An Asymptotic Property of Schachermayer’s Space under Renorming

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Let $X$ be a Banach space with closed unit ball $B$. Given $k \in \mathbb{N}$, $X$ is said to be $k$-$\beta$, respectively, $(k + 1)$-nearly uniformly convex ($((k + 1)\text{-NUC})$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\varepsilon$-separated sequence $\{x_n\} \subseteq B$ there are indices $(i_n)_{n=1}^{k+1}$, respectively, $(i_n)_{n=1}^{k+2}$, such that $(1/(k+1)) \sum_{n=1}^{k+1} \|x_n\| \leq 1 - \delta$, respectively, $(1/(k+1)) \sum_{n=1}^{k+2} \|x_n\| \leq 1 - \delta$. It is shown that a Banach space constructed by Schachermayer is $2$-$\beta$, but is not isomorphic to any $2$-NUC Banach space. Modifying this example, we also show that there is a $2$-NUC Banach space which cannot be equivalently renormed to be $1$-$\beta$.

Key Words: nearly uniform convexity; renorming; Schachermayer’s space.

1. INTRODUCTION

In [4], Huff introduced the notion of nearly uniform convexity (NUC). A Banach space $X$ with closed unit ball $B$ is said to be NUC if for any $\varepsilon > 0$ there exists $\delta < 1$ such that for every $\varepsilon$-separated sequence in $B$, $\text{co}((x_n)) \cap \delta B \neq \emptyset$. Here $\text{co}(A)$ denotes the convex hull of a set $A$; a sequence $(x_n)$
is \( \varepsilon \)-separated if \( \inf\{\|x_n - x_m\| : m \neq n\} \geq \varepsilon \). Huff showed that a Banach space is NUC if and only if it is reflexive and has the uniform Kadec–Klee property (UKK). Recall that a Banach space \( X \) with closed unit ball \( B \) is said to be UKK if for any \( \varepsilon > 0 \) there exists \( \delta < 1 \) such that for every \( \varepsilon \)-separated sequence \( (x_n) \) in \( B \) which converges weakly to some \( x \in X \) we have \( \|x\| \leq \delta \). A recent result of Knaust \textit{et al.} [5] gives an isomorphic characterization of spaces having NUC. They showed that a separable reflexive Banach space \( X \) is isomorphic to a UKK space if and only if \( X \) has a finite Szlenk index. More recent results concerning Szlenk indices and renormings are to be found in [2, 3].

Another property related to NUC is the property \( (\beta) \) introduced by Rolewicz [11]. In [6], building on the work of Prus [9, 10], the first author showed that a separable Banach space \( X \) is isomorphic to a space with \( (\beta) \) if and only if both \( X \) and \( X^* \) are isomorphic to NUC spaces. In [7], a sequence of properties lying in between \( (\beta) \) and NUC is defined. Let \( X \) be a Banach space with closed unit ball \( B \). Given \( k \in \mathbb{N} \), \( X \) is said to be \( k \)-\( \beta \)-, respectively, \( (k + 1) \)-NUC, if for every \( \varepsilon > 0 \) there exists \( \delta < 1 \), so that for every \( \varepsilon \)-separated sequence \( (x_n) \subseteq B \) there are indices \( (n_i)_{i=1}^{k+1} \), respectively \( (n_i)_{i=1}^{k+1} \), such that

\[
\frac{1}{k+1} \left\| x + \sum_{i=1}^{k} x_{n_i} \right\| \leq 1 - \delta,
\]

respectively

\[
\frac{1}{k+1} \left\| \sum_{i=1}^{k+1} x_{n_i} \right\| \leq 1 - \delta.
\]

It follows readily from the definitions that every \( k \)-\( \beta \) space is \( (k + 1) \)-NUC, every \( (k + 1) \)-NUC space is \( (k + 1) \)-\( \beta \), and that every \( k \)-\( \beta \) space (or \( (k + 1) \)-NUC space) is NUC. It is proved in [7] that property \( 1 \)-\( \beta \) is equivalent to the property \( (\beta) \) of Rolewicz. It is worth noting that the “non-uniform” version of property \( k \)-NUC has been well-studied. For \( k \geq 2 \), a Banach space \( X \) is said to have property \( (kR) \) if every sequence \( (x_n) \) in \( X \) which satisfies \( \lim_{n_1} \ldots \lim_{n_k} \|x_{n_1} + \ldots + x_{n_k}\| = k \lim_{n} \|x_n\| \) is convergent [1]. It is clear that the property \( (kR) \) implies property \( ((k + 1)R) \). It follows from James’ characterization of reflexivity that every \( (kR) \) space is reflexive. A recent result of Odell and Schlumprecht [8] shows that a separable Banach space is reflexive if and only if it can be equivalently renormed to have property \( (2R) \). Thus, all the properites \( (kR) \) are isomorphically equivalent. Similarly, “non-asymptotic” properties known as \( k \)-uniform rotundity have been studied [13]. These properites are also isomorphically equivalent to each other as they are all equivalent to superreflexivity. In this paper, we find that
the situation is different for the properties $k$-NUC and $k$-$\beta$. To be precise, we use the space constructed by Schachermayer in [12] and a variant to distinguish the properties $1$-$\beta$, $2$-NUC, and $2$-$\beta$ isomorphically.

