The Riemann Integral Revisited

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ABSTRACT

The Riemann integral is well-known. Here, we study it using an approach that is slightly different from traditional ones. We begin by introducing an integral, originally discovered by Henstock and Kurzweil, and show how the Riemann integral follows as a special case of this integral, which we shall call the Henstock integral. We further illustrate this by re-defining the Riemann integral in terms of the Henstock integral. We then use this integral to study, from the perspective of Henstock integration, some properties of the Riemann integral.

A BRIEF HISTORY OF THE INTEGRAL

The Riemann theory of integration is perhaps the most familiar, as it is the first theory of integration taught to students. The simplicity and intuitive definition of the integral — formulated in the 1850s by Bernhard Riemann — is perhaps one reason. The various applications of this integral in many physical problems also made its study an imperative even among non-mathematicians.

Yet, it is well-known that the Riemann integral is not sufficient for mathematical purposes. In particular, the class of Riemann integrable functions is severely restricted (Bartle and Sherbert, 2000). This led to the definition of many other integrals in an attempt to remedy the shortcomings of the Riemann integral.

By far the most successful definition was that due to Lebesgue. Lee and Výborný (2000) noted that the Lebesgue integral is the suitable integral for almost all mathematical uses. However, they also noted that it is not without its limitations: in 1914, for instance, Perron defined yet another integral which included, among others, improper integrals that are not Lebesgue integrals.

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A few decades later, an elementary definition of an integral, equivalent to the Perron definition (see Gordon (1994) for a proof of this), was discovered independently by Kurzweil, in the 1950s, and Henstock, in the 1960s. This integral is what we shall call here the Henstock integral, although it is known by many other names, such as the Kurzweil-Henstock integral, the generalised Riemann integral (Bartle, 2001), or the gauge integral (Swartz, 2001).

Names like the "Riemann-complete" integral, under which this integral was first introduced, or the "generalised Riemann" integral, serve to highlight the foundations of this integral. It is this close relationship between the Riemann and Henstock integrals that concerns us here.

THE RIEMANN AND HENSTOCK INTEGRALS

We define, briefly, the two integrals that will be used here, namely the Riemann and Henstock integrals. Both definitions of the integral require the notion of dividing an interval in $\mathbb{R}$ into subintervals, which we explain here.

Given a closed and bounded interval I in $\mathbb{R}$, we can subdivide I into non-overlapping subintervals

$$P = \{I_i\}_{i=1}^{n},$$

with $I_i = [u_i, v_i]$. P is then known as a partition of I.

This partition P may then be equipped with tags $\{\xi_i\}_{i=1}^{n}$, where ξ_i is a point chosen from the subinterval I_i. We call this tagged partition a division. The division D is also associated with a value

$$\|D\| = \max\{|v_1 - u_1|, |v_2 - u_2|, \cdots, |v_n - u_n|\},$$

which is known as the norm of the division.

The Riemann integral of a function $f : I \to \mathbb{R}$ is then defined as the number A such that, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that, for any division of I,

$$D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n},$$

with $\|D\| < \delta$, we have

$$|S(f, D, \delta) - A| < \varepsilon,$$

where

$$S(f, D, \delta) = \sum_{i=1}^{n} f(\xi_i)(v_i - u_i)$$

is the Riemann sum, defined on the division D, of the function f. We remark that the tags ξ_i may be arbitrarily chosen from any point in $[u_i, v_i]$.

The Henstock definition of the integral departs from the Riemann definition in not using the norm of the division. Rather, a strictly positive function $\delta : I \to \mathbb{R}^+$, called a gauge, is defined. A division is then said to be δ-fine if the condition

$$\xi_i \in [u_i, v_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$$

is satisfied. We note that the nature of the division, as defined here, is different from the division used in the Riemann definition, since the choice of the tags ξ_i now determines the subintervals $[u_i, v_i]$.

The Henstock integral of a function f may then be defined to be the number A such that, for all $\varepsilon > 0$, there exists a gauge $\delta > 0$ such that, for any δ-fine division

$$D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n},$$

we have

$$|S(f, D, \delta) - A| < \varepsilon.$$

where

$$S(f, D, \delta) = \sum_{i=1}^{n} f(\xi_i)(v_i - u_i)$$

is, again, the Riemann sum of the function, which is defined here on the δ-fine division D.

This apparently minor departure from the definition of the Riemann integral, however, allows the Henstock integral to integrate a much larger class of functions than its predecessor. We illustrate this by computing the Henstock integral of a function that is known to be not Riemann integrable.

THE HENSTOCK–RIEMANN INTEGRAL

We next define an integral, which we shall call the Henstock-Riemann integral (or the HR-integral).

Following the definition of the Henstock integral, the HR-integral is defined to be the number A such that for all $\varepsilon > 0$, there exists a constant gauge $\delta > 0$ such that for any δ-fine division,

$$D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n},$$

we have

$$|S(f, D, \delta) - A| < \varepsilon.$$

The equivalence of the HR-integral with the Riemann integral, which we prove, then allows us to study some properties of the Riemann integral within the framework of Henstock integration. These include the Cauchy

Criterion for the HR-integral; the Saks–Henstock Lemma, which is a useful result in Henstock integration, adapted for the HR-integral; and, a condition for HR-integrability.

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