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The Riemann Integral Revisited

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Contents

1 Introduction

The Riemann integral is perhaps the most familiar integral. Formulated in the 1850s by Bernhard Riemann, the Riemann integral is, without doubt, the first integral — if not the only integral, for most non-mathematicians — to be introduced in the study of the calculus. This is due in part to its intuitive formulation as the "area under the curve", and its many applications outside of mathematics.

Yet, it is well-known that the Riemann integral is not sufficient for mathematical purposes. As Bartle[1] points out, the class of Riemann-integrable functions is severely limited. This led to the definition of many other integrals in an attempt to remedy the shortcomings of the Riemann integral.

By far the most successful integral was that formulated by Henri Lebesgue. Lee and Výborný[4] noted that the Lebesgue integral is the suitable integral for almost all mathematical uses. However, they also noted that it is not without its limitations: in 1914, for instance, Perron defined yet another integral which included, among others, improper integrals that are not Lebesgue integrals.

Yet, the Perron integral shares a similar drawback with the Lebesgue integral: both are notoriously difficult to define without a level of mathematical maturity substantially beyond undergraduate analysis.

The Henstock integral, which we shall introduce shortly, requires no such level of mathematical maturity, since it uses concepts very similar to those used in the Riemann integral. Discovered independently by Kurzweil, in the 1950s, and Henstock, in the 1960s, it was quickly recognized to be as powerful as the Lebesgue integral, if not more so.

Subsequently, the Henstock integral — known variously as the Kurzweil-Henstock integral, the generalised Riemann integral, or the gauge integral — was found to be equivalent to the Perron integral (a proof of this is given in Gordon[3]). For a Riemann-type integral of such simplicity in definition to possess such power, this is both surprising and exciting indeed.

Here, however, our concern will be in the comparison of the Riemann integral with the Henstock integral. To do so, we shall define an integral, equivalent to the Riemann integral, in the framework of Henstock integration. This integral shall aid us in studying some properties and limitations of the Riemann integral from the perspective of Henstock integration.

2 The Riemann and Henstock Integrals

We begin with an introduction to the Riemann and Henstock integrals. Next, we shall define an integral, equivalent to the Riemann integral, using concepts due to Henstock integration. We shall prove this equivalence and use this integral in the subsequent sections.

2.1 The Riemann Integral

Traditionally, there are two methods of defining the Riemann integral: using Riemann sums or using the Darboux upper and lower sums. We shall adopt the first approach here.

First, we describe the construction of a partition of an interval.

Definition 2.1.1. Let $I = [a, b]$ be a closed and bounded interval in $\mathbb{R}$. Let $\mathcal{A} = (x_0, x_1, \cdots, x_{n-1}, x_n)$ be a finite ordered set of points in I such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and these points of $\mathcal{A}$ divide I into non-overlapping closed subintervals

$$I_1 = [x_0, x_1],$$

$$I_2 = [x_1, x_2],$$

$$\vdots$$

$$I_n = [x_{n-1}, x_n].$$

Then, $P = \{[x_{i-1}, x_i]\}_{i=1}^n$ is then said to be a **partition** of I.

Remark 2.1.1. This is an useful definition, but we need not restrict I to be a contiguous closed and bounded interval. If $I = \bigcup_{j=1}^m I_j$ is a union of disjoint intervals that are closed and bounded, we may define

$$\mathcal{A} = (u_1, v_1, u_2, v_2 \cdots, u_n, v_n)$$

such that these points of $\mathcal{A}$ divide I into non-overlapping closed subintervals. Then $P = \{[u_i, v_i]\}_{i=1}^n$ is also a partition of I. We shall adopt this notation, which is convenient in the definition of partitions for more general cases.

Definition 2.1.2. A **tag** of a subinterval I_i is a point ξ_i chosen from the subinterval I_i. The set of ordered pairs of the subintervals of a partition P and the associated tags,

$$D = \{(I_i, \xi_i)\}_{i=1}^n,$$

is called a **division** of I.

To highlight the structure of the subintervals, we may write

$$D = \{([u_i, v_i], \xi_i)\}_{i=1}^n.$$

If D' is a division of $J \subseteq I$, we call it a **partial division** of I.

Next, we define a way to measure the "fineness" of the division we have constructed on the interval.

Definition 2.1.3. The **norm** (or **mesh**) of a division D is defined to be

$$\|D\| = \max\{|I_1|, |I_2|, \cdots, |I_n|\},$$

where $|I_i| = |v_i - u_i|$.

We now define the Riemann sum of a function associated with a division.

Definition 2.1.4. The **Riemann sum** of a function $f : I \to \mathbb{R}$, corresponding to the division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ with $\|D\| < \delta$, is the number

$$S(f, D, \delta) = \sum_{i=1}^{n} f(\xi_i)(v_i - u_i).$$

Sometimes, we may also denote the Riemann sum by

$$\sum_D f(\xi_i)(v_i - u_i).$$

We now have all that is needed to define the Riemann integral of a function $f : I \to \mathbb{R}$ on I.

