A gauge approach to an ordinal index of Baire one functions

by

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Abstract. We develop a calculus for the oscillation index of Baire one functions using gauges analogous to the modulus of continuity.

1. Introduction. Let $X$ be a metrizable space. A real-valued function $f$ is said to be of Baire class one (or simply, a Baire 1 function) if it is the pointwise limit of a sequence of continuous functions on $X$. The Baire Characterization Theorem states that if $X$ is Polish (separable completely metrizable), then $f : X \to \mathbb{R}$ is of Baire class one if and only if $f|_F$ has a point of continuity for every nonempty closed subset $F$ of $X$. Recently, Lee, Tang and Zhao \[5\] provided a characterization of Baire 1 functions in terms of gauges analogous to the modulus of continuity for continuous functions.

Theorem 1 \([5]\). Suppose that $f : X \to \mathbb{R}$ is a real-valued function on a complete separable metric space $(X,d)$. Then the function $f$ is of Baire class one if and only if for any $\varepsilon > 0$, there exists a positive function $\delta$ on $X$ such that

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad d(x, y) < \delta(x) \land \delta(y).$$

The Baire Characterization Theorem can be naturally quantified in terms of the oscillation index of Baire 1 functions \[2\]. This ordinal index was used in \[3\] to give a fine classification of Baire 1 functions. This line of investigation was continued by various authors: see e.g., \[1\], \[4\], \[6\], \[7\]. In this paper, we develop a method to compute the oscillation index of a Baire 1 function. The advantage of this approach is that it provides an easy-to-use calculus for the oscillation index that enables us to recover and refine all previously known results.

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Let $C$ denote the collection of all closed subsets of $X$. A derivation on $C$ is a map $D : C \to C$ such that (i) $D(P) \subseteq P$ for all $P \in C$ and (ii) $D(P) \subseteq D(Q)$ if $P \subseteq Q$. A derivation $D$ may be iterated in the usual manner. Let $D^0(P) = P$. For all $\alpha < \omega_1$, let

$$D^{\alpha+1}(P) = D(D^\alpha(P)).$$

If $\alpha$ is a countable limit ordinal, set

$$D^\alpha(P) = \bigcap_{\gamma<\alpha} D^\gamma(P).$$

The index of $D$, denoted by $\iota(D)$, is the least countable ordinal $\alpha$ such that $D^\alpha(X) = \emptyset$, if such an $\alpha$ exists, and $\omega_1$ otherwise.

Let $\varepsilon > 0$ and a function $f : X \to \mathbb{R}$ be given. For any $P \in C$, let $D(f,\varepsilon,P)$ be the set of all $x \in P$ such that for any neighborhood $U$ of $x$, there exist $x_1, x_2 \in P \cap U$ such that $d(f(x_1), f(x_2)) \geq \varepsilon$. For fixed $f$ and $\varepsilon > 0$, $D(f,\varepsilon,\cdot)$ is clearly a derivation on $C$. The oscillation index of $f$ is $\beta(f, \varepsilon) = \iota(D(f,\varepsilon,\cdot))$. The Baire Characterization Theorem that a real-valued function on a complete separable metric space is Baire 1 if and only if its oscillation index is countable. For a countable ordinal $\xi$, let $\mathcal{B}^\xi_1(X)$ denote the set of all Baire 1 functions on $X$ with $\beta(f) \leq \omega^\xi$.

Let $\pi : X \to \mathbb{R}$ be a function that is never zero. For any closed subset $H$ of $X$ let $Z(\pi,H)$ be the set of all $x \in H$ such that for any neighborhood $U$ of $x$, $\inf\{|\pi(y)| : y \in U \cap H\} = 0$. Clearly, given a fixed $\pi$, $Z(\pi,\cdot)$ is a derivation on $C$. We define the zero index $o(\pi)$ of $\pi$ to be the index of the derivation $Z(\pi,\cdot)$.

We conclude this section by stating two simple facts concerning derivations that will be used below. They are easily verified by using transfinite induction.

