

Some Problems on Linear Preservers

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INTRODUCTION

Let $\mathbf{M}(\mathcal{F})$ be a space of matrices over the field \mathcal{F} and $T: \mathbf{M}(\mathcal{F}) \rightarrow \mathbf{M}(\mathcal{F})$ be a linear operator. A common problem considered in linear algebra is called a preserver problem. That is, characterize the linear operators which preserve a function or a set. We say T preserves a function $f: \mathbf{M}(\mathcal{F}) \rightarrow \mathcal{F}$ if $f(\mathbf{A}) = f(T(\mathbf{A}))$ for all $\mathbf{A} \in \mathbf{M}(\mathcal{F})$. We say T preserves a subset $\mathcal{K} \in \mathbf{M}(\mathcal{F})$ if $T(\mathbf{A}) \in \mathcal{K}$ for all $\mathbf{A} \in \mathcal{K}$. In this report, we will review the history of linear preserver, and give two new theorem on determinant preservers.

LINEAR RANK-1 PRESERVERS ON $\mathbf{M}_n(\mathbb{C})$

Linear Rank-1 Preservers

Let T be a linear operator on $\mathbf{M}_n(\mathbb{C})$. T is called a rank-1 preserver if $\text{rank}(\mathbf{A}) = 1$ whenever $\text{rank}(T(\mathbf{A})) = 1$. In 1959, Marcus and Moyls characterized the general form of it: *Suppose T is a linear rank-1 preserver on $\mathbf{M}_n(\mathbb{C})$. Then there exist invertible matrices \mathbf{P} and \mathbf{Q} , such that either $T(\mathbf{A}) = \mathbf{P}\mathbf{A}\mathbf{Q}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$, or $T(\mathbf{A}) = \mathbf{P}\mathbf{A}^T\mathbf{Q}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$.* This is one of the most powerful theorem on preserver problems, and we will use it to prove some interesting results of linear preservers on $\mathbf{M}_n(\mathbb{C})$

Linear Determinant Preservers

Let T be a linear operator on $\mathbf{M}_n(\mathbb{C})$. T is called a determinant preserver if $\det(\mathbf{A}) = \det(T(\mathbf{A}))$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$. In fact, this is the first problem on preserver problems, which is proved by Ferdinand Georg Frobenius (1849-1917) on 1897. He proved that: *Let T be a linear determinant preserver on $\mathbf{M}_n(\mathbb{C})$. Then there exist invertible matrix \mathbf{P} and \mathbf{Q} , with $\det(\mathbf{P}\mathbf{Q}) = 1$, such that either $T(\mathbf{A}) = \mathbf{P}\mathbf{A}\mathbf{Q}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$, or $T(\mathbf{A}) = \mathbf{P}\mathbf{A}^T\mathbf{Q}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$.* In this section, we claimed that a linear determinant is also a rank-1 preserver, and then proved it by using the theorem of Marcus and Moyls on linear rank-1 preserver. We also proved that a linear operator T on $\mathbf{M}_n(\mathbb{C})$

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is a determinant-trace preserver if and only if it is a characteristic polynomial preserver. Here, the linear operator T is called a determinant-trace preserver if \mathbf{A} and $T(\mathbf{A})$ have the same determinant and trace; and it is called a characteristic polynomial preserver if \mathbf{A} and $T(\mathbf{A})$ have the same characteristic polynomial. Frobenius is also the first mathematician who proved this theorem: *Suppose T is a linear characteristic polynomial preserver on $\mathbf{M}_n(\mathbb{C})$. Then there exists an invertible matrix \mathbf{P} such that, either $T(\mathbf{A}) = \mathbf{PAP}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$, or $T(\mathbf{A}) = \mathbf{PA}^T \mathbf{P}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$.*

Linear Preservers of Nonnegative Matrices

The theorem of linear rank-1 preserver can be used for classification on some linear preserver problem. In this section, we introduced two of them:

1. *Suppose T is a linear determinant-trace preserver on $\mathbf{M}_n(\mathbb{C})$ which maps nonnegative matrices into nonnegative matrices. Then there exists a generalized permutation matrix \mathbf{P} such that either $T(\mathbf{A}) = \mathbf{PAP}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$, or $T(\mathbf{A}) = \mathbf{PA}^T \mathbf{P}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$.*
2. *Suppose T is a linear determinant-trace preserver on $\mathbf{M}_n(\mathbb{C})$ which maps nonnegative integer matrices into nonnegative integer matrices. Then there exists a permutation matrix \mathbf{P} such that either $T(\mathbf{A}) = \mathbf{PAP}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$, or $T(\mathbf{A}) = \mathbf{PA}^T \mathbf{P}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$.*

DETERMINANT PRESERVERS WITH $\det(\mathbf{A} + \lambda\mathbf{B}) = \det(T(\mathbf{A}) + \lambda T(\mathbf{B}))$

Determinant Preservers on $\mathbf{M}_n(\mathbb{C})$ with $\det(\mathbf{A} + \lambda\mathbf{B}) = \det(T(\mathbf{A}) + \lambda T(\mathbf{B}))$

In the theorem of linear determinant preservers by Frobenius, it requires that the operator T is linear. However, the linearity is very strong as we use it heavily throughout the proof. An interesting result is that “can we replace the ‘linearity’ by a weaker condition?” In 2002, Dolinar and Šemrl proved that *if T is a surjective operator on $\mathbf{M}_n(\mathbb{C})$ satisfying $\det(\mathbf{A} + \lambda\mathbf{B}) = \det(T(\mathbf{A}) + \lambda T(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathbf{M}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$, then T is linear.* In this section, we weaken this condition again. We proved that we can remove that “surjective” assumption. That is, *if T is an operator on $\mathbf{M}_n(\mathbb{C})$ satisfying $\det(\mathbf{A} + \lambda\mathbf{B}) = \det(T(\mathbf{A}) + \lambda T(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathbf{M}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$, then T is linear.*