Definition 2.1.5. A function $f : I \to \mathbb{R}$ is **Riemann integrable** on I, and its **Riemann integral** equal to the number A, if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for any division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of I, with $\|D\| < \delta$,

$$|S(f, D, \delta) - A| < \varepsilon.$$

We note that the choice of ξ_i from the subinterval $[u_i, v_i]$ is arbitrary.

Certainly, the Riemann integral may also be written as $\int_a^b f$, if $I = [a, b]$, and $\int_I f$ if I is a union of closed and bounded intervals.

2.2 The Henstock Integral

The definition of the Henstock integral is very similar to that of the Riemann integral. The variation made in the Henstock integral is that δ now becomes a function of the tags of a division, instead of a constant. As δ is allowed to vary, this also changes the way we define the division and measure its "fineness".

First, we define this function δ, known as a gauge.

Definition 2.2.1. A **gauge** δ on I is defined to be a strictly positive function $\delta : I \to \mathbb{R}^+$.

Next, we modify the way we measure how "fine" the division is.

Definition 2.2.2. If a division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ satisfies the condition that

$$\xi_i \in [u_i, v_i] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)),$$

then it is said to be **δ-fine**.

Remark 2.2.1. The Riemann sum of a function for a corresponding δ-fine division D is identical to that in Definition 2.1.4, except that the condition that $\|D\| < \delta$ is replaced by the requirement that D be δ-fine. We shall employ the same notation for both types of Riemann sums.

Definition 2.2.3. A function $f : I \to \mathbb{R}$ is **Henstock integrable** on I, and its **Henstock integral** equal to the number A, if for all $\varepsilon > 0$, there exists a gauge $\delta > 0$ such that for any δ-fine division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of I,

$$|S(f, D, \delta) - A| < \varepsilon.$$

We now show that any Riemann-integrable function also has a Henstock integral with the same value, following Bartle and Sherbert[2].

Theorem 2.2.1. *If f is Riemann integrable on I, with integral A, then f is also Henstock integrable on I with the same integral.*

Proof. Let f be Riemann integrable on I, and A be the Riemann integral of f on I. Then, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ with $\|D\| < \delta$, then

$$|S(f, D, \delta) - A| < \varepsilon.$$

Define the gauge $\delta'(\xi) = \delta/2$ for all $\xi \in I$. If we let $D' = \{([u_i, v_i], \xi_i^{'})\}_{i=1}^n$ be any δ'-fine division on I, then we have, for all i,

$$[u_i, v_i] \subset (\xi_i^{'} - \delta'(\xi_i^{'}), \xi_i^{'} + \delta'(\xi_i^{'})) = (\xi_i^{'} - \delta/2, \xi_i^{'} + \delta/2),$$

and therefore

$$|v_i - u_i| < \delta.$$

Hence, D' is also a division with $\|D'\| < \delta$ and

$$|S(f, D', \delta) - A| < \varepsilon.$$

Consequently, every δ'-fine division $\mathcal{D}$ also satisfies $|S(f, \mathcal{D}, \delta) - A| < \varepsilon$, and we have shown that f is Henstock integrable with integral equal to A. □

However, the Henstock integral can also integrate functions that are not Riemann integrable.

Example 2.2.1. The Dirichlet function f on $[0, 1]$, defined by

$$f(x) = \begin{cases} 1 & x \in [0,1] \text{ and } x \text{ is rational} \\ 0 & x \in [0,1] \text{ and } x \text{ is irrational} \end{cases}$$

is Henstock integrable with integral 0, but not Riemann integrable.

Proof. The proof follows Bartle and Sherbert[2]. Since the rational numbers in $[0, 1]$ are denumerable, we may list them as a sequence $\{r_i\}_{i=1}^{\infty}$. For any $\varepsilon > 0$, we define a gauge δ on $[0, 1]$ such that

$$\delta(x) = \begin{cases} \varepsilon/2^{i+2} & x = r_i \\ 1 & x \text{ is irrational} \end{cases}$$

If a division $D = \{([u_j, v_j], \xi_j)\}_{j=1}^n$ is δ-fine, then $|v_j - u_j| \leq 2\delta(\xi_j)$. It is clear that if ξ_j is irrational, $f(\xi_j)(v_j - u_j) = 0$. For $\xi_j = r_i$,

$$0 < f(r_i)(v_j - u_j) \leq 1 \cdot \frac{2\varepsilon}{2^{i+2}} = \frac{\varepsilon}{2^{i+1}}$$

Since each tag may occur in two subintervals, but not more, we have

$$0 \leq S(f, D, \delta) < \sum_{i=1}^{\infty} \frac{2\varepsilon}{2^{i+1}} = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Therefore, f is Henstock integrable with $\int_0^1 f = 0$.