**Proposition 2.** Let $D$ and $E$ be derivations.

1. If $DP \subseteq EP$ for all $P$, then $D^\alpha P \subseteq E^\alpha P$ for all $P$ and all $\alpha < \omega_1$. Hence $\iota(D) \leq \iota(E)$.
2. Suppose that $D(P \cup Q) \subseteq D(P) \cup D(Q)$ for all $P$ and $Q$. Then $D^\alpha(P \cup Q) \subseteq D^\alpha(P) \cup D^\alpha(Q)$ for all $P,Q$ and all $\alpha < \omega_1$.

**2. Gauges and their zero indices.** Let $f$ be a real-valued function on a separable complete metric space $(X,d)$ and let $\varepsilon > 0$. A positive function $\delta : X \to \mathbb{R}^+$ is called an $\varepsilon$-gauge of $f$ if for any $x, y \in X$,

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad d(x,y) < \delta(x) \land \delta(y).$$
One can easily check that if \( \delta \) is an \( \varepsilon \)-gauge of \( f \), then \( D(f; \varepsilon, P) \subseteq Z(\delta, P) \) for every closed set \( P \). Applying Proposition 1, we obtain the following immediately.

**Proposition 3.** Let \( \varepsilon > 0 \) and a Baire 1 function \( f : X \to \mathbb{R} \) be given. If \( \delta : X \to \mathbb{R}^+ \) is an \( \varepsilon \)-gauge of \( f \), then \( \beta(f; \varepsilon) \leq o(\delta) \).

The utility of the gauge approach comes from the fact that the inequality in Proposition 3 can be made into an equality.

**Theorem 4.** Let \( f : X \to \mathbb{R} \) be a Baire 1 function. Then for any \( \varepsilon > 0 \) there exists an \( \varepsilon \)-gauge \( \delta \) of \( f \) such that \( o(\delta) = \beta(f; \varepsilon) \).

**Proof.** For simplicity, write \( D^\alpha \) for \( D^\alpha(f; \varepsilon, X) \) for all \( \alpha < \omega_1 \). For all \( \alpha < \beta(f) \) and \( x \in D^\alpha \setminus D^{\alpha+1} \), define \( \delta(x) \) to be the supremum of all \( \delta \) such that \( \delta \leq \min\{1, d(x, D^{\alpha+1})\} \) and \( |f(x_1) - f(x_2)| < \varepsilon \) for all \( x_1, x_2 \in B(x, \delta) \cap D^\alpha \). By definition of \( D^\alpha \) and \( D^{\alpha+1} \), \( \delta(x) > 0 \). We first show that \( \delta \) is an \( \varepsilon \)-gauge of \( f \). Indeed, for \( x, y \in X \) with \( d(x, y) < \delta(x) \wedge \delta(y) \), we may assume that \( x \in D^\alpha \setminus D^{\alpha+1} \), \( y \in D^\gamma \setminus D^{\gamma+1} \), with \( \alpha \leq \gamma \). Then \( d(x, y) < \delta(x) \leq d(x, D^{\alpha+1}) \) and hence \( y \notin D^{\alpha+1} \). Thus \( \alpha = \gamma \). Choose \( \delta_0 \) such that \( d(x, y) < \delta_0 < \delta(x) \). We have \( x, y \in B(x, \delta_0) \cap D^\alpha \) and hence \( |f(x) - f(y)| < \varepsilon \). This proves that \( \delta \) is an \( \varepsilon \)-gauge for \( f \). By Proposition 3, \( \beta(f; \varepsilon) \leq o(\delta) \).

We prove the reverse inequality by showing inductively that \( Z^\alpha(\delta, X) \subseteq D^\alpha \) for all \( \alpha \leq \beta(f) \). The claim is obvious if \( \alpha \) is 0 or a limit ordinal. Suppose that \( Z^\alpha(\delta, X) \subseteq D^\alpha \) for some \( \alpha < \beta(f) \). Take any \( x \in Z^\alpha(\delta, X) \setminus D^{\alpha+1} \). By the inductive hypothesis, \( x \in D^\alpha \setminus D^{\alpha+1} \). Choose \( \delta_0 \) so that \( 0 < \delta_0 < \delta(x) \). For any \( y \in B(x, \delta_0) \cap Z^\alpha(\delta, X) \subseteq D^\alpha \setminus D^{\alpha+1} \), let \( \delta' = \delta_0 - d(x, y) \). Then \( 0 < \delta' \leq \delta(x) \leq 1 \). Also,

\[
  d(y, D^{\alpha+1}) \geq d(x, D^{\alpha+1}) - d(x, y) > \delta_0 - d(x, y) = \delta'.
\]