Let $T = \bigcup_{n=0}^{\infty} \{0, 1\}^n$ be the dyadic tree. If $\varphi = (\varepsilon_i)_{i=1}^m$ and $\psi = (\delta_i)_{i=1}^n$ are nodes in $T$, we say that $\varphi \preceq \psi$ if $m \leq n$ and $\varepsilon_i = \delta_i$ for $1 \leq i \leq m$.

Also, $\emptyset \preceq \varphi$ for all $\varphi \in T$. Two nodes $\varphi$ and $\psi$ are said to be comparable if either $\varphi \preceq \psi$ or $\psi \preceq \varphi$; they are incomparable otherwise. Let $\varphi \in T$, denote by $T_\varphi$ or $T(\varphi)$ the subtree rooted at $\varphi$, i.e., the subtree consisting of all nodes $\psi$ such that $\varphi \preceq \psi$. A node $\varphi \in T$ has length $n$ if $\varphi \in \{0, 1\}^n$.

The length of $\varphi$ is denoted by $|\varphi|$. Given $\varphi = (\varepsilon_i)_{i=1}^m \in T$, let $S_\varphi$ be the set consisting of all nodes $\psi = (\delta_i)_{i=1}^m$ such that $m \geq n$, $\delta_i = \varepsilon_i$ if $1 \leq i \leq n$, and $\delta_i = 0$ otherwise. Say that a subset $A$ of $T$ is admissible, respectively, acceptable, if there exists $n \in \mathbb{N} \cup \{0\}$ such that (a) $A \subseteq \bigcup_{|\varphi|=n} T_\varphi$ and (b) $|A \cap T_\varphi| \leq 1$ for all $\varphi$ with $|\varphi| = n$, respectively, (a') $A \subseteq \bigcup_{|\varphi|=n} S_\varphi$, and (b') $|A \cap S_\varphi| \leq 1$ for all $\varphi$ with $|\varphi| = n$. For subsets $A$ and $B$ of $T$, say that $A \preceq B$ if $\max\{|\varphi|: \varphi \in A\} < \min\{|\varphi|: \varphi \in B\}$. Let $c_0(T)$ be the space of all finitely supported real-valued functions defined on $T$. For $x \in c_0(T)$, let

$$
\|x\|_X = \sup \left( \sum_{i=1}^{k} \left( \sum_{\varphi \in A_i} |x(\varphi)| \right)^2 \right)^{1/2},
$$

where the sup is taken over all $k \in \mathbb{N}$ and all sequences of admissible subsets $A_1 \preceq A_2 \preceq \cdots \preceq A_k$ of $T$. The norm $\|\cdot\|_Y$ is defined similarly except that the sup is taken over all sequences of acceptable subsets $A_1 \preceq A_2 \preceq \cdots \preceq A_k$ of $T$. Schachermayer’s space $X$ is the completion of $c_0(T)$ with respect to the norm $\|\cdot\|_X$. The completion of $c_0(T)$ with respect to $\|\cdot\|_Y$ is denoted by $Y$.

**Remark.** The space $X$ defined here differs from Schachermayer’s original definition and is only isomorphic to the space defined in [12].

In [7], it was shown that $X$ (with the norm given in [12]) is 8-NUC but is not isomorphic to any 1-$\beta$ space. We first show that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are 2-$\beta$ and 2-NUC respectively. We begin with a trivial lemma concerning the $\ell^2$-norm $\|\cdot\|_2$.

**Lemma 1.** If $\alpha$, $\beta$, and $\gamma$ are vectors in the unit ball of $\ell^2$, and $\|\alpha + \beta + \gamma\|_2^2 / 3 \geq 1 - \delta$, then $\max\{\|\alpha - \beta\|_2, \|\alpha - \gamma\|_2, \|\beta - \gamma\|_2\} \leq \sqrt{18}\delta$.

**Proposition 2.** $(X, \|\cdot\|_X)$ is 2-$\beta$.

**Proof.** Let $x$ and $x_n$, $n \geq 1$, be elements in the unit ball of $X$ such that $(x_n)$ is $\varepsilon$-separated. Choose $\delta > 0$ such that

$$
(1 - 3\delta)^2 + \left[ (1 - 24\delta)^{1/2} - (1 - \varepsilon^2/9)^{1/2} \right]^2 > 1.
$$

(1)
Without loss of generality, we may assume that \((x_n)\) converges pointwise (as a sequence of functions on \(T\)) to some \(y_0 : T \to \mathbb{R}\). It is clear that if \(y, z \in X\) and \(y \ll \text{supp } z\), then \(\|y + z\|_X \geq \|y\|_X^2 + \|z\|_X^2\). It follows easily that \(y_0 \in X\). Let \(y_n = x_n - y_0\). It may be assumed that \((\|y_n\|_X)\) converges. As \((x_n)\) is \(\epsilon\)-separated, so is \((y_n)\). We may thus further assume that \(\|y_n\|_X > \epsilon/3\) for all \(n \in \mathbb{N}\). By going to a subsequence and perturbing the vectors \(x, y_0, y_n, n \geq 1\), by as little as we please, it may be further assumed that (a) they all belong to \(c_0(T)\), (b) \(\text{supp } x \cup \text{supp } y_0 \ll \text{supp } y_1 \ll \text{supp } y_2\), and (c) \(\|y_1 x T_\alpha\|_\infty = \|y_2 x T_\alpha\|_\infty\) for all \(\phi\) such that \(|\phi| \leq M\), where \(\|\cdot\|_\infty\) is the sup norm and \(M = \max\{|\psi| : \psi \in \text{supp } x \cup \text{supp } y_0\}\).