However, f is not Riemann integrable. Let $\varepsilon = \frac{1}{2}$. If D is any division, with $\|D\| < \delta$, such that all the tags are irrational, then $S(f, D, \delta) = 0 < \varepsilon$. On the other hand, if D' is any division, with $\|D'\| < \delta$, such that all the tags are rational, then $S(f, D', \delta) = 1 > \varepsilon$. Thus, by Definition 2.1.5, f does not have a Riemann integral. □

2.3 The Henstock–Riemann Integral

We have seen that the Riemann integral is contained within the Henstock integral. We demonstrate this further by defining an integral within the framework of Henstock integration, and showing that this integral is equivalent to the Riemann integral.

Definition 2.3.1. $f : I \to \mathbb{R}$ is **Henstock–Riemann integrable** (**HR-integrable**) on I, and its **Henstock–Riemann integral** (**HR-integral**) equal to the number A, if for all $\varepsilon > 0$, there exists a constant gauge $\delta(\xi) > 0$ for $\xi \in I$ such that for any δ-fine division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$

$$|S(f, D, \delta) - A| < \varepsilon.$$

We now show that this integral is equivalent to the Riemann integral.

Theorem 2.3.1. $f : I \to \mathbb{R}$ *is Riemann integrable if and only if* f *is also HR-integrable.*

Proof. ($\Rightarrow$) This is similar to the proof of Theorem 2.2.1.

($\Leftarrow$) Conversely, suppose f is HR-integrable on I, and A be the HR-integral of f on I. Then, given any $\varepsilon > 0$, there exists a constant gauge $\delta(\xi)$ such that for any δ-fine division $D = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ of I,

$$|S(f, D, \delta) - A| < \varepsilon.$$

Let $D' = \{([u_i, v_i], \xi_i^{'})\}_{i=1}^n$ be any division of I with $\|D'\| < \delta$, that is, for any i, $|v_i - u_i| < \delta$. Since $\xi_i^{'}$ may be chosen from any point in $[u_i, v_i]$, we have

$$|\xi_i^{'} - u_i| < \delta \quad \text{and} \quad |\xi_i^{'} - v_i| < \delta.$$

It follows that, for any $\xi_i^{'} \in [u_i, v_i]$,

$$\xi_i^{'} \in [u_i, v_i] \subset (\xi_i^{'} - \delta(\xi_i^{'}), \xi_i^{'} + \delta(\xi_i^{'})),$$

so D' is also a δ-fine division. Consequently, every division $\mathcal{D}$ with $\|\mathcal{D}\| < \delta$ is also a δ-fine division and we have

$$|S(f, \mathcal{D}, \delta) - A| < \varepsilon.$$

Therefore, f is Riemann integrable on I with integral A. $\square$

We now prove that no unbounded function is HR-integrable, a result adapted from the Riemann integral.

Theorem 2.3.2. *If $f : [a,b] \to \mathbb{R}$ is HR-integrable, then f is bounded on $[a,b]$.*

Proof. Suppose f is HR-integrable and unbounded on $[a,b]$. Let $\varepsilon = 1$. Then there exists a $\delta > 0$ such that if D is any δ-fine division of $[a,b]$,

$$|S(f,D,\delta) - A| < 1$$

or

$$|S(f,D,\delta)| < |A| + 1.$$

We note that since f, and therefore $|f|$, is unbounded on $[a,b]$, then for any partition $\mathcal{P} = \{[u_i,v_i]\}_{i=1}^{n}$, there is at least one subinterval in $\mathcal{P}$ such that $|f|$ is unbounded on this subinterval.

We now construct a δ-fine division $D' = \{([u_i,v_i],\xi_i)\}_{i=1}^{n}$ such that $|f|$ is unbounded only on one subinterval $[u_k,v_k]$, for some $k = 1,2,\ldots,n$, and $\xi_k \in [u_k,v_k]$ is chosen such that

$$\Big|f(\xi_k)(v_k - u_k)\Big| > |A| + 1 + \left|\sum_{i\neq k} f(\xi_i)(v_i - u_i)\right|.$$

By the triangle inequality, it follows that

$$|S(f,D',\delta)| \geq \Big|f(\xi_k)(v_k - u_k)\Big| - \left|\sum_{i\neq k} f(\xi_i)(v_i - u_i)\right| > |A| + 1$$

contradicting the assumption that f is HR-integrable. $\square$

3 Some Properties of the HR-integral

We now study some properties of the HR-integral. These results, many of which are known for the Riemann integral, have been adapted for the HR-integral.

3.1 The Cauchy Criterion

We shall first show that any HR-integrable function satisfies the Cauchy Criterion. The proof is adapted from the proof of the Cauchy Criterion for the Riemann integral in Bartle and Sherbert[2].