If \( x_1, x_2 \in B(y, \delta') \cap D^\alpha \), then \( x_1, x_2 \in B(x, \delta_0) \cap D^\alpha \) and thus we have \( |f(x_1) - f(x_2)| < \varepsilon \). By definition, \( \delta(y) \geq \delta' = \delta_0 - d(x, y) \). This proves that \( x \notin Z^{\alpha+1}(\delta, X) \) and completes the induction.

**3. Some computational tools.** Before we see some applications of Theorem 4, we establish some computational tools for estimating the zero index. The next lemma is implicitly contained in the proof of 3 Lemma 5).

**Lemma 5.** Let \( D_1 \) and \( D_2 \) be derivations. If \( D \) is a derivation so that \( D(P \cup Q) \subseteq D(P) \cup D(Q) \) and \( D(P) \subseteq D_1(P) \cup D_2(P) \) for all \( P \) and \( Q \), then \( D_1^{\xi}(P) \subseteq D_1^{\xi}(P) \cup D_2^{\xi}(P) \) for \( \xi < \omega_1 \) and all \( P \).

**Proof.** We prove the lemma by induction on \( \xi \). By hypothesis, the statement holds for \( \xi = 0 \). Since the sequences \( (D_1^\alpha P)_\alpha, i = 1, 2 \), are non-increasing, the inductive step for a limit ordinal \( \xi \) is clear. Assume that the
statement holds for some $\xi < \omega_1$. We need to show that
\[ D_{\omega^{\xi+1}}(P) \subseteq D_{1}^{\omega^{\xi+1}}(P) \cup D_{2}^{\omega^{\xi+1}}(P). \]
To this end, we prove that
\[ (3.1) \quad D_{\omega^{\xi-2n}}(P) \subseteq D_{1}^{\omega^{\xi-n}}(P) \cup D_{2}^{\omega^{\xi-n}}(P) \]
for all $n \in \mathbb{N}$. For each $s \in 2^k = \{(\varepsilon_1, \ldots, \varepsilon_k) : \varepsilon_i = 1 \text{ or } 2\}$, $k \in \mathbb{N}$, define $P_s$ as follows:
\[ P_1 = D_{1}^{\omega^{\xi}}(P), \quad P_2 = D_{2}^{\omega^{\xi}}(P) \]
and
\[ P_{s \wedge 1} = D_{1}^{\omega^{\xi}}(P_s), \quad P_{s \wedge 2} = D_{2}^{\omega^{\xi}}(P_s). \]
To prove (3.1), we first prove that
\[ (3.2) \quad D_{\omega^{\xi-k}}(P) \subseteq \bigcup_{s \in 2^k} P_s \]
for every $k \in \mathbb{N}$. By the inductive assumption, (3.2) is true for $k = 1$. Assume that it is true for some $k \in \mathbb{N}$. Then
\[ \begin{align*}
D_{\omega^{\xi-(k+1)}}(P) &= D_{\omega^{\xi}}(D_{\omega^{\xi-k}}(P)) \subseteq D_{\omega^{\xi}}\left( \bigcup_{s \in 2^k} P_s \right) \\
&\subseteq \bigcup_{s \in 2^k} D_{\omega^{\xi}}(P_s) \quad \text{by Proposition 2(2)} \\
&\subseteq \bigcup_{s \in 2^k} (D_{1}^{\omega^{\xi}}(P_s) \cup D_{2}^{\omega^{\xi}}(P_s)) \quad \text{by the inductive hypothesis} \\
&= \left( \bigcup_{s \in 2^k} P_{s \wedge 1} \right) \cup \left( \bigcup_{s \in 2^k} P_{s \wedge 2} \right) = \bigcup_{s \in 2^{k+1}} P_s.
\end{align*} \]
Thus (3.2) is verified by induction. Therefore, for all $n \in \mathbb{N}$,
\[ D_{\omega^{\xi-2n}}(P) \subseteq \bigcup_{s \in 2^{2n}} P_s \]
\[ \subseteq \bigcup\{P_s : s \in 2^{2n} \text{ and } \text{card}\{k : s(k) = 1\} \geq n\} \]
\[ \cup \bigcup\{P_s : s \in 2^{2n} \text{ and } \text{card}\{k : s(k) = 2\} \geq n\}. \]
If $s$ takes the value 1 at least $n$ times, then $P_s \subseteq D_{1}^{\omega^{\xi-n}}(P)$. Similarly if $s$ takes the value 2 at least $n$ times, then $P_s \subseteq D_{2}^{\omega^{\xi-n}}(P)$. Thus (3.1) is proved.