**Claim.** Let \(A\) be an admissible set such that \(\min\{|\phi| : \phi \in A\} \leq M\). If \(\sum_{\phi \in A} |\phi| = c\), and \(\sum_{\phi \in A} |\psi_2(\phi)| = d\), then there exists an admissible set \(B\) such that

\[
\min\{|\phi| : \phi \in A\} \leq \min\{|\phi| : \phi \in B\} \leq \max\{|\phi| : \phi \in B\} \leq \max\{|\phi| : \phi \in A\},
\]

\(A \cap \text{supp } y_0 \subseteq B\), and \(\sum_{\psi \in B} |\psi_1(\psi)| \geq c + d\).

To prove the claim, let \(N\) be such that \(A \subseteq \bigcup_{\phi \in A} y_n T_\phi\) and let \(|A \cap T_\phi| \leq 1\) for all \(\phi\) with \(|\phi| = N\). Then \(N \leq M\). Now, for each \(\psi \in A \cap \text{supp } y_2\), \(\psi \in T_\phi\) for some \(\phi\) with \(|\phi| = N \leq M\). It follows that

\[
\|y_1 x T_\alpha\|_\infty \geq \|y_2 x T_\alpha\|_\infty \geq |\psi_2(\psi)|.
\]

Hence, there exists a \(\psi' \in T_\phi\) such that \(|\psi_1(\psi')| \geq |\psi_2(\psi)|\). Now let

\[
B = (A \cap (\text{supp } y_0 \cup \text{supp } y_1)) \cup \{\psi' : \psi \in A \cap \text{supp } y_2\}.
\]

It is easy to see that the set \(B\) satisfies the claim.

Suppose that \(\|x + x_1 + x_2\|_X/3 \geq 1 - \delta\). Let \(x + x_1 + x_2 = x + 2y_0 + y_1 + y_2\) be normed by a sequence of admissible sets \(A_1 \ll A_2 \ll \cdots \ll A_k\). Denote by \(a = (a_i)_{i=1}^k\), \(b = (b_i)_{i=1}^k\), \(\gamma = (c_i)_{i=1}^k\), and \(\eta = (d_i)_{i=1}^k\) respectively the sequences \((\sum_{\phi \in A_i} |x(\phi)|)_{i=1}^k\), \((\sum_{\phi \in A_i} |y_0(\phi)|)_{i=1}^k\), \((\sum_{\phi \in A_i} |y_1(\phi)|)_{i=1}^k\), and \((\sum_{\phi \in A_i} |y_2(\phi)|)_{i=1}^k\).

Now

\[
\|\alpha + (\beta + \gamma) + (\beta + \eta)\|_2/3 \geq \|x + x_1 + x_2\|_X/3 \geq 1 - \delta.
\]

But \(\|a\|_2 \leq \|x\|_X \leq 1\). Similarly, \(\|\beta + \gamma\|_2 \leq \|\beta + \eta\|_2 \leq 1\). By Lemma 1, we obtain that \(\|\alpha - \beta - \gamma\|_2\), \(\|\alpha - \beta - \eta\|_2\), and \(\|\gamma - \eta\|_2\) are all \(\leq \sqrt{18\delta}\).

Let \(j\) be the largest integer such that \(a_j \neq 0\). Note that this implies \(\text{supp } x \cap A_j \neq \emptyset\); hence \((\text{supp } y_1 \cup \text{supp } y_2) \cap A_i = \emptyset\) for all \(i < j\). Thus, \(c_i = d_i = 0\) for all \(i < j\). Now

\[
\|(b_{j+1} + d_{j+1}, \ldots, b_k + d_k)\|_2 \leq \|a - \beta - \eta\|_2 \leq \sqrt{18\delta}.
\]
Moreover,
\[ 1 \geq \|x_2\|_X^2 = \|y_0 + y_2\|_X^2 \geq \|y_0\|_X^2 + \|y_2\|_X^2 \geq \|\beta\|_2^2 + \|y_2\|_X^2 \]
\[ \implies \|\beta\|_2^2 \leq 1 - \varepsilon^2/9. \]  
(3)

Hence
\[ 3(1 - \delta) \leq \|\alpha\|_2 + \|\beta + \gamma\|_2 + \|\beta + \eta\|_2 \leq 2 + \|\beta + \eta\|_2 \]
\[ \implies (1 - 3\delta)^2 \leq \|\beta + \eta\|_2^2 \]
\[ \leq \|(b_1, \ldots, b_{j-1}, b_j, d_j)\|_2^2 \]
\[ + \|(b_{j+1} + d_{j+1}, \ldots, b_k + d_k)\|_2^2 \]
\[ \leq (\|(b_1, \ldots, b_{j-1}, b_j)\|_2^2 + d_j)^2 + 18\delta \]  
by (2)
\[ \leq (\|\beta\|_2 + d_j)^2 + 18\delta \]
\[ \leq ((1 - \varepsilon^2/9)^{1/2} + d_j)^2 + 18\delta. \]  
by (3)