Theorem 3.1.1. (Cauchy Criterion) *A function $f : I \to \mathbb{R}$ is HR-integrable over I if and only if given $\varepsilon > 0$, there exists a constant $\delta > 0$ such that if D and D' are any δ-fine divisions of I, then*

$$|S(f, D, \delta) - S(f, D', \delta)| < \varepsilon$$

Proof. ($\Rightarrow$) Given f is HR-integrable over I with integral A. Let D and D' be δ-fine divisions such that

$$|S(f, D, \delta) - A| < \frac{\varepsilon}{2}$$

and

$$|S(f, D', \delta) - A| < \frac{\varepsilon}{2}.$$

It follows that

$$\begin{aligned} |S(f, D, \delta) - S(f, D', \delta)| &= |S(f, D, \delta) - A + A - S(f, D', \delta)| \\ &\leq |S(f, D, \delta) - A| + |S(f, D', \delta) - A| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

($\Leftarrow$) For any $n \in \mathbb{N}$, choose $\delta_n > 0$ such that if D and D' are δ_n-fine divisions, then

$$|S(f, D, \delta_n) - S(f, D', \delta_n)| < \frac{1}{n}.$$

Here, we may assume that $(\delta_n)_{n=1}^{\infty}$ is a decreasing sequence.

For each $n \in \mathbb{N}$, let D_n be a δ_n-fine division of I . If $m > n$, then both D_m and D_n are δ_n-fine, and we have

$$|S(f, D_n, \delta_n) - S(f, D_m, \delta_m)| < \frac{1}{n}.$$

Hence, the sequence $(S(f, D_m, \delta_m))_{m=1}^{\infty}$ is Cauchy and we let

$$A = \lim_{m\to\infty} S(f, D_m, \delta_m)$$

be the limit of this convergent sequence. Then, for all $n \in \mathbb{N}$, we have

$$|S(f, D_n, \delta_n) - A| \leq \frac{1}{n}.$$

Finally, given $\varepsilon > 0$, let K be a natural number such that $1/K < \varepsilon/2$. If $\mathcal{D}$ is any δ_K-fine division, then

$$\begin{aligned} |S(f, \mathcal{D}, \delta_K) - A| &\leq |S(f, \mathcal{D}, \delta_K) - S(f, D_K, \delta_K)| + |S(f, D_K, \delta_K) - A| \\ &\leq \frac{1}{K} + \frac{1}{K} < \varepsilon. \end{aligned}$$

Therefore, f is HR-integrable over I with $\int_I f = A$. □

The next two results assure us that if f is HR-integrable on $[a, b]$, then the restriction of f to any subinterval of $[a, b]$ is also HR-integrable.

Lemma 3.1.2. *Let* $f : [a, b] \to \mathbb{R}$ *and let* $c \in [a, b]$*. If* f *is HR-integrable on* $[a, b]$*, then its restrictions to* $[a, c]$ *and* $[c, b]$ *are both HR-integrable.*

Proof. Suppose that f is HR-integrable over $[a, b]$. Given $\varepsilon > 0$, let $\delta > 0$ such that if D and D' are any δ-fine division of $[a, b]$, then

$$|S(f, D, \delta) - S(f, D', \delta)| < \varepsilon.$$

Let f_1 be the restriction of f to $[a, c]$ and let D_1 and $D_1^{'}$ be δ-fine divisions of $[a, c]$. Let D_2 be a δ-fine division of $[c, b]$. Then $D = D_1 \cup D_2$ and $D' = D_1^{'} \cup D_2$ are both δ-fine divisions of $[a, b]$, so

$$\left| S(f_1, D_1, \delta) - S(f_1, D_1^{'}, \delta) \right| = |S(f, D, \delta) - S(f, D', \delta)| < \varepsilon.$$

It follows from the Cauchy Criterion that f_1 is HR-integrable over $[a, c]$. Similarly, the restriction f_2 of f to $[c, b]$ is HR-integrable over $[c, b]$, and the proof is complete. □

Corollary 3.1.3. *If* f *is HR-integrable over* $[a, b]$*, and if* $[c, d] \subseteq [a, b]$*, then the restriction of* f *to* $[c, d]$ *is HR-integrable on* $[c, d]$*.*

Proof. As f is HR-integrable over $[a, b]$ and $c \in [a, b]$, it follows from Lemma 3.1.2 that the restriction of f to $[c, b]$ is also HR-integrable over $[c, b]$. Similarly, if $d \in [c, b]$, the restriction of f to $[c, d]$ is again HR-integrable over $[c, d]$, by the above lemma. This completes the proof. □

3.2 The Saks–Henstock Lemma

Here, we prove the Saks–Henstock Lemma for the HR-integral. The proof of the Saks–Henstock Lemma is adapted for the HR-integral from Swartz[6] and Bartle[1].

Lemma 3.2.1. (Saks–Henstock Lemma) *Let $f : I \to \mathbb{R}$ be HR-integrable over I, i.e. for $\varepsilon > 0$ let $\delta > 0$ such that whenever D is a δ–fine division of I, then*

$$\left| S(f, D, \delta) - \int_I f \right| < \frac{\varepsilon}{4}.$$

If $D' = \{([u_i, v_i], \xi_i)\}_{i=1}^n$ is any partial division of I such that D' is also δ-fine, then we have

$$\left| S(f, D', \delta) - \int_J f \right| < \frac{\varepsilon}{4} \tag{3.2.1}$$

where $J = \bigcup_{i=1}^n [u_i, v_i]$ and

$$\sum_{i=1}^n \left| f(\xi_i)(v_i - u_i) - \int_{J_i} f \right| < \frac{\varepsilon}{2} \tag{3.2.2}$$

where $J_i = [u_i, v_i]$.