Taking intersection over all $n$ in (3.1) gives
\[ D_{\omega^{\xi+1}}(P) \subseteq D_{1}^{\omega^{\xi+1}}(P) \cup D_{2}^{\omega^{\xi+1}}(P). \]
Theorem 6. If \( \pi_1, \pi_2 : X \to \mathbb{R} \) are positive functions with \( o(\pi_1) \leq \omega^\xi \) and \( o(\pi_2) \leq \omega^\xi \) for some \( \xi < \omega_1 \), then \( o(\pi_1 \land \pi_2) \leq \omega^\xi \).

Proof. Let \( \pi = \pi_1 \land \pi_2 \). Consider the derivations \( D(P) = Z^\alpha(\pi, P) \), \( D_i(P) = Z^\alpha(\pi_i, P) \), \( i = 1, 2 \). It is easy to see that these derivations satisfy the hypotheses of Lemma 5. Therefore, the conclusion follows by the same lemma. ■

For any \( \alpha < \omega_1 \), set \([\alpha] = \inf \{\omega^\xi : \alpha \leq \omega^\xi\}\).

Remark. For any Baire 1 function \( f \), we may apply Theorems 4 and 6 to obtain \( \delta_f : \mathbb{R} \times X \to \mathbb{R}^+ \) such that

(a) \( \delta_f(\varepsilon, \cdot) \) is an \( \varepsilon \)-gauge for \( f \),
(b) \( \delta_f(\cdot, x) \) is a nondecreasing function for all \( x \in X \),
(c) \( o(\delta_f(\varepsilon, \cdot)) \leq [\beta(f)] \).

Indeed, according to Theorem 4, for each \( n \in \mathbb{N} \), there is an \( 1/n \)-gauge \( \delta_n \) of \( f \) such that \( o(\delta_n) = \beta(f, 1/n) \leq [\beta(f)] \). Set \( \delta_f(1/n, \cdot) = \delta_1 \land \cdots \land \delta_n \). For all \( x \in X \), \( (\delta_f(1/n, x))_n \) is nonincreasing. Clearly, \( \delta_f(1/n, \cdot) \) is an \( \varepsilon \)-gauge for \( f \) if \( 1/n \leq \varepsilon \). Also by Theorem 6, \( o(\delta_f(1/n, \cdot)) \leq [\beta(f)] \). Given \( \varepsilon > 0 \), set \( \delta_f(\varepsilon, \cdot) = \delta_f(1/n, \cdot) \) if \( \varepsilon \in [1/n, 1/(n-1)] \) \( (1/0 = +\infty) \). Then \( \delta_f \) has the desired properties. A function \( \delta_f \) satisfying (a)–(c) will be called a \( B \)-gauge for \( f \).

Proposition 7. Let \( \phi \) and \( \psi \) be positive functions.

1. If there is \( 0 < c < \infty \) such that \( \psi \leq c\phi \), then \( o(\phi) \leq o(\psi) \).
2. \( o(\phi^2) = o(\phi) \).
3. \( o(\phi \psi) \leq [o(\phi)] \lor [o(\psi)] \).

Proof. (1) and (2) are clear. For (3), observe that \( (\phi \land \psi)^2 \leq \phi \psi \). So by (1), (2) and Theorem 6,

\[ o(\phi \psi) \leq o((\phi \land \psi)^2) = o(\phi \land \psi) \leq [o(\phi)] \lor [o(\psi)]. \]

Let \( f \) be a real-valued function on \( X \). For any closed subset \( H \) of \( X \), let \( U(f, H) = \{ x \in H : \limsup_{y \to x, y \in H} |f(y)| = \infty \} \).