Therefore,
\[ d_j \geq (1 - 24\delta)^{1/2} - (1 - \varepsilon^2/9)^{1/2}. \]  
(4)

Note that by the first part of the argument above we also obtain that
\[ \|\beta + \gamma\|_2 \geq 1 - 3\delta. \]  
(5)

Since \( A_j \cap \text{supp } x \neq \emptyset \), we may apply the claim to obtain an admissible set \( B \). Using the sequence of admissible sets \( A_1 \ll \cdots \ll A_{j-1} \ll B \ll A_{j+1} \ll \cdots \ll A_k \) to norm \( x_1 = y_0 + y_1 \) yields
\[ 1 \geq \|y_0 + y_1\|_X^2 \geq \|(b_1, \ldots, b_{j-1}, b_j + c_j, d_j, b_{j+1} + c_{j+1}, \ldots, b_k + c_k)\|_2^2 \]
\[ \geq \|(b_1, \ldots, b_{j-1}, b_j + c_j, b_{j+1} + c_{j+1}, \ldots, b_k + c_k)\|_2^2 + d_j^2 \]
\[ = \|\beta + \gamma\|_2^2 + d_j^2 \]
\[ \geq (1 - 3\delta)^2 + \left((1 - 24\delta)^{1/2} - (1 - \varepsilon^2/9)^{1/2}\right)^2 \]
by (5) and (4). As the last expression is \( > 1 \) by (1), we have reached a contradiction.

**Remark.** The same method can be used to show that \( X \) is 2-\( \beta \) with the norm given in \([12]\).

**Proposition 3.** \( (Y, \|\cdot\|_Y) \) is 2-NUC.
Proof. Let \((x_n)\) be an \(\varepsilon\)-separated sequence in the unit ball of \(Y\). Choose \(\delta > 0\) so that
\[ \delta' = 12\delta + 2\sqrt{8\delta} \leq \frac{\varepsilon^2}{18} \] (6)
and
\[ 1 - 2\delta - (2 + \sqrt{8})\sqrt{\delta} > \sqrt{1 - (\varepsilon/3)^2}. \] (7)

As in the proof of the previous proposition, it may be assumed that there exists a sequence \((y_n)_{n=0}^{\infty}\) in \(Y\) such that \(x_n = y_0 + y_n\), \(\text{supp } y_{n-1} \subset \text{supp } y_n\) for all \(n \in \mathbb{N}\), and \(\|y_j\chi_{S_t}\|_\infty = \|y_k\chi_{S_t}\|_\infty\) whenever \(|\varphi| \leq M\) and \(j, k > i\), where \(M_t = \max\{|\psi| : \psi \in \text{supp } y_j\}\). We may also assume that \((\|y_n\|_Y)\) converges. Since \((y_n)_{n=0}^{\infty}\) is \(\varepsilon\)-separated, \(\eta = \lim \|y_n\|_Y \geq \varepsilon/2\). The choice of \(\delta'\) in (6) guarantees that \(4(\eta^2 - \delta')^{1/2} > 7\eta/2 \geq 3\eta + \sqrt{\delta}\). Hence there exist \(\eta_+ > \eta > \eta_- > \varepsilon/3\) such that
\[ 4\theta \geq 3\eta_+ + \sqrt{(\eta_+)^2 - (\eta_-)^2 + \delta'}. \] (8)
where \(\theta = \sqrt{(\eta_-)^2 - \delta'}\). We may now further assume that \(\eta_+ \geq \|y_n\|_Y \geq \eta_-\) for all \(n \in \mathbb{N}\). Now suppose that \(\|x_m + x_n\|_Y/2 > 1 - \delta\) for all \(m, n \in \mathbb{N}\).

Claim. For all \(m < n\) in \(\mathbb{N}\), there exists an acceptable set \(A\) such that \(\sum_{\varphi \in A} |y_\varphi| \geq \theta\) for \(i = m, n\).