Also

$$\left| \sum_{i=1}^n \left\{ |f(\xi_i)(v_i - u_i)| - \left| \int_{J_i} f \right| \right\} \right| < \frac{\varepsilon}{2}. \tag{3.2.3}$$

Proof. We let $\{K_j\}_{j=1}^m$ denote the set of the closure of each disjoint subinterval in $[a, b] \backslash J$.

By Corollary 3.1.3, f is HR-integrable over each K_j. Then, for any $\gamma > 0$, there exists a constant $\delta > 0$, such that whenever P_j is a δ-fine division of K_j, then

$$\left| S(f, P_j, \delta) - \int_{K_j} f \right| < \frac{\gamma}{m+1}.$$

Then $D = D' \cup P_1 \cup \cdots \cup P_m$ is a δ-fine division of $[a, b]$. If $J = \bigcup_{i=1}^n [u_i, v_i]$, then

$$\left| S(f, D, \delta) - \int_a^b f \right| = \left| S(f, D', \delta) - \int_J f + \sum_{j=1}^m \Big[S(f, P_j, \delta) - \int_{K_j} f \Big] \right| < \frac{\varepsilon}{4}$$

and hence

$$\begin{aligned}\left|S(f,D',\delta)-\int_J f\right| &< \frac{\varepsilon}{4}+\sum_{j=1}^{m}\left|S(f,P_i,\delta)-\int_{K_i} f\right|\\ &= \frac{\varepsilon}{4}+m(\frac{\gamma}{m+1})\\ &< \frac{\varepsilon}{4}+\gamma.\end{aligned}$$

Since $\gamma > 0$ was arbitrary, we have inequality (3.2.1):

$$\left|S(f,D',\delta)-\int_J f\right| < \frac{\varepsilon}{4}.$$

For inequality (3.2.2), let D'_+ be those $([u_i,v_i],\xi_i)$ in D' such that

$$f(\xi_i)(v_i-u_i)-\int_{u_i}^{v_i} f \geq 0.$$

Then, we obtain

$$0 \leq \sum_{D'_+}\Big(f(\xi_i)(v_i-u_i)-\int_{u_i}^{v_i} f\Big) = \sum_{D'_+}\left|f(\xi_i)(v_i-u_i)-\int_{u_i}^{v_i} f\right| < \frac{\varepsilon}{4}.$$

Similarly, let D'_- be those pairs $([u_i,v_i],\xi_i)$ in D' such that

$$f(\xi_i)(v_i-u_i)-\int_{u_i}^{v_i} f < 0.$$

We then have

$$0 \leq -\sum_{D'_-}\Big(f(\xi_i)(v_i-u_i)-\int_{u_i}^{v_i} f\Big) = \sum_{D'_-}\left|f(\xi_i)(v_i-u_i)-\int_{u_i}^{v_i} f\right| < \frac{\varepsilon}{4}.$$

Adding the two inequalities gives us the required inequality:

$$\sum_{i=1}^{n}\left|f(\xi_i)(v_i-u_i)-\int_{u_i}^{v_i} f\right| < \frac{\varepsilon}{2}.$$

The final inequality is a consequence of the triangle inequality. We have

$$\begin{aligned}\left|\sum_{i=1}^{n}\left\{|f(\xi_i)(v_i-u_i)|-\left|\int_{u_i}^{v_i} f\right|\right\}\right| &\leq \sum_{i=1}^{n}\left||f(\xi_i)(v_i-u_i)|-\left|\int_{u_i}^{v_i} f\right|\right|\\ &\leq \sum_{i=1}^{n}\left|f(\xi_i)(v_i-u_i)-\int_{u_i}^{v_i} f\right|\\ &< \frac{\varepsilon}{2}.\end{aligned}$$

□

Corollary 3.2.2. *Let $f : I \to \mathbb{R}$ be HR-integrable over I. Then for every $\varepsilon > 0$, there exists a constant $\delta > 0$ such that if D and D' are any δ-fine divisions of I, with*

$$D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n},$$
$$D' = \{([u_i, v_i], \xi_i^{'})\}_{i=1}^{n},$$

D and D' differing only by the choice of tags, then

$$\sum_{i=1}^{n} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| < \varepsilon.$$

Proof. We note that

$$\begin{aligned} \left| \sum_{i=1}^{n} f(\xi_i)(v_i - u_i) - \sum_{i=1}^{n} f(\xi_i^{'})(v_i - u_i) \right| &= \left| \sum_{i=1}^{n} (f(\xi_i) - f(\xi_i^{'}))(v_i - u_i) \right| \\ &\leq \sum_{i=1}^{n} \left| (f(\xi_i) - f(\xi_i^{'}))(v_i - u_i) \right| \\ &= \sum_{i=1}^{n} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| . \end{aligned}$$