Then \( U(f, \cdot) \) is clearly a derivation on \( C \). The unboundedness index \( u(f) \) of \( f \) is defined as the index of the derivation \( U(f, \cdot) \).

The following proposition follows easily from the fact that \( Z(1/(\phi + a), \cdot) = U(\phi, \cdot) \) for any \( a > 0 \).

Proposition 8. Let \( \phi \) be a positive function and \( a > 0 \). Then

\[ o\left(\frac{1}{\phi + a}\right) = u(\phi). \]
4. Applications. It was shown in [3, Section 2] that $B^\xi_1(K)$, the space of bounded Baire 1 functions $f$ on a compact metric space $K$ with $\beta(f) \leq \omega^\xi$, is a Banach algebra under pointwise operations. In [6], extension to unbounded Baire 1 functions was considered. It was found that for a compact metric space $K$, if $f \in B^\xi_1(K)$ and $g \in B^{\xi_2}_1(K)$ then $f + g \in B^{\xi_1 \lor \xi_2}_1(K)$ and $fg \in B^{\xi_1}_1(K)$, where $\xi = \max\{\xi_1 + \xi_2, \xi_2 + \xi_1\}$. (See [6], Proposition 4.4 and Theorem 6.5 respectively.) In this section, we show that generalized versions of these results on a Polish space may be obtained via the gauge approach using Theorem 4 and the computational tools in §3. In fact, the gauge approach allows us to obtain estimates of the oscillation index of fairly general combinations of two Baire 1 functions.

Lemma 9. Suppose that $f : X \to \mathbb{R}$ is a Baire 1 function and $\delta_f$ is a $B$-gauge for $f$. Let $\xi = \sup_{\varepsilon > 0} o(\delta_f(\varepsilon, \cdot))$. If $\gamma : X \to \mathbb{R}$ is a positive function, then

$$o(\delta_f(\gamma(\cdot), \cdot)) \leq \xi \cdot o(\gamma).$$

Proof. If $x \notin Z(\gamma, H)$, then there exist $a > 0$ and a neighborhood $U$ of $x$ such that $\gamma(y) > a$ for all $y \in U \cap H$. It follows from the monotonicity of $\delta_f$ in the first variable that

$$U \cap Z(\delta_f(\gamma(\cdot), \cdot), H) \subseteq Z(\delta_f(a, \cdot), H).$$

By induction, we have

$$U \cap Z^\alpha(\delta_f(\gamma(\cdot), \cdot), H) \subseteq Z^\alpha(\delta_f(a, \cdot), H)$$

for all $\alpha < \omega_1$. In particular,

$$U \cap Z^\xi(\delta_f(\gamma(\cdot), \cdot), H) = \emptyset.$$

It follows that $x \notin Z^\xi(\delta_f(\gamma(\cdot), \cdot), H)$. Hence

$$Z^\xi(\delta_f(\gamma(\cdot), \cdot), H) \subseteq Z(\gamma, H).$$

By Proposition 2(1), we have

$$Z^{\xi \cdot o(\gamma)}(\delta_f(\gamma(\cdot), \cdot), H) \subseteq Z^{\alpha}(\gamma, H)$$

for all $\alpha < \omega_1$. In particular,

$$Z^{\xi \cdot o(\gamma)}(\delta_f(\gamma(\cdot), \cdot), X) = \emptyset.$$

It follows that $o(\delta_f(\gamma(\cdot), \cdot)) \leq \xi \cdot o(\gamma)$. 