First observe that there are acceptable sets \(A_1 \ll A_2 \ll \cdots \ll A_k\) such that \(\sum_{i=1}^{k} (\sum_{\varphi \in A_i} |y_\varphi|)^2 > 4(1 - \delta)^2\). Let \(\alpha = (a_j)_{j=1}^{\infty}, \beta = (b_j)_{j=1}^{\infty}, \gamma = (c_j)_{j=1}^{\infty}\) be the sequences \((\sum_{\varphi \in A} |y_\varphi|)\) for \(j = 0, m, n\), respectively. Then \(2(\alpha + \beta + \gamma)_{2} > 2(1 - \delta)\) and \(\|\alpha + \beta + \gamma\|_{2} \leq \|y_0 + y_m\|_Y = \|x_m\|_Y \leq 1\). Similarly, \(\|\alpha + \gamma\|_{2} \leq \|y_0 + y_m\|_Y = \|x_m\|_Y \leq 1\). It follows from the parallelogram law that \(\|\beta - \gamma\|_{2} < 4 - 4(1 - \delta)^2 \leq 8\delta\). Note also that \(\|\alpha + \beta + \gamma\|_{2} > 2\alpha + \beta + \gamma\|_{2} - \|\alpha + \gamma\|_{2} > 1 - 2\delta\). Similarly, \(\|\alpha + \gamma\|_{2} > 1 - 2\delta\). Let \(j_1\) and \(j_2\) be respectively the largest \(j\) such that \(a_j \neq 0\), respectively \(b_j \neq 0\). Since \(\text{supp } y_0 \cap A_j \neq \varnothing\), \(b_1 = \cdots = b_{j_1 - 1} = 0\). Similarly, \(c_1 = \cdots = c_{j_2 - 1} = 0\). Moreover, \(j_1 \leq j_2\). Let us show that \(j_1 \leq j_2\). For otherwise, \(j_1 = j_2 = j\). Then
\[ |b_j - c_j| \leq \|\beta - \gamma\|_{2} < \sqrt{8\delta}. \] (9)
Consider the set \(A_j\). Choose \(p \in \mathbb{N} \cup \{0\}\) such that \(A_j \subseteq \bigcup_{|\varphi| = p} S_\varphi\) and \(|A_j \cap S_\varphi| \leq 1\) for all \(\varphi\) with \(|\varphi| = p\). Note that \(p \leq M_0\). Let \(G = \{\varphi : \varphi = p, A_j \cap S_\varphi \subset \text{supp } y_m \neq \varnothing\}\). If \(\varphi \in G\), \(\|y_\varphi\chi_{S_t}\|_\infty = \|y_m\chi_{S_t}\|_\infty\). Hence there exists \(\psi_\varphi \in S_\varphi \cap \text{supp } y_n\) such that \(\|y_n(\psi_\varphi)| = \|y_m\chi_{S_t}\|_\infty\). It is easy to see that the set \(B = \{\psi_\varphi : \varphi \in G\} \cup (A_j \cap \text{supp } y_0) \cup (A_j \cap \text{supp } y_n)\)
is acceptable and that min\{|\varphi| : \varphi \in A_j\} \leq \min\{|\varphi| : \varphi \in B\}. Hence \( A_1 \ll \cdots \ll A_{j-1} \ll B \). Thus

\[
1 \geq \|x_n\|_Y^2 = \|y_0 + y_n\|_Y^2 \geq \sum_{i=1}^{j-1} |a_i|^2 + \left( \sum_{\varphi \in B} |(y_0 + y_n)(\varphi)| \right)^2
\]

\[
\geq \sum_{i=1}^{j-1} |a_i|^2 + \left( \sum_{\varphi \in A_j} |y_0(\varphi)| + \sum_{\varphi \in A_j} |y_n(\varphi)| + \sum_{\varphi \in G} |y_n(\varphi)| \right)^2
\]

\[
\geq \sum_{i=1}^{j-1} |a_i|^2 + \left( |a_j| + |c_j| + \sum_{\varphi \in G} \|y_n(x_S)\|_\infty \right)^2
\]

\[
\geq \sum_{i=1}^{j-1} |a_i|^2 + \left( |a_j| + |c_j| + \sum_{\varphi \in A_j} |y_m(\varphi)| \right)^2
\]

\[
\geq \left\| (a_1, \ldots, a_{j-1}, a_j + b_j + c_j) \right\|^2_2
\]

\[
\geq \left\| (a_1, \ldots, a_{j-1}, a_j + b_j) \right\|^2_2 + |c_j|^2
\]

\[
\geq \|a + \beta\|^2_2 + (|b_j| - \sqrt{8\delta})^2
\]

by (9),

\[
> (1 - 2\delta)^2 + \left( \|\beta\|_2 - \sqrt{8\delta} \right)^2.
\]

Therefore, \( \|\beta\|_2 < (2 + \sqrt{8})\sqrt{\delta} \). It follows that

\[
\|a\|_2 \geq \|a + \beta\|_2 - \|\beta\|_2 > 1 - 2\delta - \left( 2 + \sqrt{8} \right)\sqrt{\delta}.
\]  \hspace{1cm} (10)

However,

\[
\|a\|^2_2 \leq \|y_0\|^2_{Y^2} \leq \|x_m\|^2_{Y^2} - \|y_m\|^2_{Y^2} \leq 1 - (\eta_-)^2 < 1 - (\kappa/3)^2.
\]  \hspace{1cm} (11)

Combining (10) and (11) with the choice of \( \delta \) in (7) yield a contradiction. This shows that \( j_1 < j_2 \). Applying the facts that \( \|a + \beta\|_2 > 1 - 2\delta \) and \( \|(b_1, \ldots, b_{j-1})\|_2 \leq \|\beta - \gamma\|_2 < \sqrt{8\delta} \), we obtain that

\[
|b_{j_2}|^2 > (1 - 2\delta)^2 - \left( \|a\|_2 + \sqrt{8\delta} \right)^2
\]

\[
\geq (1 - 2\delta)^2 - \left( \sqrt{1 - (\eta_-)^2 + \sqrt{8\delta}} \right)^2 \geq \theta^2.
\]
Thus

Similarly,\[ (1 - 2\delta)^2 < \|\alpha + \gamma\|_2^2 = \|\alpha\|_2^2 + |c_{j_1}|^2 + \left(\sum_{j=j_1+1}^\infty |c_j|^2\right)^2 \]
\[ \leq \|\alpha\|_2^2 + |c_{j_1}|^2 + \|\beta - \gamma\|_2^2 \]
\[ \leq 1 - (\eta_\pm)^2 + |c_{j_1}|^2 + 8\delta. \]

Hence \(|c_{j_1}| > \theta\). Thus the set \(A = A_{j_2}\) satisfies the requirements of the claim.