On the other hand, by the Saks–Henstock Lemma (Lemma 3.2.1), given $\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$\begin{aligned} \sum_{i=1}^{n} \left| (f(\xi_i) - f(\xi_i^{'}))(v_i - u_i) \right| &= \sum_{i=1}^{n} \left| (f(\xi_i) - f(\xi_i^{'}))(v_i - u_i) - \int_{u_i}^{v_i} f + \int_{u_i}^{v_i} f \right| \\ &\leq \sum_{i=1}^{n} \left| f(\xi_i)(v_i - u_i) - \int_{u_i}^{v_i} f \right| + \\ &\quad \sum_{i=1}^{n} \left| f(\xi_i^{'})(v_i - u_i) - \int_{u_i}^{v_i} f \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This yields the inequality

$$\begin{aligned} \sum_{i=1}^{n} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| &= \sum_{i=1}^{n} \left| (f(\xi_i) - f(\xi_i^{'}))(v_i - u_i) \right| \\ &< \varepsilon \end{aligned}$$

and the proof is complete. □

3.3 Sets of Content Zero

We define the notion of a set having (Jordan) content zero, as per Parzynski and Zipse [5]. This will be useful when we attempt the characterisation of the set of discontinuities of a HR-integrable function.

Definition 3.3.1. A set E is of (Jordan) **content zero** if for every $\varepsilon > 0$, there exist a finite number of open intervals $\{I_j\}_{j=1}^n$ such that

$$E \subseteq \bigcup_{j=1}^{n} I_j \quad \text{and} \quad \sum_{j=1}^{n} |I_j| < \varepsilon.$$

Remark 3.3.1. In contrast, a set E is of (Lebesgue) measure zero if, in the above definition, we replace "a finite number of open intervals" with "a countable set of open intervals".

Lemma 3.3.1. *A set E is of content zero if and only if for every $\varepsilon > 0$, there exist a finite number of closed intervals $\{I_j\}_{j=1}^n$ such that*

$$E \subseteq \bigcup_{j=1}^{n} I_j \quad \textit{and} \quad \sum_{j=1}^{n} |I_j| < \varepsilon.$$

Proof. ($\Rightarrow$) If E is content zero, we adjoin the endpoints of each I_j to the interval itself. Then E is contained within the union of these n closed intervals with

$$\sum_{j=1}^{n} |I_j| < \varepsilon.$$

($\Leftarrow$) Conversely, suppose there exist a finite number of closed intervals $\{I_j\}_{j=1}^n$ such that

$$E \subseteq \bigcup_{j=1}^{n} I_j \quad \text{and} \quad \sum_{j=1}^{n} |I_j| < \frac{\varepsilon}{2}.$$

with $I_j = [u_j, v_j]$. We define the open subintervals

$$K_j = (u_j - \frac{\varepsilon}{4n}, v_j + \frac{\varepsilon}{4n}).$$

Then E is contained within the union of $\{K_j\}_{i=1}^n$ and

$$\begin{aligned} \sum_{j=1}^{n} |K_j| &< \frac{\varepsilon}{2} + n \cdot \frac{\varepsilon}{2n} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, E is of content zero. □

Remark 3.3.2. We note that by Lemma 3.3.1, E is of content zero if and only if the closure of E, $\overline{E}$, is of content zero. Certainly, this result does not hold for sets of measure zero, e.g. the set of all rational numbers $\mathbb{Q}$ is a set of measure zero, but $\overline{\mathbb{Q}} = \mathbb{R}$ is not of measure zero.

Example 3.3.1. Clearly, every set with a finite number of elements is of content zero. However, there are infinite sets of content zero.

Consider the set $S = \left\{\frac{1}{k}\right\}_{k=1}^{\infty}$. Then S is of content zero.

Proof. Let $\varepsilon > 0$ be given and choose M such that $\dfrac{1}{M} < \dfrac{\varepsilon}{4} < \dfrac{1}{M-1}$. Then

$$S \subset \bigcup_{j=1}^{M} I_j$$

where

$$I_0 = (-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}) \quad \text{and} \quad I_k = (\frac{1}{k-1} - \frac{\varepsilon}{4M}, \frac{1}{k-1} + \frac{\varepsilon}{4M})$$

for $k = 2, 3, \ldots, M$, and we have

$$\begin{aligned}\sum_{j=1}^{M} |I_j| &= \frac{\varepsilon}{2} + (M-1)\frac{\varepsilon}{2M} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, S is of content zero. □

Sets of content zero are not necessarily countably infinite, however.

Example 3.3.2. We define the Cantor set $\mathcal{C}$ on $[0, 1]$ to be the intersection of the sets C_n, $n \in \mathbb{N}$, where

$$C_1 = [0, 1] \backslash (\frac{1}{3}, \frac{2}{3}),$$

is the set obtained by removing the open middle third of $[0, 1]$;

$$C_2 = C_1 \backslash \{(\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9})\}$$

is the set obtained by removing the open middle thirds of the two intervals in C_1; and each C_n recursively defined using this procedure. Thus, C_n is the union of 2^n closed intervals of the form $[k/3^n, (k+1)/3^n]$, for some $k \in \mathbb{N}$.

