$$\Leftrightarrow J(x) \cap J(y) \neq \emptyset.$$

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All Interval Graphs are triangulated

Proof: Assume that there exists an interval that is not triangulated. This implies that there we can create a cycle of length greater than 3 which does not contain a chord.
There are only a few ways to construct an interval representation of a P_3. Let the 3 vertices be a, b and c, with b being the vertex that is connected to both a and c.

J(a) J(c)
J(b)

J(a) J(c)
J(b)

J(a) J(c)
J(b)

Without loss of generality, these are the only 3 interval representations of a P_3.
In the latter two cases, any interval that overlaps with J(c) will also overlap with J(b). Thus the vertex it represents will be adjacent to b. In the first case, since there is a path from c to a, one of the intervals representing this path must overlap with J(b) and hencethere is a chord as well. In order to create a true 4-cycle, an interval (or a series of them for a chordless cycle of length greater than 4) has to be created that overlaps J(a) and J(c) but not J(b). From the representations above, we see that it is not possible. Hence, there is no chordless cycle that is an interval graph, which implies that all interval graphs are triangulated.

DIRAC'S THEOREM

A graph is triangulated iff every minimal vertex separator of G is a clique.

Necessity: Let the graph G be triangulated and S be a minimal separator of G. Let G_A and G_B be 2 distinct components of G\S. Since S is a minimal separator, every vertex x in S must be adjacent to some vertex of G_A and some vertex of G_B. Hence, for **any** pair x, y in S, there exist paths P_1: $xa_1 \ldots a_r y$ and P_2: $xb_1 \ldots b_s y$ where each a_i º $V(G_A)$ and each b_i º $V(G_B)$. Assuming also that P_1 and P_2 are chosen to be of the shortest length, $xa_1 \ldots a_r y b_s \ldots b_1 x$ is a cycle of length at least 4, and so (as G is triangulated) must contain a chord. However, as P_1 and P_2 are chosen to be of the shortest length, the chord must be xy. Thus, **every** pair x, y in S are adjacent and S is a clique.

Sufficiency: We now have to prove that if every minimal separator of G is a clique, every cycle of length at least 4 in G contains a chord. Assume that every minimal separator of G is a clique. Let $axby_1y_2 \ldots y_r a$ be a cycle C of length ≥ 4 in G. If ab were not a chord of C, denote by S a minimal seperator that puts a and b in distinct components of G\S.Then S must contain x and

y_j for some j. By hypothesis, S is a clique, and hence xy_j is an edge of G, and therefore a chord in C. Thus, G is triangulated.

DIRAC'S CHARACTERIZATION

Every triangulated graph G has a simplicial vertex. Moreover, if G is not complete, it has 2 nonadjacent simplicial vertices

Proof: If G is either complete or has just 2 or 3 vertices, the lemma is trivial.
Thus, we assume that G is not complete. We shall prove the lemma by induction.
Assume that the lemma is true for all graphs with fewer vertices than G. Let S be a minimal ab-separator, and let G_A and G_B be components of G\S containing a and b, respectively, and with vertex sets A and B respectively. By the induction hypothesis, if
G [A U S] is not complete; it has 2 nonadjacent simplicial vertices. This way, since G[S] is complete; at least one of the 2 simplicial vertices must be in A. Such a vertex is simplicial in G because none of its neighbours is in B. Furthermore, if G[A U S] is complete, then any vertex of A is a simplicial vertex of G. Thus, in both cases, we see that there exists at least one simplicial vertex in A. Using the same argument, we can see that there exist also at least one simplicial vertex in B. Hence, as A is disconnected with B in G\S, we see that if G is not a complete triangulated graph , it has at least 2 non-adjacent simplicial vertices.

MOPLEXES

A **module** is a subset A of V (the vertex set of a graph) such that for all a_i and a_j in A, $N(a_i) \cap N(A) = N(a_j) \cap N(A) = N(A)$, i.e. every vertex of N(A) is adjacent to every vertex in A.
A single vertex is a *trivial* module. For a module that is a clique, all its neighbours are adjacent to every single vertex in the clique itself.

$A \subset V$ is a maximal clique module if and only if A is **both a module and a clique**, and A is **maximal** for both properties.

A **moplex** is a maximal clique module whose neighbourhood is a minimal separator. A moplex is *simplicial* iff its neighbourhood is a clique, and it is *trivial* iff it has only 1 vertex.

MOPLEX THEOREM 1

Any non-clique triangulated graph has at least 2 non-adjacent simplicial moplexes.