Taking \(m = 1, n = 2, m = 2, n = 3\), respectively, we obtain acceptable sets \(A\) and \(A'\) from the claim. Since \(A \cap \text{supp} y_1 \neq \emptyset\), if \(\varphi \in A \cap \text{supp} y_2\), \(\varphi \in S_{\sigma}\) for some \(\sigma\) such that \(|\varphi| \leq M_1\). This implies that there exists \(\psi_\varphi \in S_{\sigma}\) such that \(|y_1(\psi_\varphi)| = \|y_1x_{S_{\sigma}}\|_\infty = \|y_2x_{S_{\sigma}}\|_\infty \geq |y_2(\varphi)|\). Let \(q = \min\{|\varphi| : \varphi \in \text{supp} y_3\} \) and \(\Phi = \{\sigma \in T : |\sigma| = q\}\). For \(\sigma \in \Phi\), define \(s(\sigma) = |y_3(\psi_\varphi)|\) if there exists \(\varphi \in A \cap \text{supp} y_2\) such that \(\psi_\varphi \in S_{\sigma}\); otherwise, let \(s(\sigma) = 0\). Also, let \(t(\sigma) = |y_3(\varphi)|\) if there exists \(\varphi \in A' \cap \text{supp} y_3 \cap S_{\sigma}\); otherwise, let \(t(\sigma) = 0\). Finally, let \(r(\sigma) = \|y_3x_{S_{\sigma}}\|_\infty\) for all \(\sigma \in \Phi\). Then \(r(\sigma) \geq s(\sigma) \geq 0\) for all \(\sigma \in \Phi\), \(\sum_{\sigma} r(\sigma) \leq \|y_3\|_Y < \eta_+\), and \(\sum_{\sigma} s(\sigma) > \theta\). Hence \(\sum_{\sigma} r(\sigma) - s(\sigma) < \eta_+ - \theta\). Similarly, \(\sum_{\sigma} r(\sigma) - t(\sigma) < \eta_+ - \theta\). Therefore, \(\sum_{\sigma} |t(\sigma) - s(\sigma)| < 2(\eta_+ - \theta)\). Let \(B\) be the set of all nodes in \(A \cap \text{supp} y_2\) that are comparable with some node in \(A' \cap \text{supp} y_3\). Then
\[ \sum_{\varphi \in B} |y_2(\varphi)| \leq \sum_{\varphi \in A' \cap B} |y_3(\psi_\varphi)| \leq \sum_{\sigma} |t(\sigma) - s(\sigma)| < 2(\eta_+ - \theta). \]

Hence \(\sum_{\varphi \in B} |y_2(\varphi)| > \theta - 2(\eta_+ - \theta) = 3\theta - 2\eta_+\). Now let \(l = \min\{|\varphi| : \varphi \in A' \cap \text{supp} y_2\}\). Divide \(B\) into \(B_1 = \{\varphi \in B : |\varphi| < l\}\) and \(B_2 = \{\varphi \in B : |\varphi| \geq l\}\). Since \(B_1\) and \(A' \cap \text{supp} y_2\) are acceptable sets such that \(B_1 \ll A' \cap \text{supp} y_2\),
\[ \left(\eta_+\right)^2 \geq \|y_2\|_Y^2 \geq \left(\sum_{\varphi \in B_1} |y_2(\varphi)|\right)^2 + \left(\sum_{\varphi \in A'} |y_3(\varphi)|\right)^2 \]
\[ > \left(\sum_{\varphi \in B_1} |y_2(\varphi)|\right)^2 + \theta^2. \]
Thus
\[ \sum_{\varphi \in B_2} |y_2(\varphi)| > 3\theta - 2\eta_+ - \sqrt{(\eta_+)^2 - \theta^2}. \]
Finally, since $B_2 \cup (A' \cap \supp y_2)$ is acceptable,

$$\eta_+ > \| y_2 \|_Y \geq \sum_{\varphi \in B_2} |y_2(\varphi)| + \sum_{\varphi \in A' \cap \supp y_2} |y_2(\varphi)|$$

$$> 3\theta - 2\eta_+ - \sqrt{(\eta_+)^2 - \theta^2 + \theta}.$$  

This contradicts inequality (8). \[\square\]

Before proceeding further, let us introduce some more notation. A branch in $T$ is a maximal subset of $T$ with respect to the partial order $\leq$. If $\gamma$ is a branch in $T$ and $n \in \mathbb{N} \cup \{0\}$, let $\varphi_n^\gamma$ be the node of length $n$ in $\gamma$. A collection of pairwise distinct branches is said to have separated at level $L$ if for any pair of distinct branches $\gamma$ and $\gamma'$ in the collection the nodes of length $L$ belonging to $\gamma$ and $\gamma'$ respectively are distinct. Finally, if $(\gamma_1, \ldots, \gamma_k)$ is a sequence of pairwise distinct branches which have separated at a certain level $L$, we say that a sequence of nodes $(\varphi_1, \ldots, \varphi_k) \in S(\gamma_1, \ldots, \gamma_k; L)$ if $\varphi_i \in T(\varphi_i^\gamma)$, $1 \leq i \leq k$. Let us note that in this situation $\|x_{\{\varphi_i : 1 \leq i \leq k\}}\|_X = k$.