It is well-known that the Cantor set is an uncountable set. It is also of content zero, however.

Proof. In general, C_n consists of the union of 2^n intervals, with each interval of length $1/3^n$. Hence, C_n has a total length of $(2/3)^n$.

Let $\varepsilon > 0$ be given. Choose $n \in \mathbb{N}$ such that

$$\left(\frac{2}{3}\right)^n < \varepsilon.$$

Since $\mathcal{C} \subset C_n$ and C_n is the union of 2^n intervals, $\mathcal{C}$ is contained in the union of a finite set of closed intervals with total length less than ε.

Hence, the Cantor set $\mathcal{C}$ has content zero. $\square$

Next, we give a characterisation of a set having content zero, using the HR-integral.

Definition 3.3.2. The characteristic function of a set E, 1_E, is given by

$$1_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Theorem 3.3.2. *A set $E \subset [a, b]$ has content zero if and only if $\int_a^b 1_E = 0$.*

Proof. ($\Rightarrow$) Suppose E has content zero. Given $\varepsilon > 0$, there exists a finite number of intervals $\{I_j\}_{j=1}^n$, where $I_j = [u_j, v_j]$, such that

$$E \subseteq \bigcup_{j=1}^n I_j \quad \text{and} \quad \sum_{j=1}^n |I_j| < \frac{\varepsilon}{2}.$$

We want to construct two δ-fine partial division of $[a, b]$, D_1 and D_2, such that D_1 contains tags that are in E, and D_2 has no tags belonging to E. Further, if $D = D_1 \cup D_2$, then D is a δ-fine division of $[a, b]$.

Choose $\delta = \varepsilon/4n > 0$. To construct the δ-fine partial division D_1, we first choose tags ξ_i such that $\xi_i \in E$. This implies that for each tag in the division D_1, $\xi_i \in I_j$ for some j.

Let $D_1 = \{([x_i, y_i], \xi_i)\}_{i=1}^m$ be such a δ-fine partial division. Since D_1 is δ-fine, for each i,

$$[x_i, y_i] \subset (u_j - \delta, v_j + \delta) \quad \text{for some } j.$$

Then we have

$$\bigcup_{i=1}^m [x_i, y_i] \subset \bigcup_{j=1}^n (u_j - \delta, v_j + \delta)$$

and therefore

$$\begin{aligned}\sum_{i=1}^{m}|y_i-x_i| &\le \sum_{j=1}^{n}|v_j-u_j|+2n\delta\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Next, let D_2 be a δ-fine partial division of $[a,b]$ satisfying the conditions that $\xi_i^{'} \notin E$, where $\xi_i^{'}$ is any tag in D_2, and $D = D_1 \cup D_2$ is a δ-fine division of $[a,b]$. It is clear that $S(1_E, D_2, \delta) = 0$, since no points of E are in D_2, so we have

$$\begin{aligned}|S(1_E, D, \delta)| &= |S(1_E, D_1, \delta) + S(1_E, D_2, \delta)|\\ &= |S(1_E, D_1, \delta)|\\ &= \left|\sum_{i=1}^{m} 1_E(\xi_i)(y_i - x_i)\right|\\ &\le \sum_{i=1}^{m}|y_i - x_i|\\ &< \varepsilon\end{aligned}$$

Therefore, we conclude that $\int_a^b 1_E = 0$.

($\Leftarrow$) Conversely, suppose $\int_a^b 1_E = 0$. Then given $\varepsilon > 0$, there exists a constant $\delta > 0$ such that if $D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n}$ is any δ-fine division of $[a,b]$, then

$$\left|\sum_{i=1}^{n} 1_E(\xi_i)(v_i - u_i)\right| = |S(1_E, D, \delta)| < \varepsilon. \tag{3.3.1}$$

We now construct two δ-fine partial divisions of $[a,b]$ such that one of them contains all points in E and the other does not intersect E at any point.

Let $D_1 = \{([x_k, y_k], \xi_k)\}_{k=1}^{m}$ be a δ-fine partial division of $[a,b]$ such that we have, for all k,

$$\xi_k \in E \subset \bigcup_{k=1}^{m}[x_k, y_k].$$

Without loss of generality, we may assume that for all k, $[x_k, y_k] \cap E \neq \emptyset$ and $x_k, y_k \notin E$.

Let D_2 be a δ-fine division of $[a,b]\backslash \bigcup_{k=1}^{m}[x_k, y_k]$. It is readily seen, from the above construction, that $D_2 \cap E = \emptyset$, so any tag of D_2 is not in E. Note that the division $D' = D_1 \cup D_2$ is again a δ-fine division of $[a,b]$.