Special case: When N = 3, the only connected non-clique graph is a P_3 (path) of vertices, in order, a, b and c. There are 3 moplexes in this graph; b is the minimal separator, but a and c are 2 *trivial* moplexes.
Let G be a non-clique triangulated graph. Assume that the theorem is true for non-clique triangulated graphs. Let S be a minimal separator of G which is a clique by Dirac's Theorem. Let also A and B be 2 full components of CC(S).

Case 1: If $A \cup S$ is a clique, N(A) = S. This implies that A is both a module and a clique. For any $x \in S$, $A \cup \{x\}$ is not a module. For any $y \notin A \cup S$, $A \cup \{y\}$ is not a clique. Therefore, A is a maximal clique module

Case 2*:* If $A \cup S$ is not a clique, by induction hypothesis, $A \cup S$ has 2 non-adjacent moplexes. If each of these 2 moplexes are inclusive of vertices in both A and S, they will be adjacent because S is a clique, which is a contradiction. Hence, one of the moplexes (we call M) is contained in A. Thus, N(M) is a minimal separator in $A \cup S$. This implies that N(M) is also a minimal separator in G. Hence, M is a moplex in G. In either case, there is simplicial moplex which is contained in A. Similarly, there is also such a moplex contained in B.

MOPLEX THEOREM 2

A graph is triangulated iff one can repeatedly delete a simplicial moplex until the graph is a clique (i.e. there exists a 'perfect simplicial moplex elimination scheme')

Necessity*:* Let G be a triangulated graph. There exist 2 non-adjacent simplicial moplexes in G by theorem 3.5. Removing one of these 2 moplexes (call the removed moplex M), G\M is still a triangulated graph. By continuously doing so, we will obtain a clique.

Sufficiency*:* Any vertex in M is simplicial by property 3.4. Hence, a simplicial moplex elimination scheme is similar to a perfect vertex elimination scheme. By theorem 2.4, we can conclude that every graph with a simplicial moplex elimination scheme is a triangulated graph.

GENERALIZATION OF DIRAC'S THEOREM TO ANY GRAPH

LEMMA 1

Let H be a minimal triangulation of G and A be a moplex of H. Then $N_H(A) = N_G(A)$

Let A be a moplex of H and $a \in A$. It is easily seen that $N_G(A) \subseteq N_H(A)$. Assume that $N_G(A) \neq N_H(A)$. Consider a vertex z in $N_H(A)$ but not in $N_G(A)$. Since H is a *minimal* triangulation of G, by the unique chord property, az is the unique chord of some 4-cycle in H: axzya. However, since the neighborhood of A is a clique by definition, x must already be adjacent to y for any x, $y \in N_H(A)$, and hence, az cannot be the unique chord. Therefore, by contradiction, $N_G(A) = N_H(A)$.

LEMMA 2

Let H = (V, E + F) be a minimal triangulation of G = (V, E). If A is a moplex of H, then A is a moplex of G.

Let A be a moplex of H. let N(A) be the neighbourhood of A. Note that $N(A) = N_G(A) = N_H(A)$. $A \cup N(A)$ is a clique of H. All we have to do now is to show that A is also a moplex of G. Suppose $\exists$ a, $b \in A$ such that $a \notin N_G(b)$. Then ab must belong to the minimal fill-in F, so with the unique chord property, ab must be the unique chord of some 4-cycle axbya of H.

However, in H, x is adjacent to y since they are neighbors of a and A ∪ N(A) is a clique. ab cannot be the unique chord of axbya. Hence, by contradiction, **A is a clique of G.**

Assume A is not a module of G. ∃z in N(A) such that z is not adjacent to a of A in G. This edge az must then be in the minimal fill-in, which gives another contradiction because of the unique chord property. Thus, **A is a module of G.**

If s ∉ N(A), A ∪ {s} is not a clique. If s ∈ N(A), s is adjacent to some vertex in B, where B is the other full component of N(A); but the moplex containing A cannot be adjacent to a B, which gives rise to a contradiction; Thus, **A is maximal.**

MOPLEX GENERALIZATION TO ANY GRAPH

<u>Any non-clique graph G has at least 2 non-adjacent moplexes.</u>

Let the triangulation of G be H. By theorem 3.5, H contains at least 2 non-adjacent simplicial moplexes. By theorem 4.2 above, we know that a moplex of H is also a moplex of G. Hence, we conclude that any non-clique graph has at least 2 non-adjacent moplexes.

REFERNCES

1. R. Balakrishnan, K. Ranganathan.1999.A textbook of Graph Theory. New York, Springer
2. Anne Berry, J-P Bordat. 1996. Separability generalizes Dirac’s
3. Hans L. Bodlaender, Ton Kloks, Dieter Kratsch, Haiko Muller. 1998. Journal of Graph Algorithms