Suppose $\| \cdot \|$ is an equivalent norm on $X$ which is 2-NUC. It may be assumed that there exists $\varepsilon > 0$ so that $\varepsilon \|x\|_X \leq \| \|x\| \| \leq \|x\|_X$ for all $x \in X$. Let $\delta = \delta(2\varepsilon) > 0$ be the number obtained from the definition of 2-NUC for the norm $\| \cdot \|$.

**Proposition 4.** Let $n \in \mathbb{N} \cup \{0\}$. Then there are pairwise incomparable nodes $\varphi_1, \ldots, \varphi_{2^n}$ such that whenever $\gamma_i, \gamma_i'$ are distinct branches passing through $\varphi_i$, $1 \leq i \leq 2^n$, and $\{\gamma_i, \gamma_i' : 1 \leq i \leq 2^n\}$ have separated at level $L$, there is a sequence of nodes $(\psi_1, \ldots, \psi_{2^{n+1}}) \in S(\gamma_1, \gamma_1', \ldots, \gamma_{2^n}, \gamma_{2^n}'; L)$ satisfying $\|x_{\{\psi_i : 1 \leq i \leq 2^{n+1}\}}\| \leq (2(1 - \delta))^{n+1}$.

**Proof.** Assume that $n$ is the first non-negative integer where the proposition fails. Let $\varphi_1, \ldots, \varphi_{2^n-1}$ be the nodes obtained by applying the proposition for the case $n - 1$. (If $n = 0$, begin the argument with any node $\varphi_1$.) For each $i$, $1 \leq i \leq 2^{n-1}$, let $\psi_{2i-1,1}$ and $\psi_{2i,1}$ be a pair of incomparable nodes in $T_{\varphi_i}$. (If $n = 0$, let $\psi_{1,1}$ be any node in $T_{\varphi_1}$.) Since the proposition fails for the nodes $\psi_1, \ldots, \psi_{2^n-1}$, there are distinct branches $\gamma_{i,1}, \gamma_{i,1}'$ passing through $\psi_{i,1}$, $1 \leq i \leq 2^n$, and a number $L_1$ so that $\{\gamma_{i,1}, \gamma_{i,1}' : 1 \leq i \leq 2^n\}$ have separated at level $L_1$, but $\|x_{\{\xi_i : 1 \leq i \leq 2^{n+1}\}}\| > (2(1 - \delta))^{n+1}$ for any sequence of nodes $(\xi_1, \ldots, \xi_{2^{n+1}}) \in S(\gamma_{1,1}, \gamma_{1,1}', \ldots, \gamma_{2^n,1}, \gamma_{2^n,1}'; L_1)$. However, since the proposition holds for the nodes $\varphi_1, \ldots, \varphi_{2^n-1}$, we obtain a sequence of nodes $(\xi_{1,1}, \ldots, \xi_{2^n,1}) \in S(\gamma_{1,1}', \ldots, \gamma_{2^n,1}', L_1)$ such that $\|x_{\{\xi_i : 1 \leq i \leq 2^{n+1}\}}\| \leq (2(1 - \delta))^{n}$. 
(Note that the preceding statement holds trivially if \( n = 0 \).) For each \( i \), choose a node \( \psi_{i,2} \) in \( \gamma_{i,1} \) such that \( |\psi_{i,2}| > L_1 \). Then \( \psi_{2i-1,2} \) and \( \psi_{2i,2} \) are a pair of incomparable nodes in \( T_e \), and the argument may be repeated. (If \( n = 0 \), repeat the argument using the node \( \psi_{1,2} \).) Inductively, we thus obtain sequences of branches \( (\gamma_{i,r}, \gamma'_{i,r}, \ldots, \gamma_{i,2^s^r}, \gamma'_{i,2^s^r})_{s_i=1}^\infty \), a sequence of numbers \( L_1 < L_2 < \cdots \), and sequences of nodes \( (\xi_{i,r}, \ldots, \xi_{i,2^s^r}, r)_{s_i=1}^\infty \) such that

1. the branches \( \{\gamma_{i,r}, \gamma'_{i,r} : 1 \leq i \leq 2^n\} \) have separated at level \( L_r, r \geq 1 \),
2. \( \|\chi_{\{\xi_{i,r} : 1 \leq i \leq 2^n\}}\| > (2(1-\delta))^n+1 \) for any sequence of nodes \( (\xi_1, \ldots, \xi_{2^n}) \in S(\gamma_{i,r}, \gamma'_{i,r}, \ldots, \gamma_{i,2^s^r}, \gamma'_{i,2^s^r}; L_r) \),
3. \( (\xi_{1,r}, \ldots, \xi_{2^s^r}) \in S(\gamma'_{i,r}, \ldots, \gamma'_{i,2^s^r}; L_r) \), and
   \[ \|\|\chi_{\{\xi_{i,r} : 1 \leq i \leq 2^n\}}\| \leq (2(1-\delta))^{n}, r \geq 1; \]
4. \( \xi_{i,r} \in T(\varphi_L) \) whenever \( r > s \), and \( 1 \leq i \leq 2^n \).

