Undergraduate Research Opportunity Programme in Science

Gale's Vingt-et-un

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ABSTRACT

In the book "Games of No Chance" MSRI Volume 29, 1996, Richard K. Guy listed down saying -et-un (or Gale's 21) is one of the unsolved problems in combinatorial games. As such, a number of people have tried to solve this problem, but not much documentation of the investigation of this game has been done. In this project, we tried to investigate the game to see whether there is a certain strategy that the players can follow in order to win, or that this is a neutral game. We later found out that the game is too easy, and that it is biased towards the first player. Thus we varied the parameters that then led us to a higher level of investigation and to obtain a deeper understanding of the game.

RULES OF THE ORIGINAL GAME

Cards numbered 1 thru 10 are laid on the table. P1 chooses a card. Then P2 chooses cards until his total of chosen cards exceeds the card chosen by P1. Then P1 chooses until his cumulative total exceeds that of P2 etc. The first player to get 21 wins. Who is it?(The rules can be interpreted to mean either '21 exactly' or '21 or more'.)

INVESTIGATION

Before we begin our investigation, let us define some terms and conventions.
Definition:
In the notion **(n, m)** used, n represents the number of cards from 1 to n, and m represents the total points that each player has to exceed in order to win. For the rules of the game, we will stick to the

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second interpretation in this project (namely to have exactly the number of points or exceed it for a player to win).

The Original Game: Gale's-21

In this project, we analyzed the game Gal's 21 through the exhaustive method by listing down all the moves that player 1 can make and subsequent moves that player 2 can make. In our investigation, we found out that when both players play with an optimal strategy, player 1 can win only if his first move is by picking 1. If not, player 2 will always have a move to cause player 1 to lose subsequently. However no general or elegant pattern can be observed in our analysis.

The General Game (n, m)

When we analyse the results for general cases (n, m), where m <= n
→ Player 1 wins, by taking m as the first card. (obvious).
When we analyse the results for general cases in the form (n,n+k), where k=1,2,3…… by using exhaustive means, we found out that:
When k=1, player 1 wins. The winning first move for player 1 is 1.
When k=2, player 1 wins. The winning first move for player 1 is 2
When k=3, player 1 wins. The winning first move for player 1 is 1.
When k=4, player 1 wins. The winning first move for player 1 is 1
When k=5, player 1 wins. The winning first move for player 1 is 2
When k=6, player 1 wins. The winning first move for player 1 is 2
When k=7, player 1 wins if n = 7 or 8. The winning first move for player 1 is 3. Player 2 wins if n>8.
When k=8, player 1 wins. The winning first move for player 1 is 3 when n=8, and 1 when n>8.
Simple explanations are made also as to why n has to be larger than a certain value before we can have a general case.

THEOREMS AND CONJECTURES

Theorem: If (n, m) is a draw (if both players play optimally), (n, m+1) is a draw.
Conjecture: The first draw that occurs for (n, m) is when m = [n(n+1)/4 + 0.5] + 1, where [x] denotes the largest integer smaller or equal to x.

SUGGESTIONS FOR FURTHER RESEARCH

These are also some of the hypothesis we have come up, but so far, due to constraints, we are not yet able to prove them.

Hypothesis 1: When (n, m) is a draw, (n + 4, 2n + m + 5) is also a draw.
Hypothesis 2: When n → infinity, the player who wins at (n, m) is the same player who wins at (n+1, m+1)
Hypothesis 3: The occurrence of player 2 winning a certain (n, m) is almost random.

CONCLUSION

In this project, we tried to investigate on the actual game of Gale's Vingt-et-un, and found it not as complex as we first thought. Through some logical reasoning and exhaustive search algorithm, with results verified by computers, we managed to find out that player 1 wins the game if he plays optimally. As such, we went on to investigate the games with parameters (n, m) in general, and to try to formulate a sure win method for either player 1 or player 2, or to see a general trend in which the occurrence of P2 winning for a certain (n, m) or when it will turn out to be draw.

REFERENCES

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