Given more data, we can try establish a best winning strategy or a best counteracting strategy, and if possible try to come up with a formula to generate winning cards.

Explore the other possible meaning of the rule (the limit being m exactly)

As mentioned at the beginning, there are two possible interpretations of the rule. We worked on the "21 or more" option. We can work on the "21 exactly" option and see if there are any generalizations to be made.

Try to prove P1 wins for $m = n + k$ for $k \geq 9$.

We have only considered the cases for $1 \leq k \leq 8$. We can consider more k and try to make generalizations if any.

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