Let $I, J \subseteq \mathbb{R}$, and let $f : X \to I$ and $g : X \to J$. Given $\varepsilon > 0$, a function $F : I \times J \to \mathbb{R}$ is said to satisfy property $(\ast)_\varepsilon$ with respect to $(f, g)$ if there are $h_1 : I \to \mathbb{R}$ and $h_2 : J \to \mathbb{R}$ such that

$$|F(f(x), g(x)) - F(f(y), g(y))| < \varepsilon$$
whenever

\[ |f(x) - f(y)| < h_1(x) \lor h_1(y) \quad \text{and} \quad |g(x) - g(y)| < h_2(x) \lor h_2(y). \]

**Theorem 10.** Suppose that \( f,g : X \to \mathbb{R} \) are functions of Baire class one with \( f(X) \subseteq I \) and \( g(X) \subseteq J \). If \( F : I \times J \to \mathbb{R} \) satisfies property \((*)_\varepsilon\) with respect to \((f,g)\), then \( \beta(F(f,g),\varepsilon) \leq [\beta(f)][o(h_1)] \lor [\beta(g)][o(h_2)]. \)

**Proof.** Let \( \varepsilon > 0 \) and let \( h_1,h_2 \) be given by property \((*)_\varepsilon\). Set \( \delta(x) = \delta_f(h_1(x),x) \land \delta_g(h_2(x),x) \), where \( \delta_f \) and \( \delta_g \) are B-gauges of \( f \) and \( g \) respectively. We first show that \( \delta \) is an \( \varepsilon \)-gauge for \( F(f,g) \). Indeed, if \( d(x,y) < \delta(x) \land \delta(y) \), then

\[
d(x,y) < \delta_f(h_1(x),x) \land \delta_f(h_1(y),y) \\
\leq \delta_f(h_1(x) \lor h_1(y),x) \land \delta_f(h_1(x) \lor h_1(y),y).
\]

Therefore,

\[
|f(x) - f(y)| < h_1(x) \lor h_1(y).
\]

Likewise,

\[
|g(x) - g(y)| < h_2(x) \lor h_2(y).
\]

By property \((*)_\varepsilon\),

\[
|F(f(x),g(y)) - F(f(x),g(x))| < \varepsilon.
\]

So \( \delta \) is an \( \varepsilon \)-gauge for \( F(f,g) \). It remains to estimate \( o(\delta) \). By the definitions of \( \delta_f \) and \( \delta_g \),

\[
\sup_{\varepsilon > 0} o(\delta_f(\varepsilon,\cdot)) \leq [\beta(f)] \quad \text{and} \quad \sup_{\varepsilon > 0} o(\delta_g(\varepsilon,\cdot)) \leq [\beta(g)].
\]

By Lemma \[9\]

\[
o(\delta_f(h_1(\cdot),\cdot)) \leq [\beta(f)][o(h_1)] \quad \text{and} \quad o(\delta_g(h_2(\cdot),\cdot)) \leq [\beta(g)][o(h_2)].
\]

Therefore, by Theorem \[6\]

\[
o(\delta) = o(\delta_f(h_1(\cdot),\cdot) \land \delta_g(h_2(\cdot),\cdot)) \leq [\beta(f)][o(h_1)] \lor [\beta(g)][o(h_2)].
\]

The desired conclusion follows from Proposition \[3\] \( \blacksquare \)

**Theorem 11.** If \( F : I \times J \to \mathbb{R} \) is uniformly continuous, then \( \beta(F(f,g)) \leq [\beta(f)] \lor [\beta(g)]. \)

**Proof.** Since \( F \) is uniformly continuous on \( I \times J \), for any \( \varepsilon > 0 \), there are positive constant functions \( h_1 \) and \( h_2 \) that witness the fact that \( F \) satisfies \((*)_\varepsilon\) with respect to \((f,g)\). By Theorem \[10\], \( \beta(F(f,g),\varepsilon) \leq [\beta(f)][o(h_1)] \lor [\beta(g)][o(h_2)] = [\beta(f)] \lor [\beta(g)] \), as \( o(h_1) = o(h_2) = 1 \). \( \blacksquare \)
COROLLARY 12 ([3, Section 2], [6, Proposition 4.4]). If \( f, g \in \B_\xi^1(X) \), then \( f + g, |f| \in \B_\xi^1(X) \). Furthermore, when \( f, g \) are bounded, \( fg \in \B_\xi^1(X) \).

Proof. Since \( F_1(u, v) = u + v \) and \( F_2(u, v) = |u| \) are uniformly continuous functions, the first assertion follows easily from Theorem 11. The second assertion is also clear as the product function is uniformly continuous on bounded sets. ■

The next result improves the estimate given in [6, Theorem 6.5].