Since no tag of D_2 is in E, $S(1_E, D_2, \delta) = 0$. We then have

$$\begin{aligned} |S(1_E, D', \delta)| &= |S(1_E, D_1, \delta) + S(1_E, D_2, \delta)| \\ &= |S(1_E, D_1, \delta)| \\ &< \varepsilon \end{aligned}$$

by (3.3.1) above. However, this implies that

$$|S(1_E, D_1, \delta)| = \left| \sum_{k=1}^{m} 1_E(\xi_k)(y_k - x_k) \right| = \sum_{k=1}^{m} |(y_k - x_k)| < \varepsilon.$$

Since $E \subset \bigcup_{k=1}^{m} [x_k, y_k]$, E is a set of content zero. □

3.4 An Integrability Condition in Terms of Content

We now prove a necessary condition, in terms of the Jordan content of the set of discontinuities of a function, for the function to be HR-integrable. A similar result is known for the Riemann integral (see Parzynski and Zipse[5]).

For a function f defined on I, we begin by defining, for each $n \in \mathbb{N}$, the set $\Gamma_{\frac{1}{n}}(\delta)$, where

$$\Gamma_{\frac{1}{n}}(\delta) = \left\{ x \in I \middle| \, \exists y \in I \text{ with } |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \frac{1}{n} \right\}.$$

Theorem 3.4.1. *Let $f : I \to \mathbb{R}$ be defined on I. If f is HR-integrable on I, then for each $n \in \mathbb{N}$, there exists a $\delta_n > 0$ such that*

$$\bigcap_{\delta \leq \delta_n} \Gamma_{\frac{1}{n}}(\delta)$$

is of content zero.

Proof. Given f is HR-integrable on I. Suppose to the contrary that there exists a $n \in \mathbb{N}$ such that for any $\delta_n > 0$, $\bigcap_{\delta \leq \delta_n} \Gamma_{\frac{1}{n}}(\delta)$ is not of content zero. Then, there is a $\gamma > 0$ such that for any finite set of closed subintervals $\{[u_j, v_j]\}_{j=1}^{m}$ with

$$\bigcap_{\delta \leq \delta_n} \Gamma_{\frac{1}{n}}(\delta) \subset \bigcup_{j=1}^{m} [u_j, v_j],$$

we have

$$\sum_{j=1}^{m} |v_j - u_j| \geq \gamma$$

Since f is HR-integrable, by Corollary 3.2.2, given any $\varepsilon > 0$ and any $n \in \mathbb{N}$, there exists a $\delta_n > 0$ such that if D and D' are any δ_n-fine divisions of I, with

$$D = \{([u_i, v_i], \xi_i)\}_{i=1}^{n},$$
$$D' = \{([u_i, v_i], \xi_i^{'})\}_{i=1}^{n},$$

D and D' differing only by the choice of tags, then

$$\sum_{i=1}^{n} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| < \varepsilon.$$

Let $\mathcal{F}$ be the set of subintervals in D such that for each $[u_i, v_i] \in \mathcal{F}$, $[u_i, v_i] \cap \bigcap_{\delta \leq \delta_n} \Gamma_{\frac{1}{n}}(\delta) \neq \emptyset$, and such that

$$\bigcap_{\delta \leq \delta_n} \Gamma_{\frac{1}{n}}(\delta) \subset \bigcup_{[u_j, v_j] \in \mathcal{F}} [u_j, v_j].$$

For each $[u_i, v_i] \in \mathcal{F}$, then, we may choose a corresponding $\xi_i \in [u_i, v_i]$ such that $\xi_i \in \bigcap_{\delta \leq \delta_n} \Gamma_{\frac{1}{n}}(\delta)$ with $\left| f(\xi_i) - f(\xi_i^{'}) \right| \geq \frac{1}{n}$.

We then have

$$\begin{aligned} \sum_{i=1}^{n} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| &= \sum_{[u_i, v_i] \in \mathcal{F}} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| + \\ &\quad \sum_{[u_i, v_i] \notin \mathcal{F}} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| \\ &< \varepsilon. \end{aligned} \tag{3.4.1}$$

On the other hand, since $\bigcap_{\delta \leq \delta_n} \Gamma_{\frac{1}{n}}(\delta)$ is not of content zero, we have

$$\sum_{[u_i, v_i] \in \mathcal{F}} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| \geq \frac{1}{n} \cdot \gamma > 0.$$

As $\varepsilon > 0$ is arbitrary, this contradicts the assumption that f is HR-integrable and the inequality (3.4.1), since we can find a $\varepsilon > 0$ such that

$$\begin{aligned} \sum_{i=1}^{n} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| &= \sum_{[u_i, v_i] \in \mathcal{F}} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| + \\ &\quad \sum_{[u_i, v_i] \notin \mathcal{F}} \left| f(\xi_i) - f(\xi_i^{'}) \right| |v_i - u_i| \\ &\geq \frac{\gamma}{n} \\ &> \varepsilon > 0. \end{aligned}$$

Therefore, for each $n \in \mathbb{N}$, there exists a $\delta_n > 0$ such that

$$\bigcap_{\delta \leq \delta_n} \Gamma_{\frac{1}{n}}(\delta)$$

is of content zero. $\square$

Remark 3.4.1. We note that the converse of this result, in a slightly different form, is also true. A proof is given by Parzynski and Zipse[5] using Darboux upper and lower sums and the Heine-Borel Theorem.

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