It follows that if \( r > s \), then

\[
(\xi_{1,r}, \xi_{1,s}, \ldots, \xi_{2^s^r}, \xi_{2^s^r}) \in S(\gamma_{1,r}, \gamma'_{1,r}, \ldots, \gamma_{2^s^r}, \gamma'_{2^s^r}; L_s). \tag{12}
\]

Let \( x_r = (2(1-\delta))^{-n} \chi_{\{\xi_{1,r} : 1 \leq i \leq 2^n\}}, r \geq 1 \). By Item 3, \( \|\|x_r\|| \leq 1 \). Moreover, because of (12), if \( r > s \), then

\[
\|\|x_r - x_s\|| \geq \frac{2^n + 1}{e} \|\|x_r - x_s\||_X = 2^n e/(2(1-\delta))^{n} \geq 2 e.
\]

Thus \( (x_r) \) is \( 2e \)-separated in the norm \( \|\cdot\| \). By the choice of \( \delta \), there are \( r > s \) such that \( \|\|x_r + x_s\|| \leq 1 - \delta \). Therefore, \( \|\|\chi_{\{\xi_{1,r}, \xi_{1,s}, \ldots, \xi_{2^s^r}, \xi_{2^s^r}\}}\|| \leq (2(1-\delta))^{n+1} \). But this contradicts Item 2 and the condition (12).

**THEOREM 5.** There is no equivalent \( 2\)-NUC norm on \( X \).

**Proof.** In the notation of the statement of Proposition 4, we obtain, for each \( n \), nodes \( \psi_1, \ldots, \psi_{2^{n+1}} \) such that \( \|\|\chi_{\{\psi_{1,1} : 1 \leq i \leq 2^{n+1}\}}\|| \leq (2(1-\delta))^{n+1} \) and \( \|\|\chi_{\{\psi_{1,1} : 1 \leq i \leq 2^{n+1}\}}\||_X = 2^{n+1} \). Hence \( \|\|\cdot\|| \) cannot be an equivalent norm on \( X \).  

The proof that the space \( Y \) has no equivalent \( 1\)-\( \beta \) norm follows along similar lines. Suppose that \( \|\|\cdot\|| \) is an equivalent \( 1\)-\( \beta \) norm on \( Y \). We may assume \( e\|\|\cdot\||_Y \leq \|\|\cdot\|| \leq \|\|\cdot\||_Y \) for some \( e > 0 \). Let \( \delta = \delta(e) \) be the constant obtained from the definition of \( 1\)-\( \beta \) for the norm \( \|\|\cdot\|| \). Let \( n \in \mathbb{N} \cup \{0\} \) and denote the set \( \{\varphi \in T : |\varphi| = n\} \) by \( \Phi \).

**PROPOSITION 6.** For any \( m, 0 \leq m \leq n \), any subset \( \Phi' \) of \( \Phi \) with \( |\Phi'| = 2^m \), and any \( p \in \mathbb{N} \), there exists an acceptable set \( A \subseteq \cup_{\varphi \in \Phi} S_{\varphi} \) such that \( |A| = 2^m \), \( \min\{|\varphi| : \varphi \in A\} \geq p \), and \( \|\|\chi_{\Phi'}\|| \leq 2^m (1-\delta)^m \).
Proof. The case $m = 0$ is trivial. Suppose the proposition holds for some $m$, $0 \leq m < n$. Let $\Phi' \subseteq \Phi$, $|\Phi'| = 2^{m+1}$, and let $p \in \mathbb{N}$. Divide $\Phi'$ into disjoint subsets $\Phi_1$ and $\Phi_2$ such that $|\Phi_1| = |\Phi_2| = 2^m$. By the inductive hypothesis, there exist acceptable sets $B$ and $C_j$, $j \in \mathbb{N}$, such that $B \subseteq \cup_{\varphi \in \Phi_1} S_{\varphi}$, $|B| = 2^m$, $\min\{|\varphi| : \varphi \in B\} \geq p$, and $||\chi_B|| \leq 2^m(1 - \delta)^m$; and also $C_j \subseteq \cup_{\varphi \in \Phi_2} S_{\varphi}$, $|C_j| = 2^m$, $\min\{|\varphi| : \varphi \in C_1\} \geq p$, $C_j \ll C_{j+1}$, and $||\chi_{C_j}|| \leq 2^m(1 - \delta)^m$ for all $j \in \mathbb{N}$. It is easily verified that the sequence $(2^{-m}(1 - \delta)^{-m}\chi_{C_j})$ is $\epsilon$-separated and has norm bounded by 1 with respect to $|| \cdot ||$. It follows that there exists $j_0$ such that $2^{-m}(1 - \delta)^{-m}||\chi_B + \chi_{C_{j_0}}|| \leq 2(1 - \delta)$. The induction is completed by taking $A$ to be $B \cup C_{j_0}$.

Using the same argument as in Theorem 5, we obtain

Theorem 7. There is no equivalent $1$-$\beta$ norm on $Y$.

We close with the obvious problem.

Problem. For $k \geq 3$, can every $k$-NUC Banach space, respectively, $k$-$\beta$ Banach space, be equivalently renormed to be $(k - 1)$-$\beta$, respectively, $k$-NUC?

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