THEOREM 13. \( \beta(fg) \leq [\beta(f)][u(g)] \lor [\beta(g)][u(f)]. \)

Proof. Let \( 0 < \varepsilon < 1 \). Then \( F(u, v) = uv \) has property \((\ast)_\varepsilon\) with respect to \((f, g)\) with

\[
h_1(x) = \frac{\varepsilon}{3(|g(x)| + 1)} \quad \text{and} \quad h_2(x) = \frac{\varepsilon}{3(|f(x)| + 1)}.
\]

Indeed, if

\[
|f(x) - f(y)| < h_1(x) \lor h_1(y) \quad \text{and} \quad |g(x) - g(y)| < h_2(x) \lor h_2(y),
\]

then

\[
|f(x) - f(y)| < \frac{\varepsilon}{3(|g(x)| + 1)} \lor \frac{\varepsilon}{3(|g(y)| + 1)} \leq 1
\]

and

\[
|g(x) - g(y)| < \frac{\varepsilon}{3(|f(x)| + 1)} \lor \frac{\varepsilon}{3(|f(y)| + 1)} \leq 1.
\]

Since \(|g(y)| \leq |g(y) - g(x)| + |g(x)| < 1 + |g(x)|\), it follows that

\[
|g(y)| |f(x) - f(y)| \leq \frac{\varepsilon |g(y)|}{3(|g(x)| + 1)} \lor \frac{\varepsilon |g(y)|}{3(|g(y)| + 1)} < \frac{\varepsilon}{2}.
\]

Similarly,

\[
|f(x)| |g(x) - g(y)| < \frac{\varepsilon}{2}.
\]

Therefore,

\[
|F(f(x), g(x)) - F(f(y), g(y))| = |f(x)g(x) - f(y)g(y)|
\]

\[
\leq |g(y)| |f(x) - f(y)| + |f(x)| |g(x) - g(y)|
\]

\[
< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Thus \( F \) has property \((\ast)_\varepsilon\) with respect to \((f, g)\). By Theorem 10 and Proposition 8,

\[
\beta(fg, \varepsilon) \leq [\beta(f)]\left[ o\left(\frac{\varepsilon}{3(|g| + 1)}\right) \right] \lor [\beta(g)]\left[ o\left(\frac{\varepsilon}{3(|f| + 1)}\right) \right]
\]

\[
= [\beta(f)][u(g)] \lor [\beta(g)][u(f)].
\]

Taking supremum over all \( \varepsilon > 0 \) completes the proof. ■
Indeed, if

\[ (fg) \leq \omega^\xi, \]

where \( \xi = (\xi_1 + \xi_2) \lor (\xi_2 + \xi_1) \).

**Proof.** Since \( u(f) \leq \beta(f) \) and \( u(g) \leq \beta(g) \), the result follows easily. \( \blacksquare \)

Note that an example has been constructed in [6] to show that the above result is optimal.

**Theorem 15.** If \( g(x) \neq 0 \) for all \( x \in X \), then

\[ \beta\left(\frac{f}{g}\right) \leq [\beta(f)][\omega(g)] \lor [\beta(g)]([\omega(g)] \lor [u(f)]). \]

**Proof.** Let \( 0 < \varepsilon < 1 \). Then \( F(u, v) = u/v \) has property \((\ast)_\varepsilon\) with respect to \((f, g)\) with

\[ h_1(x) = \frac{\varepsilon |g(x)|}{8} \land 1, \quad h_2(x) = \frac{\varepsilon}{3}\left(\frac{|g(x)|^2}{|f(x)| + 4} \lor |g(x)|\right). \]

Indeed, if

\[ |f(x) - f(y)| < h_1(x) \lor h_1(y) \quad \text{and} \quad |g(x) - g(y)| < h_2(x) \lor h_2(y), \]

then \( |f(x)| < |f(y)| + 1, \)

\[ |f(x) - f(y)| < \frac{\varepsilon |g(x)|}{8} \lor \frac{\varepsilon |g(y)|}{8} \quad \text{and} \quad |g(x) - g(y)| < \frac{|g(x)|}{3} \lor \frac{|g(y)|}{3}. \]