Consequences of Theorem 2.3

In the last section, we showed that the property $\det(\mathbf{A} + \lambda\mathbf{B}) = \det(T(\mathbf{A}) + \lambda T(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathbf{M}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$ is equivalent to that “ T is linear and preserves the determinant”. Therefore, by using the Frobenius’ theorem on linear determinant preserver stated in Chapter 1, we immediately can get some interesting consequences:

1. Let T be an operator on $\mathbf{M}_n(\mathbb{C})$ satisfying $\det(\mathbf{A} + \lambda\mathbf{B}) = \det(T(\mathbf{A}) + \lambda T(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathbf{M}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Suppose T is a trace preserver. Then there exists an invertible matrix \mathbf{P} such that, either $T(\mathbf{A}) = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$, or $T(\mathbf{A}) = \mathbf{P}\mathbf{A}^T\mathbf{P}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$.
2. Let T be an operator on $\mathbf{M}_n(\mathbb{C})$ satisfying $\det(\mathbf{A} + \lambda\mathbf{B}) = \det(T(\mathbf{A}) + \lambda T(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathbf{M}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Suppose T is a trace preserver, and T maps nonnegative matrices into nonnegative matrices. Then there exists a generalized permutation matrix \mathbf{P} such that either $T(\mathbf{A}) = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$, or $T(\mathbf{A}) = \mathbf{P}\mathbf{A}^T\mathbf{P}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$.
3. Let T be an operator on $\mathbf{M}_n(\mathbb{C})$ satisfying $\det(\mathbf{A} + \lambda\mathbf{B}) = \det(T(\mathbf{A}) + \lambda T(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathbf{M}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Suppose T is a trace preserver, and T maps nonnegative integer matrices into nonnegative integer matrices. Then there exists a permutation matrix \mathbf{P} such that either $T(\mathbf{A}) = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$, or $T(\mathbf{A}) = \mathbf{P}\mathbf{A}^T\mathbf{P}^{-1}$ for all $\mathbf{A} \in \mathbf{M}_n(\mathbb{C})$.

Determinant Preservers on \mathbb{U}_n with $\det(\mathbf{A} + \lambda\mathbf{B}) = \det(T(\mathbf{A}) + \lambda T(\mathbf{B}))$

It is known that if T is linear determinant preserver on \mathbb{U}_n , then $[T(\mathbf{A})]_{ii} = c_i(\mathbf{A})_{\sigma(i)\sigma(i)}$ $i = 1, \dots, n$ for some scalar c_1, \dots, c_n with $\prod_{i=1}^n c_i = 1$ and permutation σ of $\{1, \dots, n\}$. In this section, we generalized the theorem, and showed that if T is an operator on \mathbb{U}_n satisfying $\det(\mathbf{A} + \lambda\mathbf{B}) = \det(T(\mathbf{A}) + \lambda T(\mathbf{B}))$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{U}_n$ and $\lambda \in \mathbb{C}$, then there exists scalar c_1, \dots, c_n with $\prod_{i=1}^n c_i = 1$ and permutation σ of $\{1, \dots, n\}$ such that $[T(\mathbf{A})]_{ii} = c_i(\mathbf{A})_{\sigma(i)\sigma(i)}$ for all $i = 1, \dots, n$.

LINEAR RANK-1 PRESERVERS ON \mathcal{H}_n

Some Preliminary Lemmas

Not only for $\mathbf{M}_n(\mathbb{C})$, we are also interested in classify the general form of linear preservers on some other space. For example, \mathcal{H}_n , the space of hermitian matrices. In 1986, Johnson and Pierce proved that, if the linear invertible operation T is a rank-1 preserver on \mathcal{H}_n , then either $T(\mathbf{A}) = \varepsilon \mathbf{S}\mathbf{A}\mathbf{S}^H$, $\mathbf{A} \in \mathcal{H}_n$ or $T(\mathbf{A}) = \varepsilon \mathbf{S}\mathbf{A}^T\mathbf{S}^H$, $\mathbf{A} \in \mathcal{H}_n$ for some invertible \mathbf{S} and $\varepsilon \in \{1, -1\}$. In this section, we proved some preliminary lemmas.

Linear Rank-1 Preservers on \mathcal{H}_n

In this section, we generalized the result by Johnson and Pierce: *let T be a linear rank-1 preserver on \mathcal{H}_n . Suppose there is a hermitian matrix whose image is invertible. Then there exists an invertible matrix S and $\varepsilon \in \{1, -1\}$ such that either $T(A) = \varepsilon SAS^H$ for all $A \in \mathcal{H}_n$ or $T(A) = \varepsilon SA^T S^H$ for all $A \in \mathcal{H}_n$.* Then, we use this result to get a more general result, which is first proven by Raphael Loewy in 1987: *let T be a linear rank-1 preserver on \mathcal{H}_n with $\text{rank}(T) \geq 2$. Then there exists an invertible matrix S and $\varepsilon \in \{1, -1\}$ such that either $T(A) = \varepsilon SAS^H$ for all $A \in \mathcal{H}_n$ or $T(A) = \varepsilon SA^T S^H$ for all $A \in \mathcal{H}_n$.*

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