It follows that

\[ (4.1) \quad \frac{1}{2}|g(y)| \leq |g(x)| \leq 2|g(y)| \]

and thus

\[ \frac{|f(x) - f(y)|}{|g(y)|} \leq \frac{\varepsilon}{4}. \]

Also, from \( |g(x) - g(y)| \leq \frac{\varepsilon}{3} \frac{|g(x)|^2}{|f(x)| + 4} \lor \frac{\varepsilon}{3} \frac{|g(y)|^2}{|f(y)| + 4} \), it follows that

\[ |f(x)| |g(x) - g(y)| \leq \frac{\varepsilon}{3} \frac{|g(x)|^2 |f(x)|}{|f(x)| + 4} \lor \frac{\varepsilon}{3} \frac{|g(y)|^2 |f(x)|}{|f(y)| + 4} \]

\[ \leq \frac{\varepsilon}{3} |g(x)|^2 \lor \frac{\varepsilon}{3} |g(y)|^2 \]

\[ \leq \frac{2\varepsilon}{3} |g(x)| |g(y)| \quad \text{by (4.1)}. \]

Thus

\[ \frac{|f(x)| |g(x) - g(y)|}{|g(x)| |g(y)|} \leq \frac{2\varepsilon}{3}. \]
Therefore
\[
\left| \frac{f(x) - f(y)}{g(x) - g(y)} \right| \leq \left| \frac{f(x) - f(x)}{g(x) - g(y)} \right| + \left| \frac{f(x) - f(y)}{g(x) - g(y)} \right|
\]
\[
= |f(x)| \left| \frac{1}{g(x)} - \frac{1}{g(y)} \right| + \frac{1}{|g(y)|} |f(x) - f(y)|
\]
\[
= \frac{|f(x)| |g(x) - g(y)|}{|g(x)| |g(y)|} + \frac{|f(x) - f(y)|}{|g(y)|}
\]
\[
\leq 2\varepsilon/3 + \varepsilon/4 < \varepsilon.
\]

Hence \( F(u, v) = u/v \) has property \((*)_\varepsilon\) with respect to \((f, g)\). By Theorems 10 and 9 and Propositions 7 and 8,
\[
\beta \left( \frac{f}{g} \right) \leq \beta(f) \left[ o \left( \frac{|g|}{8} \wedge 1 \right) \right] \vee \beta(g) \left[ o \left( \frac{\varepsilon}{3} \left( \frac{|g|^2}{|f| + 4} \wedge |g| \right) \right) \right]
\]
\[
\leq \beta(f) [o(g)] \vee \beta(g) \left[ o \left( \frac{|g|^2}{|f| + 4} \right) \right] \vee [o(g)]
\]
\[
\leq \beta(f) [o(g)] \vee \beta(g) \left[ [o(g)] \vee [u(f)] \right] \vee [o(g)]
\]
\[
\leq \beta(f) [o(g)] \vee \beta(g) \left[ [o(g)] \vee [u(f)] \right].
\]

If \( o(\pi_1) = m \) and \( o(\pi_2) = n \) are both finite, then the proof of Theorem 6 yields \( o(\pi_1 \wedge \pi_2) \leq m + n - 1 \). Now suppose that \( f \) and \( g \) are Baire 1 functions with \( \beta(f) = m \) and \( \beta(g) = n \). By Theorem 4, for any \( \varepsilon > 0 \), there are \( \varepsilon/2 \)-gauges \( \delta_f \) and \( \delta_g \) of \( f \) and \( g \) respectively such that \( o(\delta_f) \leq m \) and \( o(\delta_g) \leq n \). It is clear that \( \delta = \delta_f \wedge \delta_g \) is an \( \varepsilon \)-gauge for \( f + g \). Since \( o(\delta) = o(\delta_f \wedge \delta_g) \leq m + n - 1 \), we see that \( \beta(f + g, \varepsilon) \leq m + n - 1 \) for all \( \varepsilon > 0 \). The same argument goes for \( f \vee g, f \wedge g, \) and for bounded \( f \) and \( g \), \( fg \). This recovers Theorem 1.3 of [1].

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